BMS Institute of Technology and Management, Bengaluru

Department of Electrical and Electronics Engineering

Study Material

18EE32

Electric Circuit Analysis

Contents

Chapter 1 Circuit Concepts and Network Simplification Techniques

1.1 Introduction

Today we live in a predominantly electrical world. Electrical technology is a driving force in the changes that are occurring in every engineering discipline. For example, surveying is now done using lasers and electronic range finders.

Circuit analysis is the foundation for electrical technology. An indepth knowledge of circuit analysis provides an understanding of such things as cause and effect, feedback and control and, stability and oscillations. Moreover, the critical importance is the fact that the concepts of electrical circuit can also be applied to economic and social systems. Thus, the applications and ramifications of circuit analysis are immense.

In this chapter, we shall introduce some of the basic quantities that will be used throughout the text. *An electric circuit or electric network is an interconnection of electrical elements linked together in a closed path so that an electric current may continuously flow.* Alternatively, an electric circuit is essentially a pipe-line that facilitates the transfer of charge from one point to another.

1.2 Current, voltage, power and energy

The most elementary quantity in the analysis of electric circuits is the electric charge. Our interest in electric charge is centered around its motion results in an energy transfer. Charge is the intrinsic property of matter responsible for electrical phenomena. The quantity of charge q can be expressed in terms of the charge on one electron. which is -1.602×10^{-19} coulombs. Thus, -1 coulomb is the charge on 6.24×10^{18} electrons. The current flows through a specified area A and is defined by the electric charge passing through that area per unit time. Thus we define q as the charge expressed in coulombs.

Charge is the quantity of electricity responsible for electric phenomena.

The time rate of change constitutes an electric current. Mathemetically, this relation is expressed as

$$i(t) = \frac{dq(t)}{dt} \tag{1.1}$$

or

$$q(t) = \int_{-\infty}^{t} i(x)dx \tag{1.2}$$

The unit of current is ampere(A); an ampere is 1 coulomb per second.

Current is the time rate of flow of electric charge past a given point.

The basic variables in electric circuits are current and voltage. If a current flows into terminal a of the element shown in Fig. 1.1, then a voltage or potential difference exists between the two terminals a and b. Normally, we say that a voltage exists across the element.

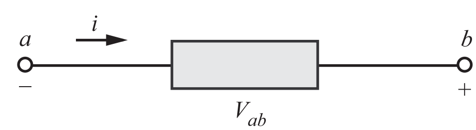


Figure 1.1 Voltage across an element

The voltage across an element is the work done in moving a positive charge of 1 coulomb from first terminal through the element to second terminal. The unit of voltage is volt, V or Joules per coulomb.

We have defined voltage in Joules per coulomb as the energy required to move a positive charge of 1 coulomb through an element. If we assume that we are dealing with a differential amount of charge and energy,

then

$$v = \frac{dw}{dq} \tag{1.3}$$

Multiplying both the sides of equation (1.3) by the current in the element gives

$$vi = \frac{dw}{dq}\left(\frac{dq}{dt}\right) \quad \Rightarrow \quad \frac{dw}{dt} = p \tag{1.4}$$

which is the time rate of change of energy or power measured in Joules per second or watts (W).

p could be either positive or negative. Hence it is imperative to give sign convention for power. If we use the signs as shown in Fig. 1.2., the current flows out of the terminal indicated by x, which shows the positive sign for the voltage. In this case, the element is said to provide energy to the charge as it moves through. Power is then provided by the element.

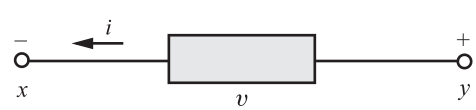


Figure 1.2 An element with the current leaving from the terminal with a positive voltage sign

Conversely, power absorbed by an element is $p = vi$, when i is entering through the positive voltage terminal.

Energy is the capacity to perform work. Energy and power are related to each other by the following equation:

$$Energy = w = \int_{-\infty}^{t} p\ dt$$

EXAMPLE 1.1

Consider the circuit shown in Fig. 1.3 with $v = 8e^{-}t$ V and $i = 20e^{-t}$ A for $t \geq 0$. Find the power absorbed and the energy supplied by the element over the first second of operation. we assume that v and i are zero for $t < 0$.

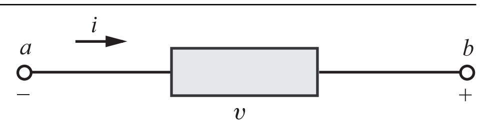


Figure 1.3

SOLUTION

The power supplied is

$$\begin{aligned} p &= vi = (8e^{-t})(20e^{-t}) \\ &= 160e^{-2t}\ \text{W} \end{aligned}$$

The element is providing energy to the charge flowing through it.

The energy supplied during the first seond is

$$\begin{aligned} w &= \int_0^1 p\ dt = \int_0^1 160e^{-2t}dt \\ &= 80(1 - e^{-2}) = \mathbf{69.17\ Joules} \end{aligned}$$

1.3 Linear, active and passive elements

A linear element is one that satisfies the principle of superposition and homogeneity.

In order to understand the concept of superposition and homogeneity, let us consider the element shown in Fig. 1.4.

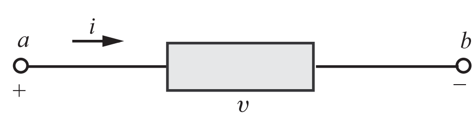


Figure 1.4 An element with excitation i and response v

The excitation is the current, i and the response is the voltage, v. When the element is subjected to a current i_1, it provides a response v_1. Furthermore, when the element is subjected to a current i_2, it provides a response v_2. If the principle of superposition is true, then the excitation $i_1 + i_2$ must produce a response $v_1 + v_2$.

Also, it is necessary that the magnitude scale factor be preserved for a linear element. If the element is subjected to an excitation βi where β is a constant multiplier, then if principle of homogencity is true, the response of the element must be βv.

We may classify the elements of a circuir into categories, passive and active, depending upon whether they absorb energy or supply energy.

1.3.1 Passive Circuit Elements

An element is said to be passive if the total energy delivered to it from the rest of the circuit is either zero or positive.

Then for a passive element, with the current flowing into the positive (+) terminal as shown in Fig. 1.4 this means that

$$w = \int_{-\infty}^{t} vi\ dt \geq 0$$

Examples of passive elements are resistors, capacitors and inductors.

1.3.1.A Resistors

Resistance is the physical property of an element or device that impedes the flow of current; it is represented by the symbol R. Resistance of a wire element is calculated using the relation:

Figure 1.5 Symbol for a resistor R

$$R = \frac{\rho l}{A} \tag{1.5}$$

where A is the cross-sectional area, ρ the resistivity, and l the length of the wire. The practical unit of resistance is ohm and represented by the symbol Ω.

An element is said to have a resistance of 1 ohm, if it permits 1A of current to flow through it when 1V is impressed across its terminals.

Ohm's law, which is related to voltage and current, was published in 1827 as

$$v = Ri \tag{1.6}$$

or

$$R = \frac{v}{i}$$

where v is the potential across the resistive element, i the current through it, and R the resistance of the element.

The power absorbed by a resistor is given by

$$p = vi = v\left(\frac{v}{R}\right) = \frac{v^2}{R} \tag{1.7}$$

Alternatively,

$$p = vi = (iR)i = i^2 R \tag{1.8}$$

Hence, the power is a nonlinear function of current i through the resistor or of the voltage v across it.

The equation for energy absorbed by or delivered to a resistor is

$$w = \int_{-\infty}^{t} p d\tau = \int_{-\infty}^{t} i^2 R\ d\tau \tag{1.9}$$

Since i^2 is always positive, the energy is always positive and the resistor is a passive element.

1.3.1.B Inductors

Whenever a time-changing current is passed through a coil or wire, the voltage across it is proportional to the rate of change of current through the coil. This proportional relationship may be expressed by the equation

$$v = L\frac{di}{dt} \tag{1.10}$$

Where L is the constant of proportionality known as inductance and is measured in Henrys (H). Remember v and i are both funtions of time.
Let us assume that the coil shown in Fig. 1.6 has N turns and the core material has a high permeability so that the magnetic fluk ϕ is connected within the area A. The changing flux creates an induced voltage in each turn equal to the derivative of the flux ϕ, so the total voltage v across N turns is

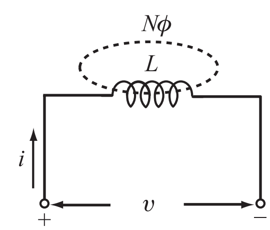


Figure 1.6 Model of the inductor

$$v = N\frac{d\phi}{dt} \tag{1.11}$$

Since the total flux $N\phi$ is proportional to current in the coil, we have

$$N\phi = Li \tag{1.12}$$

Where L is the constant of proportionality. Substituting equation (1.12) into equation(1.11), we get

$$v = L\frac{di}{dt}$$

The power in an inductor is

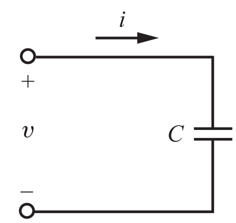
$$p = vi = L\left(\frac{di}{dt}\right)i$$

The energy stored in the inductor is

$$\begin{aligned} w &= \int_{-\infty}^{t} p \; d\tau \\ &= L\int_{i(-\infty)}^{i(t)} i \; di = \frac{1}{2}Li^2 \text{ Joules} \end{aligned} \tag{1.13}$$

Note that when $t = -\infty, i(-\infty) = 0$. Also note that $w(t) \geq 0$ for all $i(t)$, so the inductor is a passive element. The inductor does not generate energy, but only stores energy.

1.3.1.C Capacitors



1.7 Circuit symbol for a capacitor

A capacitor is a two-terminal element that is a model of a device consisting of two conducting plates seperated by a dielectric material. Capacitance is a measure of the ability of a deivce to store energy in the form of an electric field. ***Capacitance is defined as the ratio of the charge stored to the voltage difference between the two conducting plates or wires,***

$$C = \frac{q}{v}$$

The current through the capacitor is given by

$$i = \frac{dq}{dt} = C\frac{dv}{dt} \tag{1.14}$$

The energy stored in a capacitor is

$$w = \int_{-\infty}^{t} vi\ d\tau$$

Remember that v and i are both functions of time and could be written as $v(t)$ and $i(t)$.

Since
$$i = C\frac{dv}{dt}$$
we have
$$\begin{aligned} w &= \int_{-\infty}^{t} v\ C\frac{dv}{d\tau}\ d\tau \\ &= C\int_{v(-\infty)}^{v(t)} v\ dv = \frac{1}{2}Cv^2\Big|_{v(-\infty)}^{v(t)} \end{aligned}$$

Since the capacitor was uncharged at $t = -\infty$, $v(-\infty) = 0$.

Hence
$$\begin{aligned} w &= w(t) \\ &= \frac{1}{2}Cv^2(t) \text{ Joules} \end{aligned} \tag{1.15}$$

Since $q = Cv$, we may write

$$w(t) = \frac{1}{2C}q^2(t) \text{ Joules} \tag{1.16}$$

Note that since $w(t) \geq 0$ for all values of $v(t)$, the element is said to be a passive element.

1.3.2 Active Circuit Elements (Energy Sources)

An active two-terminal element that supplies energy to a circuit is a source of energy. An ideal voltage source is a circuit element that maintains a prescribed voltage across the terminals regardless of the current flowing in those terminals. Similarly, an ideal current source is a circuit element that maintains a prescribed current through its terminals regardless of the voltage across those terminals.

These circuit elements do not exist as practical devices, they are only idealized models of actual voltage and current sources.

Ideal voltage and current sources can be further described as either independent sources or dependent sources. An independent source establishes a voltage or current in a circuit without relying on voltages or currents elsewhere in the circuit. The value of the voltage or current supplied is specified by the value of the independent source alone. In contrast, a dependent source establishes a voltage or current whose value depends on the value of the voltage or current elsewhere in the circuit. We cannot specify the value of a dependent source, unless you know the value of the voltage or current on which it depends.

The circuit symbols for ideal independent sources are shown in Fig. 1.8.(a) and (b). Note that a circle is used to represent an independent source. The circuit symbols for dependent sources are shown in Fig. 1.8.(c), (d), (e) and (f). A diamond symbol is used to represent a dependent source.

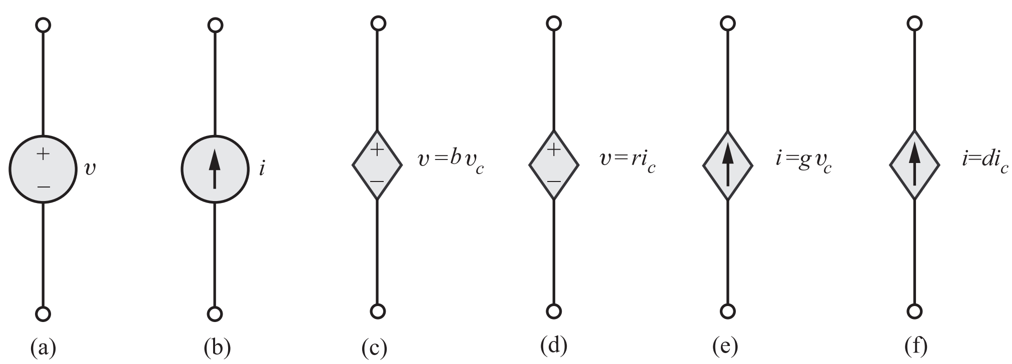


Figure 1.8 (a) An ideal independent voltage source
(b) An ideal independent current source
(c) voltage controlled voltage source
(d) current controlled voltage source
(e) voltage controlled current source
(f) current controlled current source

1.4 Unilateral and bilateral networks

A Unilateral network is one whose properties or characteristics change with the direction. An example of unilateral network is the semiconductor diode, which conducts only in one direction.

A bilateral network is one whose properties or characteristics are same in either direction. For example, a transmission line is a bilateral network, because it can be made to perform the function equally well in either direction.

1.5 Network simplification techniques

In this section, we shall give the formula for reducing the networks consisting of resistors connected in series or parallel.

1.5.1 Resistors in Series

When a number of resistors are connected in series, the equivalent resistance of the combination is given by

$$R = R_1 + R_2 + \cdots + R_n \tag{1.17}$$

Thus the total resistance is the algebraic sum of individual resistances.

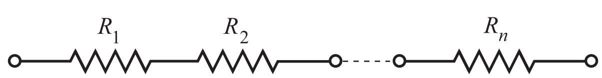


Figure 1.9 Resistors in series

1.5.2 Resistors in Parallel

When a number of resistors are connected in parallel as shown in Fig. 1.10, then the equivalent resistance of the combination is computed as follows:

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + ... + \frac{1}{R_n} \tag{1.18}$$

Thus, the reciprocal of a equivalent resistance of a parallel combination is the sum of the reciprocal of the individual resistances. Reciprocal of resistance is conductance and denoted by G. Consequently the equivalent conductance,

$$G = G_1 + G_2 + \cdots + G_n$$

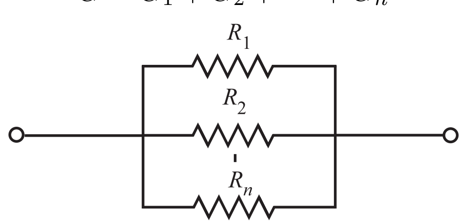


Figure 1.10 Resistors in parallel

1.5.3 Division of Current in a Parallel Circuit

Consider a two branch parallel circuit as shown in Fig. 1.11. The branch currents I_1 and I_2 can be evaluated in terms of total current I as follows:

$$I_1 = \frac{IR_2}{R_1 + R_2} = \frac{IG_1}{G_1 + G_2} \tag{1.19}$$

$$I_2 = \frac{IR_1}{R_1 + R_2} = \frac{IG_2}{G_1 + G_2} \tag{1.20}$$

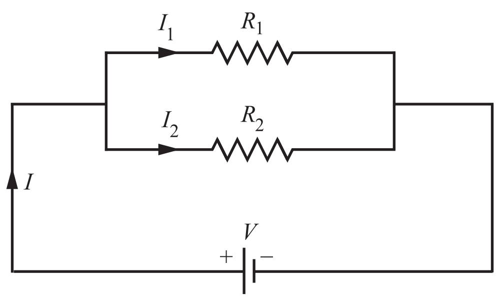


Figure 1.11 Current division in a parallel circuit

That is, current in one branch equals the total current multiplied by the resistance of the other branch and then divided by the sum of the resistances.

EXAMPLE 1.2

The current in the 6Ω resistor of the network shown in Fig. 1.12 is 2A. Determine the current in all branches and the applied voltage.

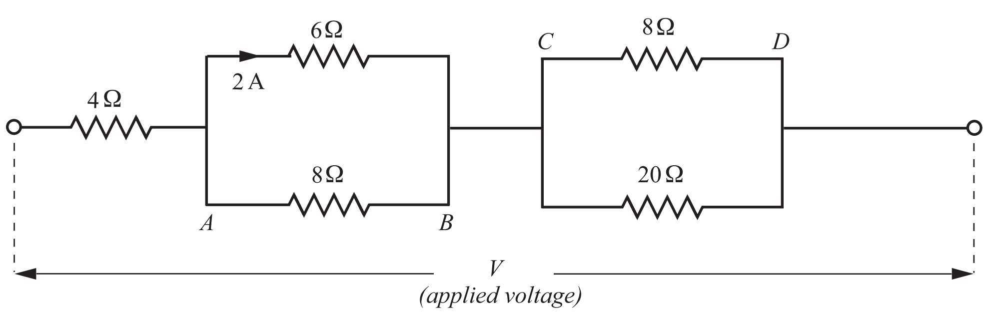
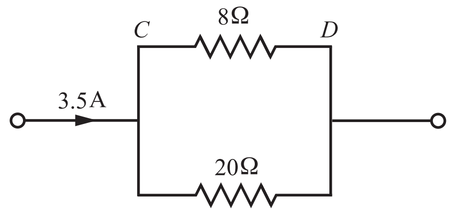


Figure 1.12

SOLUTION

$$\text{Voltage across} \quad 6\Omega = 6 \times 2$$
$$= 12 \text{ volts}$$



Since 6Ω and 8Ω are connected in parallel, voltage across 8Ω = 12 volts.

$$\left.\begin{array}{r}\text{Therefore, the current through}\\ 8\Omega\,\text{(between A and B)}\end{array}\right\} = \frac{12}{8} = \mathbf{1.5\ A}$$

Total current in the circuit $= 2 + 1.5 = \mathbf{3.5\ A}$

Current in the 4Ω branch = 3.5 A

$$\text{Current through } 8\Omega \text{ (betwen C and D)} = 3.5 \times \frac{20}{20+8}$$
$$= \mathbf{2.5\ A}$$
$$\text{Therefore, current through } 20\Omega = 3.5 - 2.5$$
$$= \mathbf{1A}$$
$$\text{Total resistance of the circuit} = 4 + \frac{6 \times 8}{6+8} + \frac{8 \times 20}{8+20}$$
$$= 13.143\Omega$$
$$\text{Therefore applied voltage, } V = 3.5 \times 13.143 \quad (\because\ V = IR)$$
$$= \mathbf{46\ Volts}$$

EXAMPLE 1.3

Find the value of R in the circuit shown in Fig. 1.13.

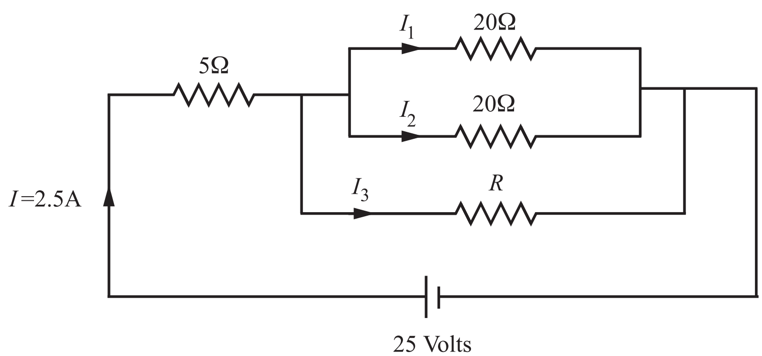


Figure 1.13

SOLUTION

Voltage across 5Ω = 2.5 × 5 = 12.5 volts

Hence the voltage across the parallel circuit = 25 − 12.5 = 12.5 volts

$$\text{Current through } 20\Omega = I_1 \text{ or } I_2$$
$$= \frac{12.5}{20} = 0.625\text{A}$$

$$
\begin{aligned}
\text{Therefore, current through} \quad R &= I_3 = I - I_1 - I_2 \\
&= 2.5 - 0.625 - 0.625 \\
&= 1.25 \text{ Amps} \\
\text{Hence,} \quad \mathrm{R} &= \frac{12.5}{1.25} = 10\Omega
\end{aligned}
$$

1.6 Kirchhoff's laws

In the preceeding section, we have seen how simple resistive networks can be solved for current, resistance, potential etc using the concept of Ohm's law. But as the network becomes complex, application of Ohm's law for solving the networks becomes tedious and hence time consuming. For solving such complex networks, we make use of Kirchhoff's laws. Gustav Kirchhoff (1824-1887), an eminent German physicist, did a considerable amount of work on the principles governing the behaviour of eletric circuits. He gave his findings in a set of two laws: (i) current law and (ii) voltage law, which together are known as Kirchhoff's laws. Before proceeding to the statement of these two laws let us familarize ourselves with the following definitions encountered very often in the world of electrical circuits:

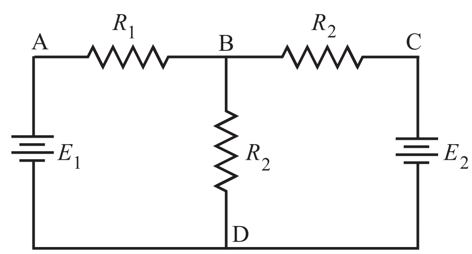


Figure 1.14 A simple resistive network for difining various circuit terminologies

(i) *Node*: A node of a network is an equi-potential surface at which two or more circuit elements are joined. Referring to Fig. 1.14, we find that A,B,C and D qualify as nodes in respect of the above definition.

(ii) *Junction*: A junction is that point in a network, where three or more circuit elements are joined. In Fig. 1.14, we find that B and D are the junctions.

(iii) *Branch*: A branch is that part of a network which lies between two junction points. In Fig. 1.14, BAD,BCD and BD qualify as branches.

(iv) *Loop*: A loop is any closed path of a network. Thus, in Fig. 1.14, ABDA,BCDB and ABCDA are the loops.

(v) *Mesh*: A mesh is the most elementary form of a loop and cannot be further divided into other loops. In Fig. 1.14, ABDA and BCDB are the examples of mesh. Once ABDA and BCDB are taken as meshes, the loop ABCDA does not qualify as a mesh, because it contains loops ABDA and BCDB.

1.6.1 Kirchhoff's Current Law

The first law is Kirchhoff's current law(KCL), which states that the algebraic sum of currents entering any node is zero.

Let us consider the node shown in Fig. 1.15. The sum of the currents entering the node is

$$-i_a + i_b - i_c + i_d = 0$$

Note that we have $-i_a$ since the current i_a is leaving the node. If we multiply the foregoing equation by -1, we obtain the expression

$$i_a - i_b + i_c - i_d = 0$$

which simply states that the algebraic sum of currents leaving a node is zero. Alternately, we can write the equation as

$$i_b + i_d = i_a + i_c$$

which states that the sum of currents entering a node is equal to the sum of currents leaving the node. If the sum of the currents entering a node were not equal to zero, then the charge would be accumulating at a node. However, a node is a perfect conductor and cannot accumulate or store charge. Thus, the sum of currents entering a node is equal to zero.

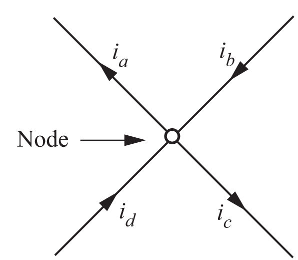


Figure 1.15 Currents at a node

1.6.2 Kirchhoff's Voltage Law

Kirchhoff's voltage law(KVL) states that the algebraic sum of voltages around any closed path in a circuit is zero.

In general, the mathematical representation of Kirchhoff's voltage law is

$$\sum_{j=1}^{N} v_j(t) = 0$$

where $v_j(t)$ is the voltage across the j^{th} branch (with proper reference direction) in a loop containing N voltages.

In Kirchhoff's voltage law, the algebraic sign is used to keep track of the voltage polarity. In other words, as we traverse the circuit, it is necessary to sum the increases and decreases in voltages to zero. Therefore, it is important to keep track of whether the voltage is increasing or decreasing as we go through each element. We will adopt a policy of considering the increase in voltage as *negative* and a decrease in voltage as *positive*.

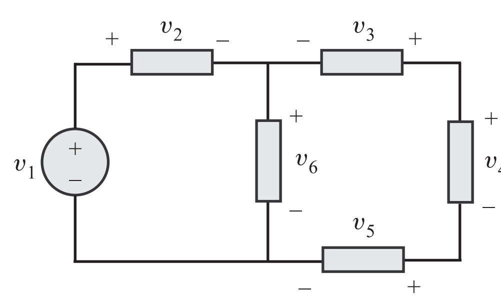


Figure 1.16 Circuit with three closed paths

Consider the circuit shown in Fig. 1.16, where the voltage for each element is identified with its sign. The ideal wire used for connecting the components has zero resistance, and thus the voltage across it is equal to zero. The sum of voltages around the loop incorporating v_6, v_3, v_4 and v_5 is

$$-v_6 - v_3 + v_4 + v_5 = 0$$

The sum of voltages around a loop is equal to zero. A circuit loop is a conservative system, meaning that the work required to move a unit charge around any loop is zero.

However, it is important to note that not all electrical systems are conservative. Example of a nonconservative system is a radio wave broadcasting system.

EXAMPLE 1.4

Consider the circuit shown in Fig. 1.17. Find each branch current and voltage across each branch when $R_1 = 8\Omega, v_2 = -10$ volts $i_3 = 2$A and $R_3 = 1\Omega$. Also find R_2.

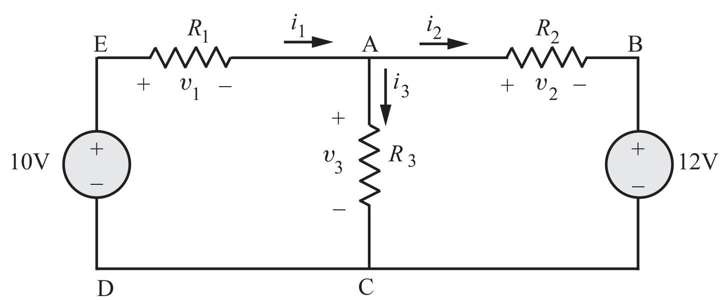


Figure 1.17

SOLUTION

Applying KCL (Kirchhoff's Current Law) at node A, we get

$$i_1 = i_2 + i_3$$

and using Ohm's law for R_3, we get

$$v_3 = R_3 i_3 = 1(2) = \mathbf{2V}$$

Applying KVL (Kirchhoff's Voltage Law) for the loop EACDE, we get

$$-10 + v_1 + v_3 = 0$$

$$\Rightarrow \quad v_1 = 10 - v_3 = \mathbf{8V}$$

Ohm's law for R_1 is

$$v_1 = i_1 R_1$$

$$\Rightarrow \quad i_1 = \frac{v_1}{R_1} = \mathbf{1A}$$

Hence,

$$i_2 = i_1 - i_3$$
$$= 1 - 2 = -\mathbf{1A}$$

From the circuit,

$$v_2 = R_2 i_2$$

$$\Rightarrow \quad R_2 = \frac{v_2}{i_2} = \frac{-10}{-1} = \mathbf{10\Omega}$$

EXAMPLE 1.5

Referring to Fig. 1.18, find the following:

(a) i_x if $i_y = 2\text{A}$ and $i_z = 0\text{A}$

(b) i_y if $i_x = 2\text{A}$ and $i_z = 2i_y$

(c) i_z if $i_x = i_y = i_z$

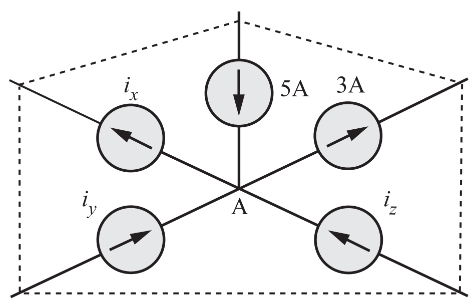


Figure 1.18

SOLUTION

Applying KCL at node A, we get

$$5 + i_y + i_z = i_x + 3$$

(a) $i_x = 2 + i_y + i_z$

$= 2 + 2 + 0 = \mathbf{4A}$

(b) $i_y = 3 + i_x - 5 - i_z$

$= -2 + 2 - 2i_y$

$\Rightarrow \quad i_y = \mathbf{0A}$

(c) This situation is not possible, since i_x and i_z are in opposite directions. The only possibility is $i_z = 0$, and this cannot be allowed, as KCL will not be satisfied ($5 \neq 3$).

EXAMPLE 1.6

Refer the Fig. 1.19.

(a) Calculate v_y if $i_z = -3\text{A}$

(b) What voltage would you need to replace 5 V source to obtain $v_y = -6$ V if $i_z = 0.5\text{A}$?

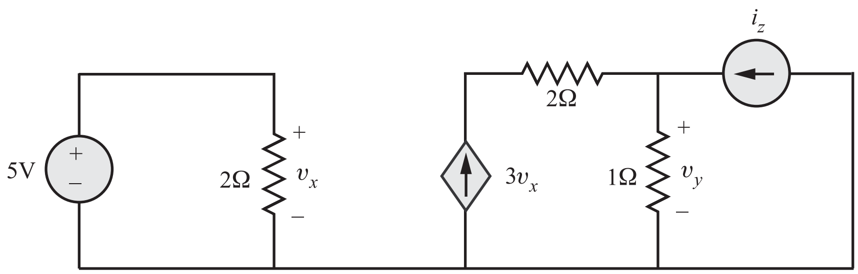


Figure 1.19

SOLUTION

(a) $v_y = 1\ (3\ v_x + i_z\ )$

Since $v_x = 5\text{V}$ and $i_z = -3\text{A}$,

we get $v_y = 3(5) - 3 = \mathbf{12V}$

(b)
$$v_y = 1\ (3\ v_x + i_z) = -6$$
$$= 3\ v_x + 0.5$$
$$\Rightarrow \quad 3\ v_x = -6.5$$

Hence, $v_x = -\mathbf{2.167\ volts}$

EXAMPLE 1.7

For the circuit shown in Fig. 1.20, find i_1 and v_1, given $R_3 = 6\Omega$.

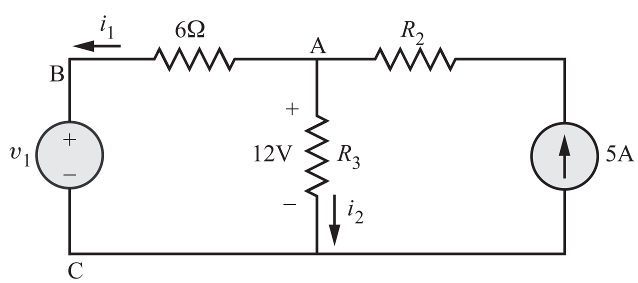


Figure 1.20

SOLUTION

Applying KCL at node A, we get

$$-i_1 - i_2 + 5 = 0$$

From Ohm's law, $12 = i_2 R_3$

$$\Rightarrow \quad i_2 = \frac{12}{R_3} = \frac{12}{6} = 2\text{A}$$

Hence, $i_1 = 5 - i_2 = \mathbf{3A}$

Applying KVL clockwise to the loop CBAC, we get

$$-v_1 - 6i_1 + 12 = 0$$

$$\Rightarrow \qquad v_1 = 12 - 6i_1$$

$$= 12 - 6(3) = -\mathbf{6volts}$$

EXAMPLE 1.8

Use Ohm's law and Kirchhoff's law to evaluate (a) v_x, (b) i_{in}, (c) I_s and (d) the power provided by the dependent source in Fig 1.21.

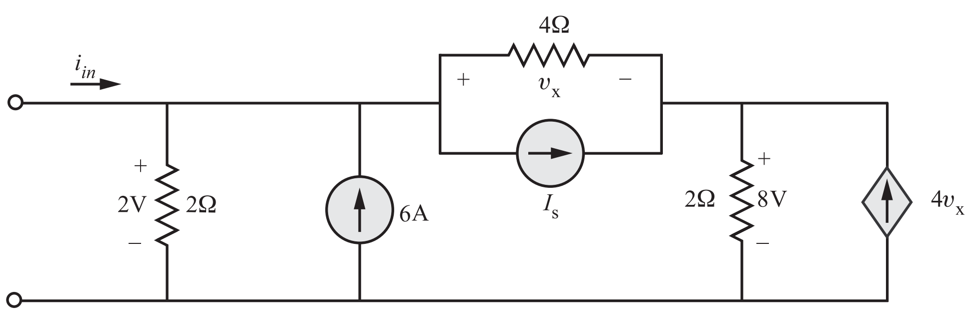


Figure 1.21

SOLUTION

(a) *Applying KVL*, (Referring Fig. 1.21 (a)) we get

$$-2 + v_x + 8 = 0$$

$$\Rightarrow \qquad v_x = -\mathbf{6V}$$

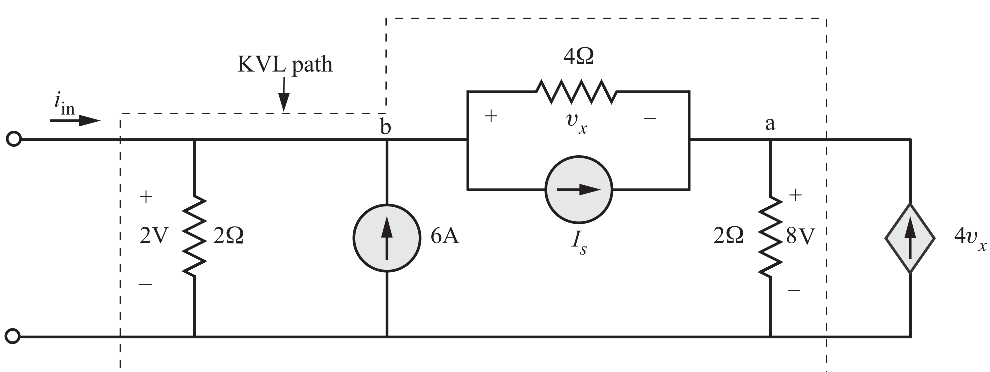


Figure 1.21(a)

(b) *Applying KCL at node a*, we get

$$I_s + 4v_x + \frac{v_x}{4} = \frac{8}{2}$$

$$\Rightarrow \qquad I_s + 4(-6) - \frac{6}{4} = 4$$

$$\Rightarrow \qquad I_s - 24 - 1.5 = 4$$

$$\Rightarrow \qquad I_s = \mathbf{29.5A}$$

(c) *Applying KCL at node b*, we get

$$i_{in} = \frac{2}{2} + I_s + \frac{v_x}{4} - 6$$

$$\Rightarrow \qquad i_{in} = 1 + 29.5 - \frac{6}{4} - 6 = \mathbf{23A}$$

(d) The power supplied by the dependent current source = $8\,(4v_x) = 8 \times 4 \times -6 = \mathbf{-192W}$

EXAMPLE 1.9

Find the current i_2 and voltage v for the circuit shown in Fig. 1.22.

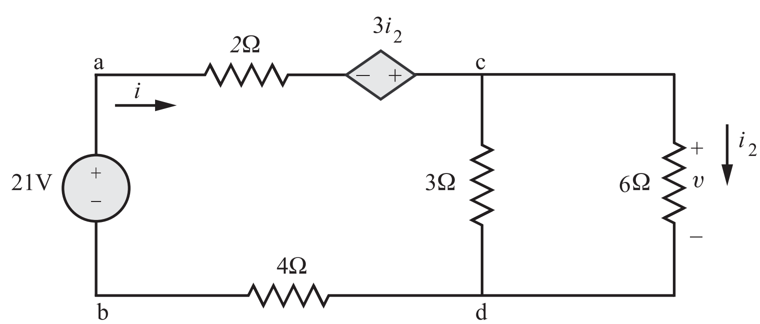


Figure 1.22

SOLUTION

From the network shown in Fig. 1.22, $i_2 = \dfrac{v}{6}$

The two parallel resistors may be reduced to

$$R_p = \frac{3 \times 6}{3 + 6} = 2\Omega$$

Hence, the total series resistance around the loop is

$$R_s = 2 + R_p + 4$$
$$= 8\Omega$$

Applying KVL around the loop, we have

$$-21 + 8i - 3i_2 = 0 \tag{1.21}$$

Using the principle of current division,

$$i_2 = \frac{iR_2}{R_1 + R_2} = \frac{i \times 3}{3+6}$$
$$= \frac{3i}{9} = \frac{i}{3}$$
$$\Rightarrow \quad i = 3i_2 \tag{1.22}$$

Substituting equation (1.22) in equation (1.21), we get

$$-21 + 8(3i_2) - 3i_2 = 0$$

Hence, $$i_2 = \mathbf{1A}$$

and $$v = 6i_2 = \mathbf{6V}$$

EXAMPLE 1.10

Find the current i_2 and voltage v for resistor R in Fig. 1.23 when $R = 16\Omega$.

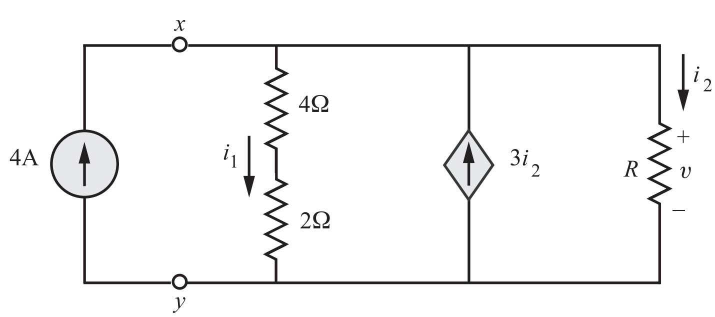


Figure 1.23

SOLUTION

Applying KCL at node x, we get

$$4 - i_1 + 3i_2 - i_2 = 0$$

Also, $$i_1 = \frac{v}{4+2} = \frac{v}{6}$$

$$i_2 = \frac{v}{R} = \frac{v}{16}$$

Hence, $$4 - \frac{v}{6} + 3 \times \frac{v}{16} - \frac{v}{16} = 0$$

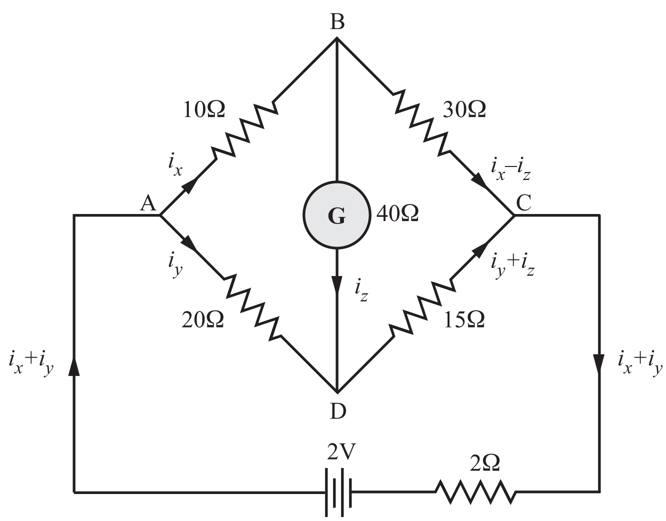
$$\Rightarrow \quad v = \mathbf{96volts}$$

and $$i_2 = \frac{v}{6} = \frac{96}{16} = \mathbf{6A}$$

EXAMPLE 1.11

A wheatstone bridge ABCD is arranged as follows: AB = 10Ω, BC = 30Ω, CD = 15Ω and DA = 20Ω. A 2V battery of internal resistance 2Ω is connected between points A and C with A being positive. A galvanometer of resistance 40Ω is connected between B and D. Find the magnitude and direction of the galvanometer current.

SOLUTION



Applying KVL clockwise to the loop ABDA, we get

$$10i_x + 40i_z - 20i_y = 0$$

$$\Rightarrow \quad 10i_x - 20i_y + 40i_z = 0 \tag{1.23}$$

Applying KVL clockwise to the loop BCDB, we get

$$30(i_x - i_z) - 15(i_y + i_z) - 40i_z = 0$$

$$\Rightarrow \quad 30i_x - 15i_y - 85i_z = 0 \tag{1.24}$$

Finally, *applying KVL clockwise to the loop ADCA,* we get

$$20i_y + 15(i_y + i_z) + 2(i_x + i_y) - 2 = 0$$

$$\Rightarrow \quad 2i_x + 37i_y + 15i_z = 2 \tag{1.25}$$

Putting equations (1.23),(1.24) and (1.25) in matrix form, we get

$$\begin{bmatrix} 10 & -20 & 40 \\ 30 & -15 & -85 \\ 2 & 37 & 15 \end{bmatrix} \begin{bmatrix} i_x \\ i_y \\ i_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

Using Cramer's rule, we find that

$$i_z = \textbf{0.01 A (Flows from B to D)}$$

1.7 Multiple current source networks

Let us now learn how to reduce a network having multiple current sources and a number of resistors in parallel. Consider the circuit shown in Fig. 1.24. We have assumed that the upper node is $v(t)$ volts positive with respect to the lower node. Applying KCL to upper node yields

$$i_1(t) - i_2(t) - i_3(t) + i_4(t) - i_5(t) - i_6(t) = 0$$

$$\Rightarrow \quad i_1(t) - i_3(t) + i_4(t) - i_6(t) = i_2(t) + i_5(t) \tag{1.26}$$

$$\Rightarrow \quad i_o(t) = i_2(t) + i_5(t) \tag{1.27}$$

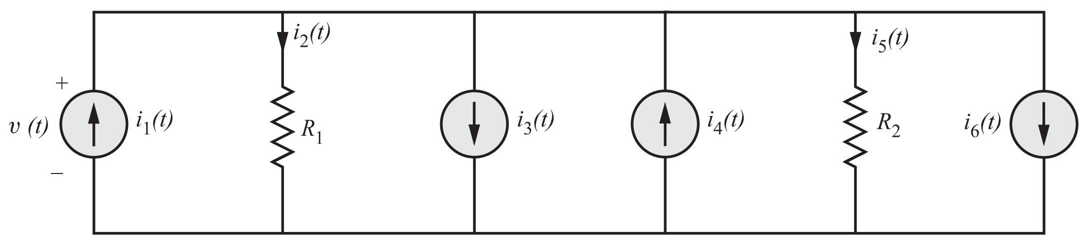


Figure 1.24 Multiple current source network

where $i_o(t) = i_1(t) - i_3(t) + i_4(t) - i_6(t)$ is the algebraic sum of all current sources present in the multiple source network shown in Fig. 1.24. As a consequence of equation (1.27), the network of Fig. 1.24 is effectively reduced to that shown in Fig. 1.25. Using Ohm's law, the currents on the right side of equation (1.27) can be expressed in terms of the voltage and individual resistance so that KCL equation reduces to

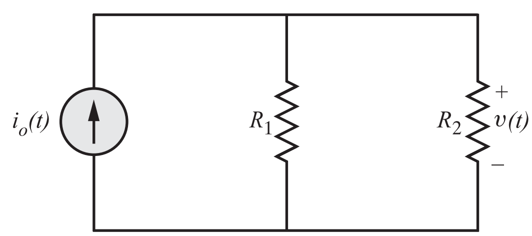


Figure 1.25 Equivalent circuit

$$i_o(t) = \left[\frac{1}{R_1} + \frac{1}{R_2}\right] v(t)$$

Thus, we can reduce a multiple current source network into a network having only one current source.

1.8 Source transformations

Source transformation is a procedure which transforms one source into another while retaining the terminal characteristics of the original source.

Source transformation is based on the concept of equivalence. An equivalent circuit is one whose terminal characteristics remain identical to those of the original circuit. The term equivalence as applied to circuits means an identical effect at the terminals, but not within the equivalent circuits themselves.

We are interested in transforming the circuit shown in Fig. 1.26 to a one shown in Fig. 1.27.

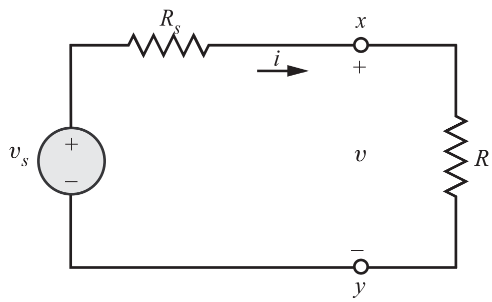


Figure 1.26 Voltage source connected to an external resistance R

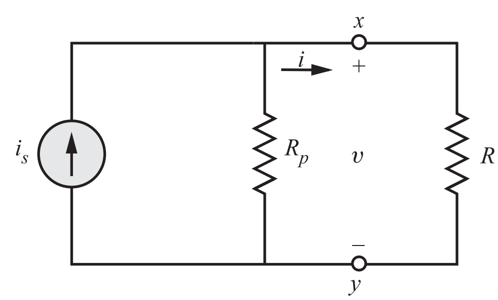


Figure 1.27 Current source connected to an external resistance R

We require both the circuits to have the equivalence or same characteristics between the terminals x and y for all values of external resistance R. We will try for equivanlence of the two circuits between terminals x and y for two limiting values of R namely $R = 0$ and $R = \infty$. When $R = 0$, we have a short circuit across the terminals x and y. It is obligatory for the short circuit to be same for each circuit. The short circuit current of Fig. 1.26 is

$$i_s = \frac{v_s}{R_s} \tag{1.28}$$

The short circuit current of Fig. 1.27 is i_s. This enforces,

$$i_s = \frac{v_s}{R_s} \tag{1.29}$$

When $R = \infty$, from Fig. 1.26 we have $v_{xy} = v_s$ and from Fig. 1.27 we have $v_{xy} = i_s R_p$. Thus, for equivalence, we require that

$$v_s = i_s R_p \tag{1.30}$$

Also from equation (1.29), we require $i_s = \dfrac{v_s}{R_s}$. Therefore, we must have

$$v_s = \left(\frac{v_s}{R_s}\right) R_p$$

$$\Rightarrow \qquad R_s = R_p \tag{1.31}$$

Equations(1.29) and (1.31) must be true simulaneously for both the circuits for the two sources to be equivalent. We have derived the conditions for equivalence of two circuits shown in Figs. 1.26 and 1.27 only for two extreme values of R, namely $R = 0$ and $R = \infty$. However, the equality relationship holds good for all R as explained below.

Applying KVL to Fig. 1.26, we get

$$v_s = iR_s + v$$

Dividing by R_s gives

$$\frac{v_s}{R_s} = i + \frac{v}{R_s} \tag{1.32}$$

If we use KCL for Fig. 1.27, we get

$$i_s = i + \frac{v}{R_p} \tag{1.33}$$

Thus two circuits are equal when

$$i_s = \frac{v_s}{R_s} \text{ and } R_s = R_p$$

Transformation procedure: If we have embedded within a network, a current source i in parallel with a resistor R can be replaced with a voltage source of value $v = iR$ in series with the resistor R.

The reverse is also true; that is, a voltage source v in series with a resistor R can be replaced with a current source of value $i = \dfrac{v}{R}$ in parallel with the resistor R. Parameters within the circuit are unchanged under these transformation.

EXAMPLE 1.12

A circuit is shown in Fig. 1.28. Find the current i by reducing the circuit to the right of the terminals $x - y$ to its simplest form using source transformations.

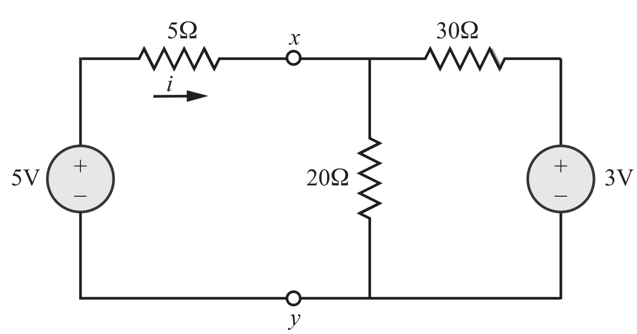
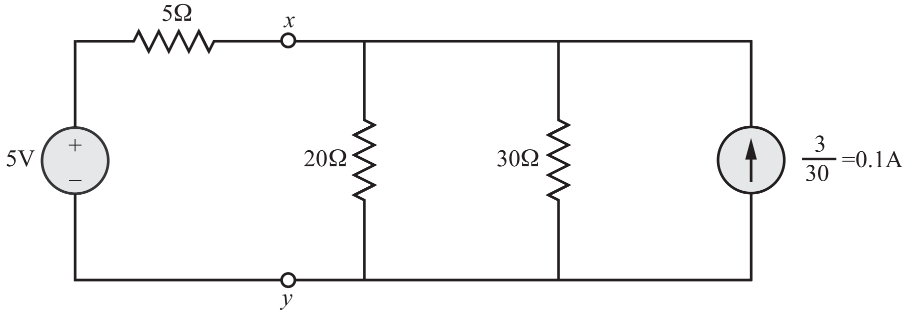


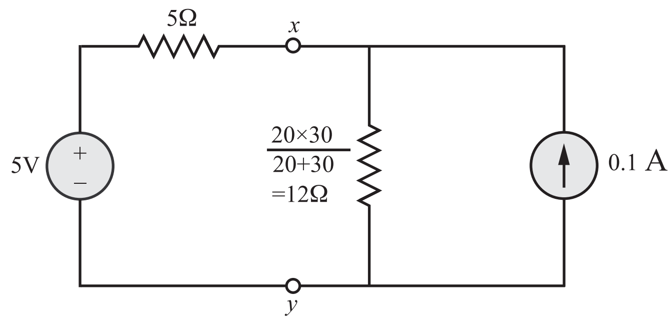
Figure 1.28

SOLUTION

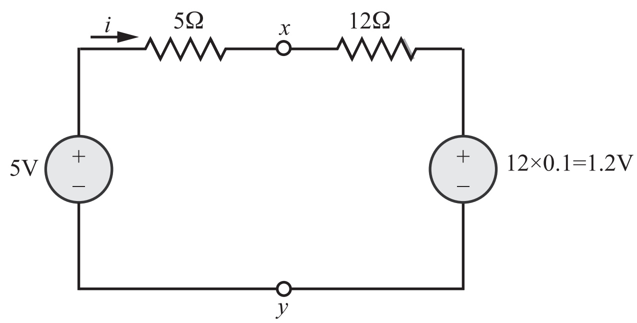
The first step in the analysis is to transform 30 ohm resistor in series with a 3 V source into a current source with a parallel resistance and we get:



Reducing the two parallel resistances, we get:



The parallel resistance of 12Ω and the current source of 0.1A can be transformed into a voltage source in series with a 12 ohm resistor.



Applying KVL, we get

$$5i + 12i + 1.2 - 5 = 0$$

$$\Rightarrow \quad 17i = 3.8$$

$$\Rightarrow \quad \boldsymbol{i = 0.224\text{A}}$$

EXAMPLE 1.13

Find current i_1 using source transformation for the circuit shown Fig. 1.29.

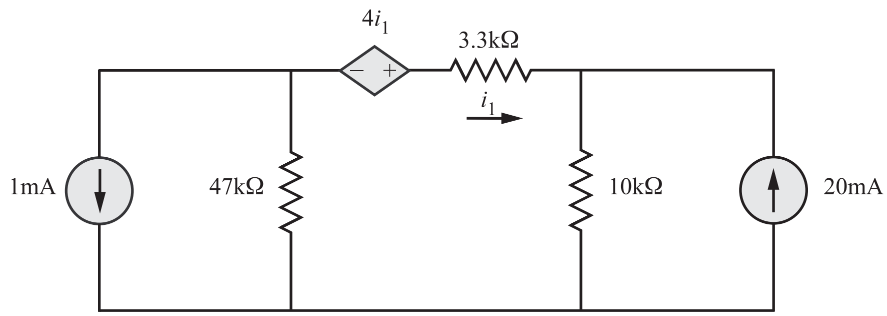


Figure 1.29

SOLUTION

Converting 1 mA current source in parallel with 47kΩ resistor and 20 mA current source in parallel with 10kΩ resistor into equivalent voltage sources, the circuit of Fig. 1.29 becomes the circuit shown in Fig. 1.29(a).

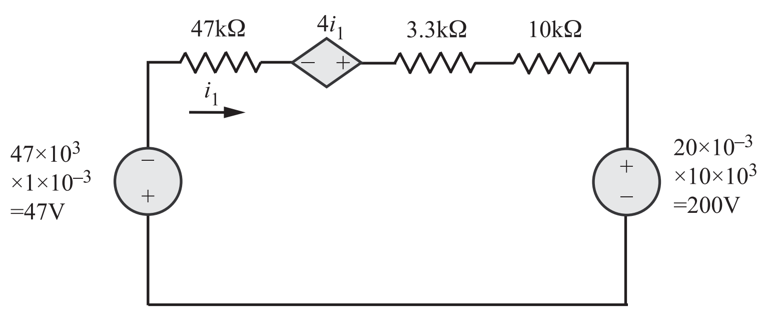


Figure 1.29(a)

Please note that for each voltage source, "+" corresponds to its corresponding current source's arrow head.

Using KVL to the above circuit,

$$47 + 47 \times 10^3 i_1 - 4i_1 + 13.3 \times 10^3 i_1 + 200 = 0$$

Solving, we find that

$$i_1 = \mathbf{-4.096\ mA}$$

EXAMPLE 1.14

Use source transformation to convert the circuit in Fig. 1.30 to a single current source in parallel with a single resistor.

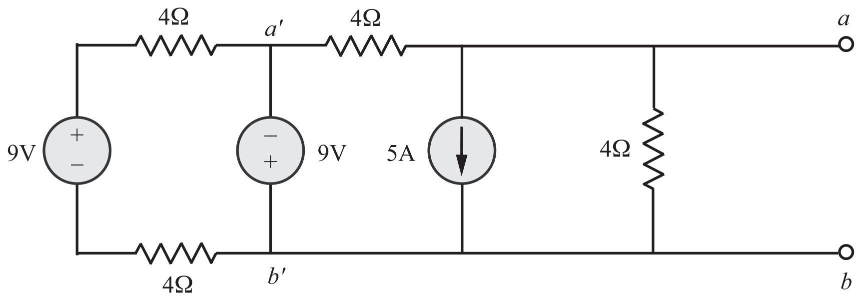
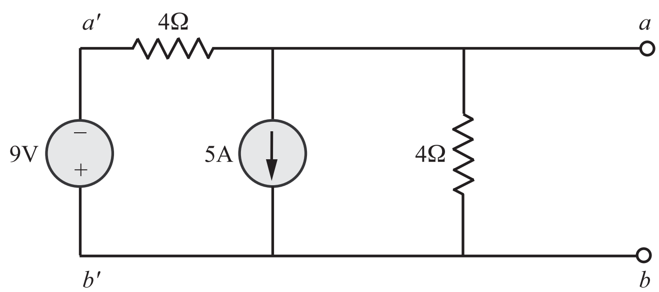


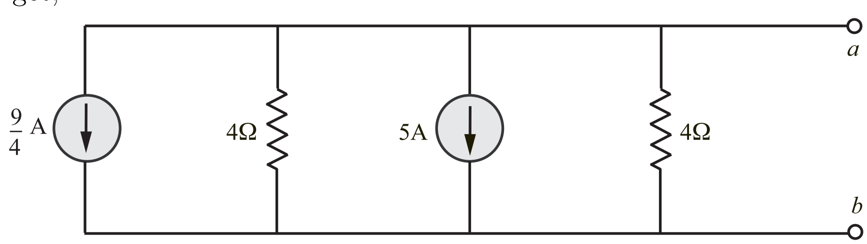
Figure 1.30

SOLUTION

The 9V source across the terminals a' and b' will force the voltage across these two terminals to be 9V regardless the value of the other 9V source and 8Ω resistor to its left. Hence, these two components may be removed from the terminals, a' and b' without affecting the circuit condition. Accordingly, the above circuit reduces to,



Converting the voltage source in series with 4Ω resistor into an equivalent current source, we get,



Adding the current sources in parallel and reducing the two 4 ohm resistors in parallel, we get the circuit shown in Fig. 1.30 (a):

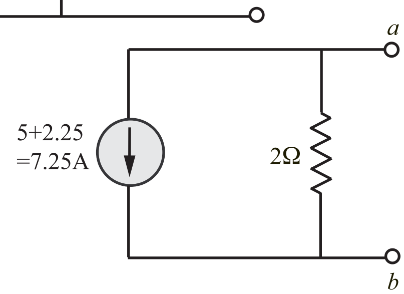


Figure 1.30 (a)

1.8.1 Source Shift

The source transformation is possible only in the case of practical sources. ie $R_s \neq \infty$ and $R_p \neq 0$, where R_s and R_p are internal resistances of voltage and current sources respectively. Transformation is not possible for ideal sources and source shifting methods are used for such cases.

Voltage source shift (E−shift):

Consider a part of the network shown in Fig. 1.31(a) that contains an ideal voltage source.

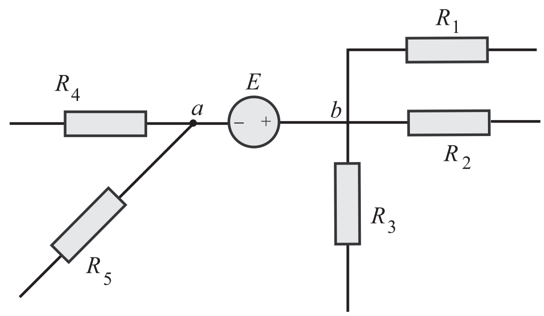


Figure 1.31(a) Basic network

Since node b is at a potential E with respect to node a, the network can be redrawn equivalently as in Fig. 1.31(b) or (c) depend on the requirements.

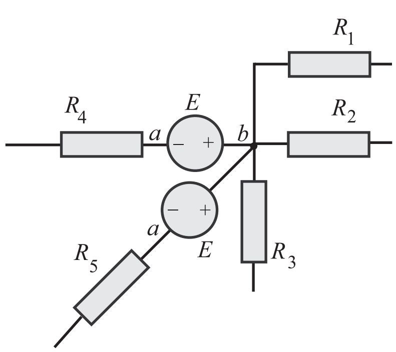


Figure 1.31(b) Networks after E-shift

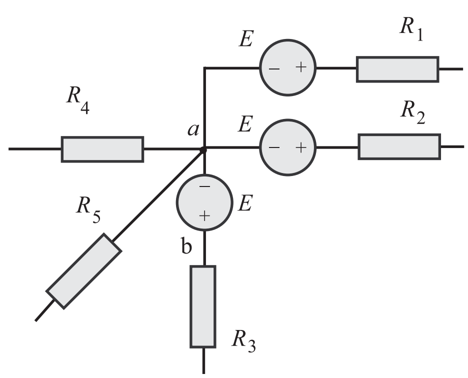


Figure 1.31(c) Network after the E-shift

Current source shift (I−shift)

In a similar manner, current sources also can be shifted. This can be explained with an example. Consider the network shown in Fig. 1.32(a), which contains an ideal current source between nodes a and c. The circuit shown in Figs. 1.32(b) and (c) illustrates the equivalent circuit after the I - shift.

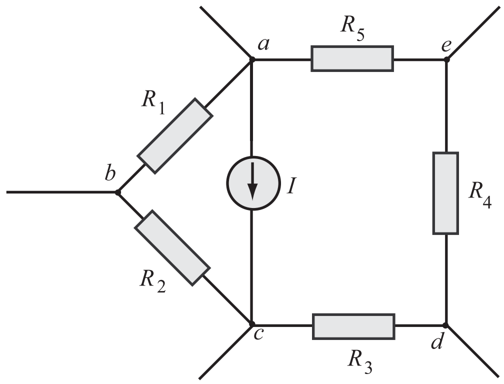


Figure 1.32(a) basic network

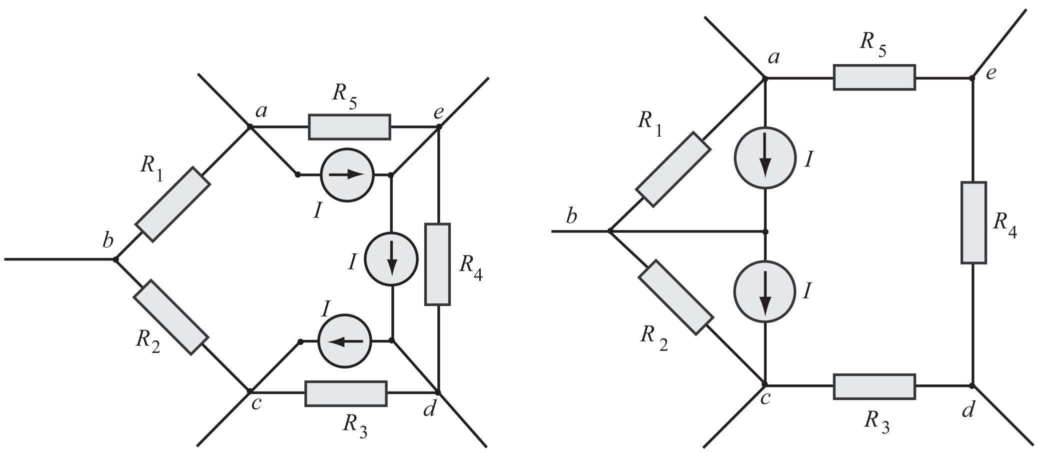


Figure 1.32(b) and (c) Networks after I--shift

EXAMPLE 1.15

Use source shifting and transformation techiniques to find voltage across 2Ω resistor shown in Fig. 1.33(a). All resistor values are in ohms.

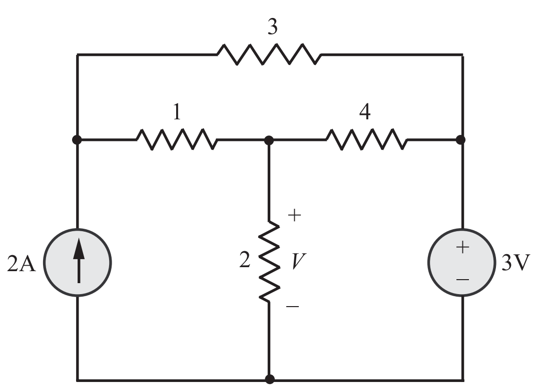
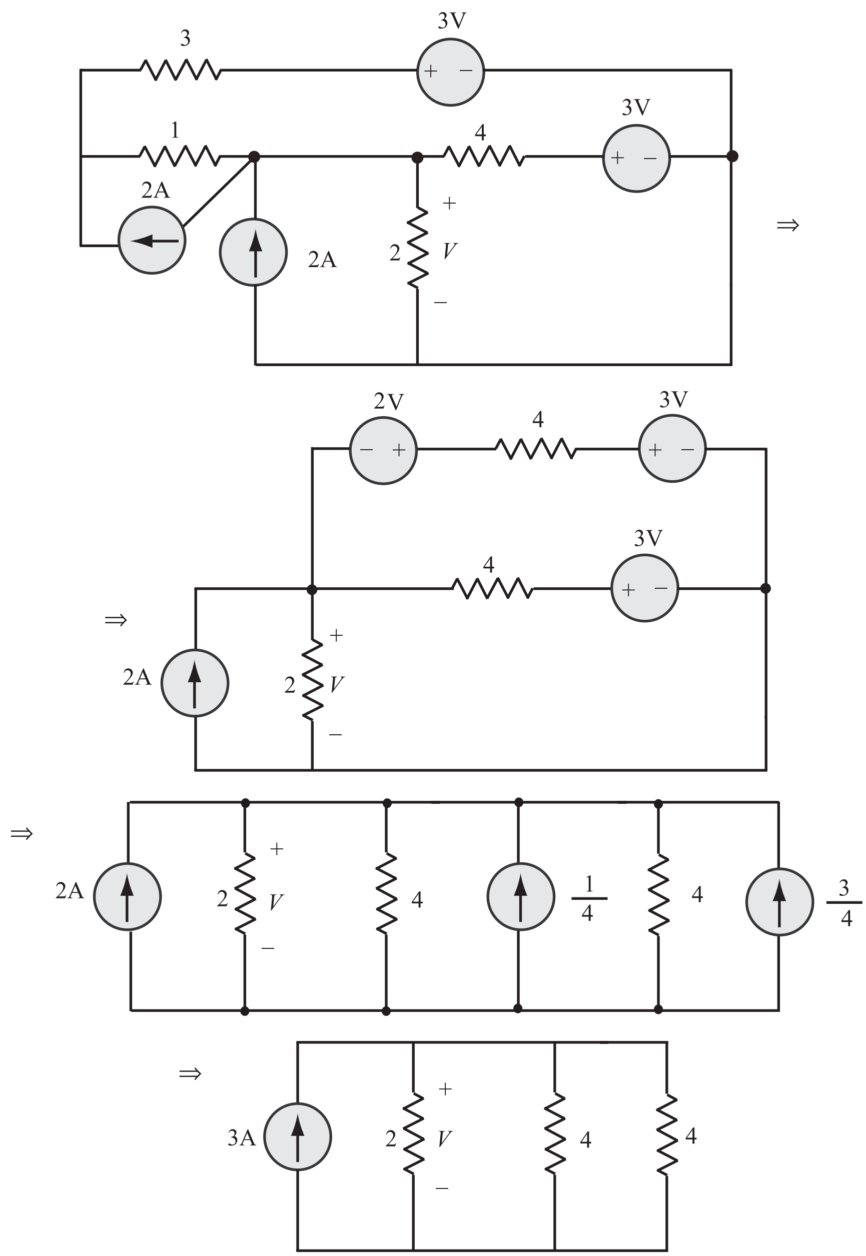


Figure 1.33(a)

SOLUTION

The circuit is redrawn by shifting 2A current source and 3V voltage source and further simplified as shown below.



Thus the voltage across 2Ω resistor is

$$V = 3 \times \frac{1}{2^{-1} + 4^{-1} + 4^{-1}} = 3\ \mathbf{V}$$

EXAMPLE 1.16

Use source mobility to calculate v_{ab} in the circuits shown in Fig. 1.34 (a) and (b). All resistor values are in ohms.

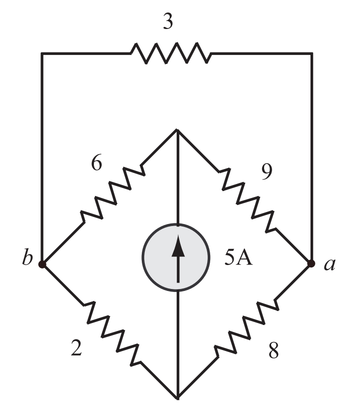


Figure 1.34(a)

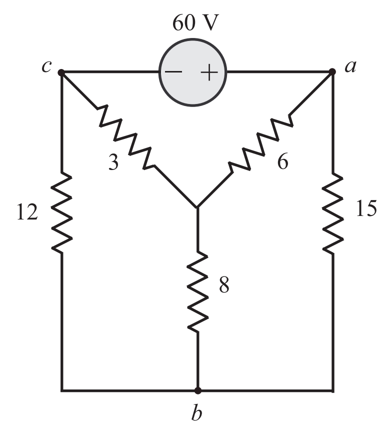
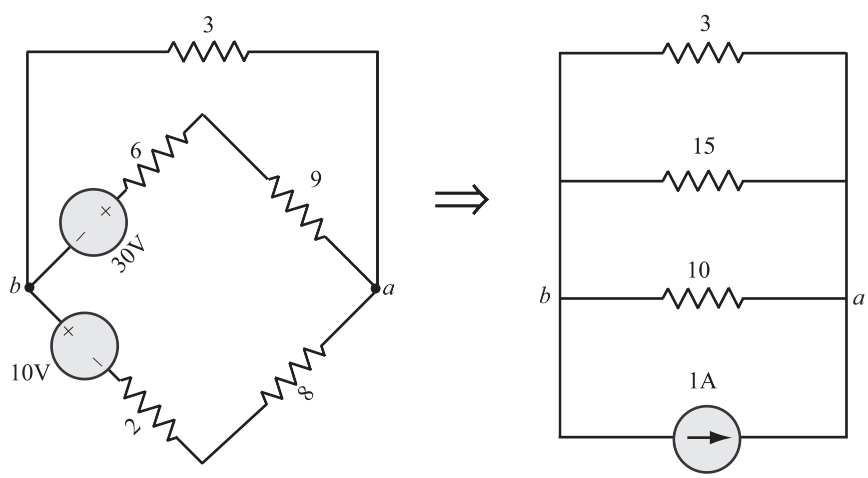


Figure 1.34(b)

SOLUTION

(a) The circuit shown in Fig. 1.34(a) is simplified using source mobility technique, as shown below and the voltage across the nodes a and b is calculated.



Voltage across a and b is

$$V_{ab} = \frac{1}{3^{-1} + 10^{-1} + 15^{-1}} = 2 \text{ V}$$

(b) The circuit shown in Fig. 1.34 (b) is reduced as follows.

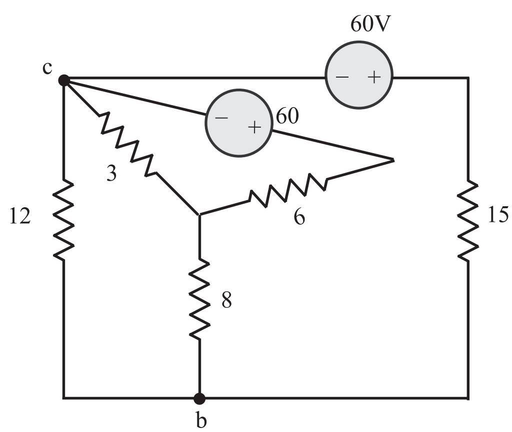


Figure 1.34(c)

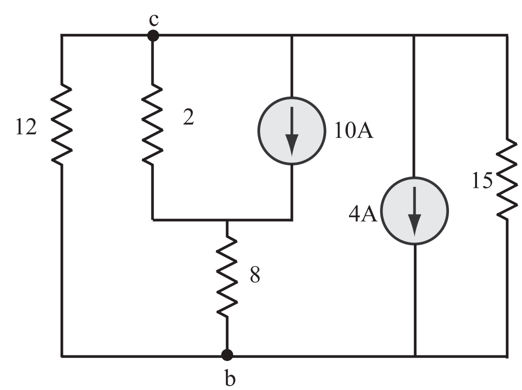


Figure 1.34(d)

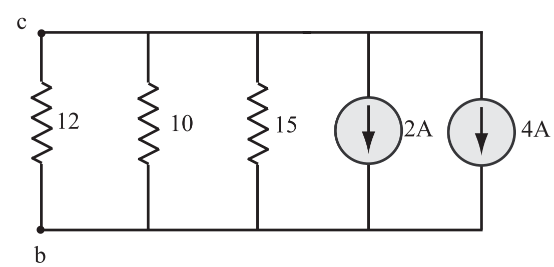


Figure 1.34(e)

From Fig. 1.34(e),

$$V_{bc} = \frac{12^{-1} \times 6}{12^{-1} + 10^{-1} + 15^{-1}} \times 12 = 24 \text{ V}$$

Applying this result in Fig. 1.34(b), we get

$$\begin{aligned} v_{ab} &= v_{ac} - v_{bc} \\ &= 60 - 24 = 36 \text{ V} \end{aligned}$$

EXAMPLE 1.17

Use mobility and reduction techniques to solve the node voltages of the network shown in Fig. 1.35(a). All resistors are in ohms.

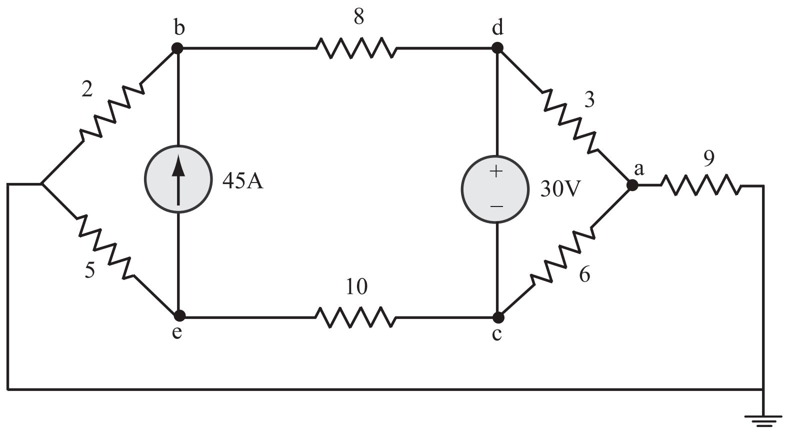


Figure 1.35(a)

SOLUTION

The circuit shown in Fig. 1.35(a) can be reduced by using desired techniques as shown in Fig. 1.35(b) to 1.35(e).

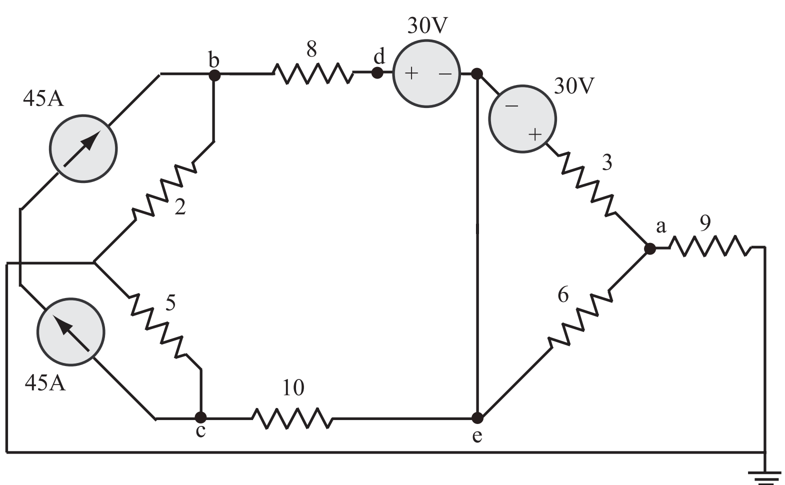


Figure 1.35(b)

From Fig. 1.35(e)

$$i = \frac{34}{17} = 2 \text{ A}$$

Using this value of i in Fig. 1.35(e),

$$V_a = -9 \times 2 = -18 \text{ V}$$

and $$V_e = V_a - 2 \times 2 - 20 = -42V$$

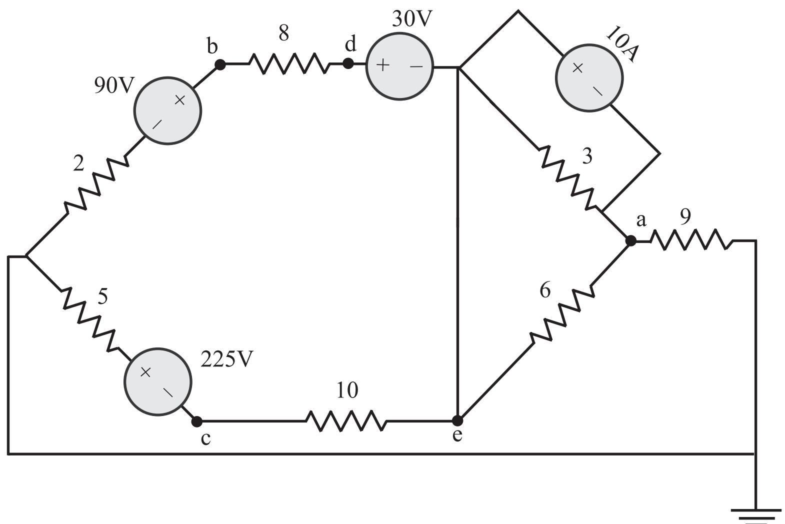


Figure 1.35(c)

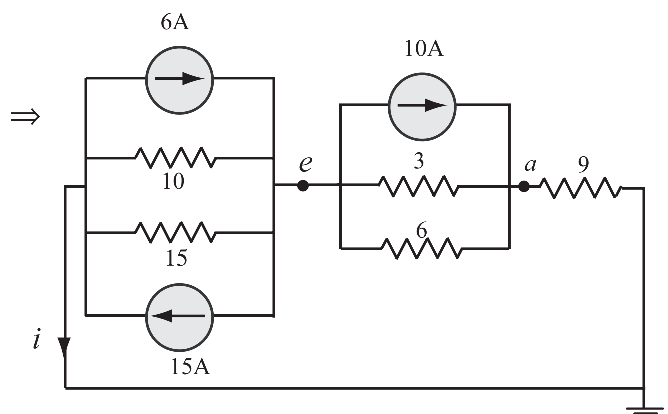


Figure 1.35(d)

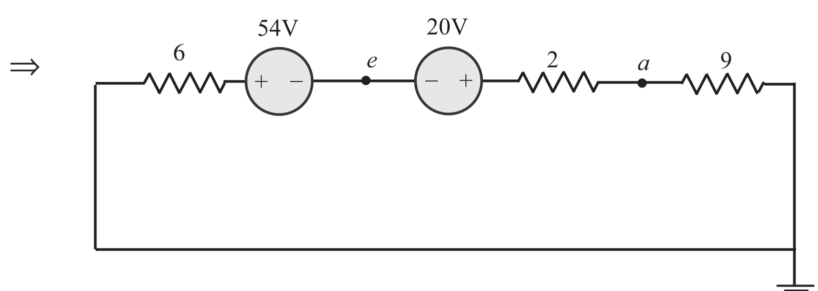


Figure 1.35(e)

From Fig 1.35(a)

$$V_d = V_e + 30 = -42 + 30 = -12\text{V}$$

Then at node b in Fig. 1.35(b),

$$\frac{V_b}{2} - 45 + \frac{V_b - V_d}{8} = 0$$

Using the value of V_d in the above equation and rearranging, we get,

$$V_b\left(\frac{1}{2} + \frac{1}{8}\right) = 45 - \frac{12}{8}$$

$$\Rightarrow \qquad V_b = 69.6 \text{ V}$$

At node c of Fig. 1.35(b)

$$\frac{V_c}{5} + 45 + \frac{V_c - V_e}{10} = 0$$

$$V_c\left(\frac{1}{5} + \frac{1}{10}\right) = -45 - \frac{42}{10}$$

$$\Rightarrow \qquad V_c = -164 \text{ V}$$

EXAMPLE 1.18

Use source mobility to reduce the network shown in Fig. 1.36(a) and find the value of V_x. All resistors are in ohms.

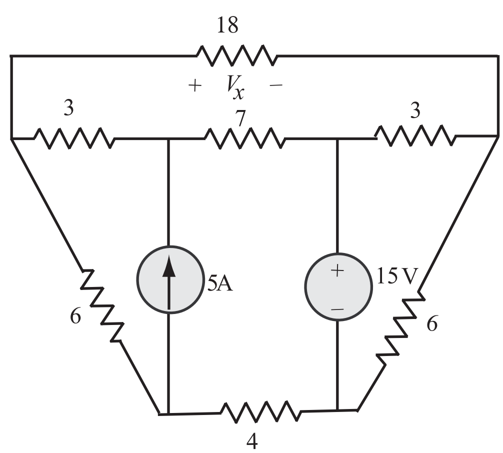


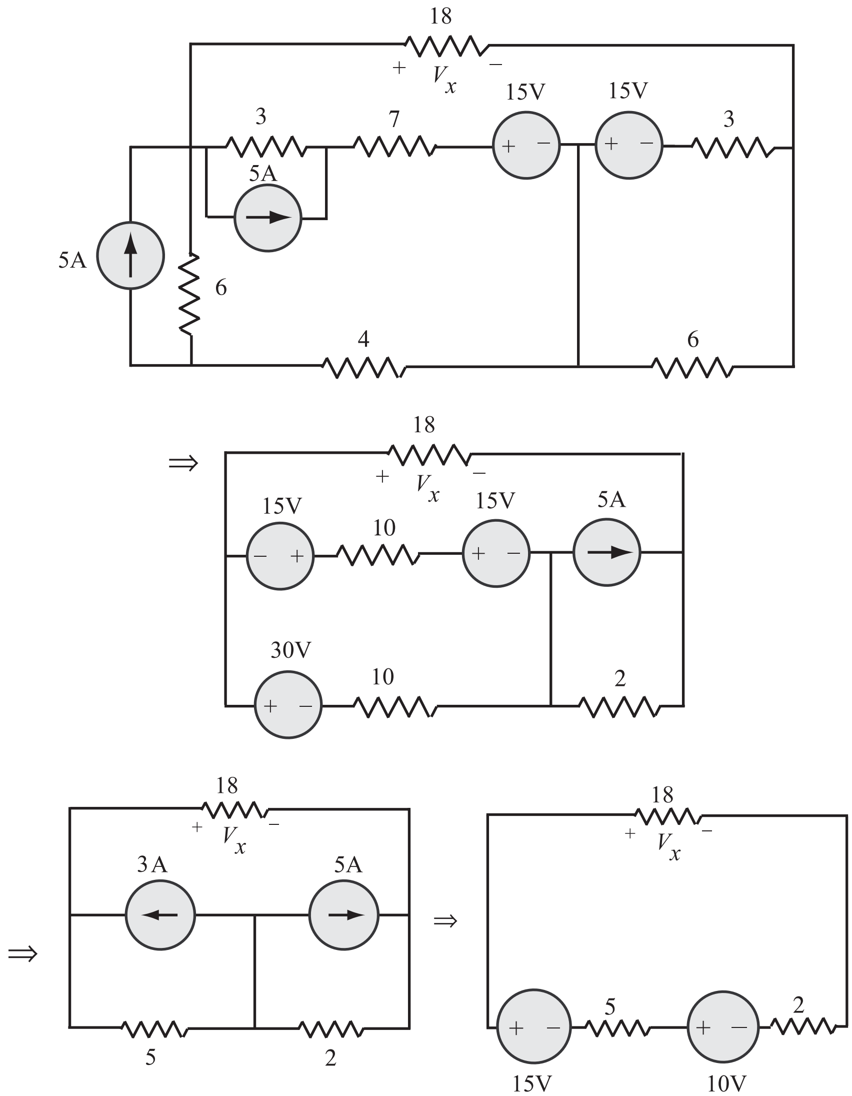
Figure 1.36(a)

SOLUTION

The circuit shown in Fig. 1.36(a) can be reduced as follows and V_x is calculated.

Thus

$$V_x = \frac{5}{25} \times 18 = 3.6\text{V}$$



1.9 Mesh analysis with independent voltage sources

Before starting the concept of mesh analysis, we want to reiterate that a closed path or a loop is drawn starting at a node and tracing a path such that we return to the original node without passing an intermediate node more than once. A mesh is a special case of a loop. A mesh is a loop that does not contain any other loops within it. The network shown in Fig. 1.37(a) has four meshes and they are identified as M_i, where $i = 1, 2, 3, 4$.

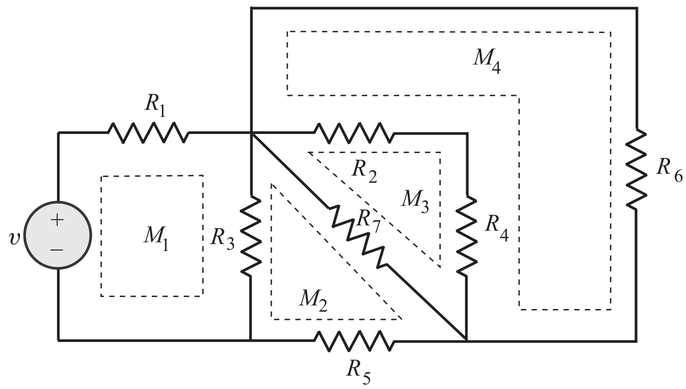


Figure 1.37(a) A circuit with four meshes. Each mesh is identified by a circuit

The current flowing in a mesh is defined as mesh current. As a matter of convention, the mesh currents are assumed to flow in a mesh in the clockwise direction.

Let us consider the two mesh circuit of Fig. 1.37(b).

We cannot choose the outer loop, $v \rightarrow R_1 \rightarrow R_2 \rightarrow v$ as one mesh, since it would contain the loop $v \rightarrow R_1 \rightarrow R_3 \rightarrow v$ within it. Let us choose two mesh currents i_1 and i_2 as shown in the figure.

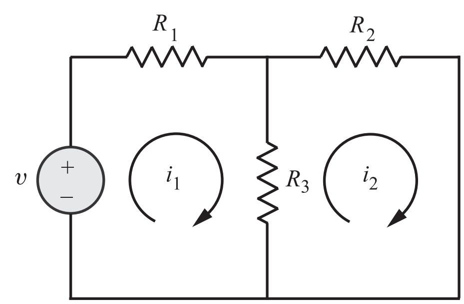


Figure 1.37(b) A circuit with two meshes

We may employ KVL around each mesh. We will travel around each mesh in the clockwise direction and sum the voltage rises and drops encountered in that particular mesh. We will adpot a convention of taking voltage drops to be *positive* and voltage rises to be *negative* . Thus, for the network shown in Fig. 1.37(b) we have

$$Mesh\ 1: -v + i_1R_1 + (i_1 - i_2)R_3 = 0 \tag{1.34}$$

$$Mesh\ 2: \qquad R_3(i_2 - i_1) + R_2i_2 = 0 \tag{1.35}$$

Note that when writing voltage across R_3 in mesh 1, the current in R_3 is taken as $i_1 - i_2$. Note that the mesh current i_1 is taken as '+ve' since we traverse in clockwise direction in mesh 1, On the other hand, the voltage across R_3 in mesh 2 is written as $R_3(i_2 - i_1)$. The current i_2 is taken as +ve since we are traversing in clockwise direction in this case too.

Solving equations (1.34) and (1.35), we can find the mesh currents i_1 and i_2.

Once the mesh currents are known, the branch currents are evaluated in terms of mesh currents and then all the branch voltages are found using Ohms's law. If we have N meshes with N mesh currents, we can obtain N independent mesh equations. This set of N equations are independent, and thus guarantees a solution for the N mesh currents.

EXAMPLE 1.19

For the electrical network shown in Fig. 1.38, determine the loop currents and all branch currents.

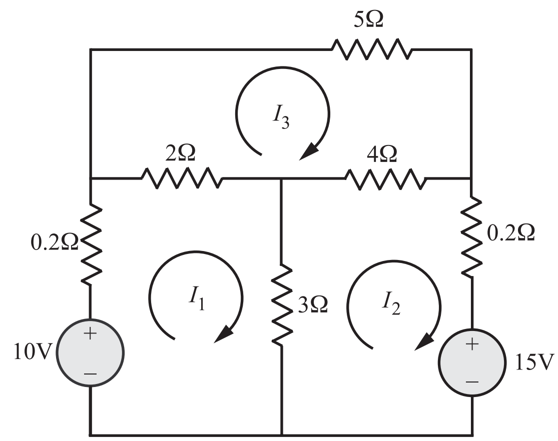


Figure 1.38

SOLUTION

Applying KVL for the meshes shown in Fig. 1.38, we have

$$\begin{aligned}
&Mesh\ 1: && 0.2I_1 + 2(I_1 - I_3) + 3(I_1 - I_2) - 10 = 0 \\
&\Rightarrow && 5.2I_1 - 3I_2 - 2I_3 = 10 \qquad (1.36) \\
&Mesh\ 2: && 3(I_2 - I_1) + 4(I_2 - I_3) + 0.2I_2 + 15 = 0 \\
&\Rightarrow && -3I_1 + 7.2I_2 - 4I_3 = -15 \qquad (1.37) \\
&Mesh\ 3: && 5I_3 + 2(I_3 - I_1) + 4(I_3 - I_2) = 0 \\
&\Rightarrow && -2I_1 - 4I_2 + 11I_3 = 0 \qquad (1.38)
\end{aligned}$$

Putting the equations (1.36) through (1.38) in matrix form, we have

$$\begin{bmatrix} 5.2 & -3 & -2 \\ -3 & 7.2 & -4 \\ -2 & -4 & 11 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 10 \\ -15 \\ 0 \end{bmatrix}$$

Using Cramer's rule, we get

$$I_1 = \mathbf{0.11A}$$

$$I_2 = -\mathbf{2.53A}$$

and

$$I_3 = -\mathbf{0.9A}$$

The various branch currents are now calculated as follows:

Current through 10V battery $= I_1 = 0.11\text{A}$

Current through 2Ω resistor $= I_1 - I_3 = 1.01\text{A}$

Current through 3Ω resistor $= I_1 - I_2 = 2.64\text{A}$

Current through 4Ω resistor $= I_2 - I_3 = -1.63\text{A}$

Current through 5Ω resistor $= I_3 = -0.9\text{A}$

Current through 15V battery $= I_2 = -2.53\text{A}$

The negative sign for I_2 and I_3 indicates that the actual directions of these currents are opposite to the assumed directions.

1.10 Mesh analysis with independent current sources

Let us consider an electrical circuit source having an independent current source as shown Fig. 1.39(a).

We find that the second mesh current $i_2 = -i_s$ and thus we need only to determine the first mesh current i_1, *Applying KVL to the first mesh*, we obtain

$$(R_1 + R_2)i_1 - R_2 i_2 = v$$

Since $\quad i_2 = -i_s,$

we get $\quad (R_1 + R_2)i_1 + i_s R_2 = v$

$$\Rightarrow \quad i_1 = \frac{v - i_s R_2}{R_1 + R_2}$$

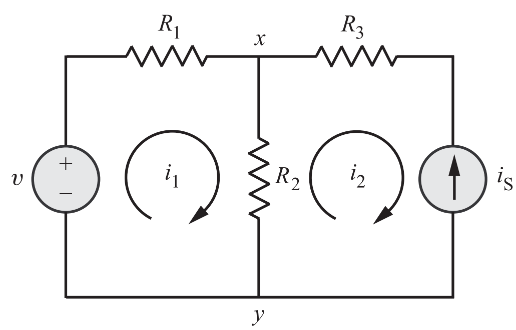


Figure 1.39(a) Circuit containing both independent voltage and current sources

As a second example, let us take an electrical circuit in which the current source i_s is common to both the meshes. This situation is shown in Fig. 1.39(b).

By *applying KCL at node x*, we recognize that, $i_2 - i_1 = i_s$

The two mesh equations (*using KVL*) are

Mesh 1 : $\quad R_1 i_1 + v_{xy} - v = 0$

Mesh 2 : $\quad (R_2 + R_3)i_2 - v_{xy} = 0$

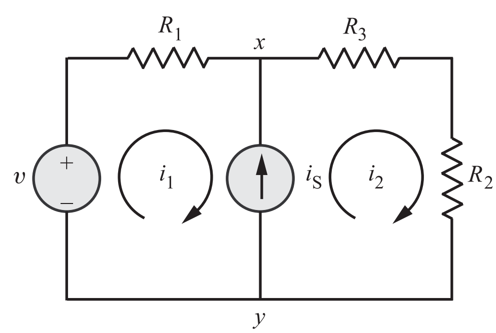


Figure 1.39(b) Circuit containing an independent current source common to both meshes

Adding the above two equations, we get

$$R_1 i_1 + (R_2 + R_3) i_2 = v$$

Substituting $i_2 = i_1 + i_s$ in the above equation, we find that

$$R_1 i_1 + (R_2 + R_3)(i_1 + i_s) = v$$

$$\Rightarrow \qquad i_1 = \frac{v - (R_2 + R_3) i_s}{R_1 + R_2 + R_3}$$

In this manner, we can handle independent current sources by recording the relationship between the mesh currents and the current source. The equation relating the mesh current and the current source is recorded as the *constraint equation.*

EXAMPLE 1.20

Find the voltage V_o in the circuit shown in Fig. 1.40.

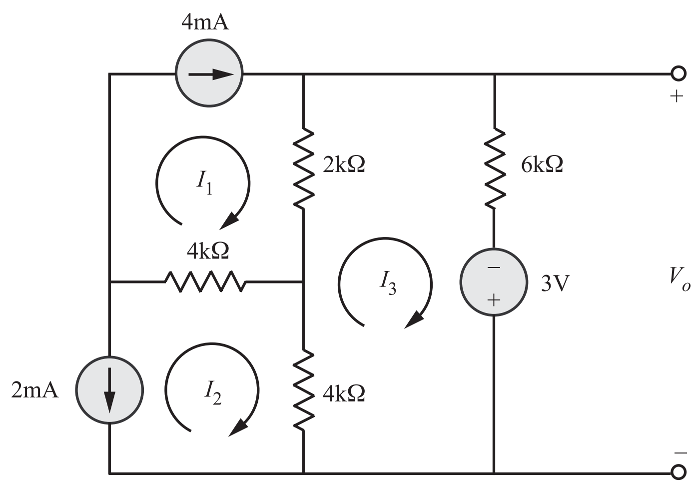


Figure 1.40

SOLUTION

Constraint equations:

$$I_1 = 4 \times 10^{-3} \text{ A}$$
$$I_2 = -2 \times 10^{-3} \text{ A}$$

Applying KVL for the mesh 3, we get

$$4 \times 10^3 [I_3 - I_2] + 2 \times 10^3 [I_3 - I_1] + 6 \times 10^3 I_3 - 3 = 0$$

Substituting the values of I_1 and I_2, we obtain

$$I_3 = 0.25 \text{ mA}$$

Hence,

$$\begin{aligned} V_o &= 6 \times 10^3 I_3 - 3 \\ &= 6 \times 10^3 (0.25 \times 10^{-3}) - 3 \\ &= \mathbf{-1.5\ V} \end{aligned}$$

1.11 Supermesh

A more general technique for mesh analysis method, when a current source is common to two meshes, involves the concept of a supermesh. A supermesh is created from two meshes that have a current source as a common element; the current source is in the interior of a supermesh. We thus reduce the number of meshes by one for each current source present. Figure 1.41 shows a supermesh created from the two meshes that have a current source in common.

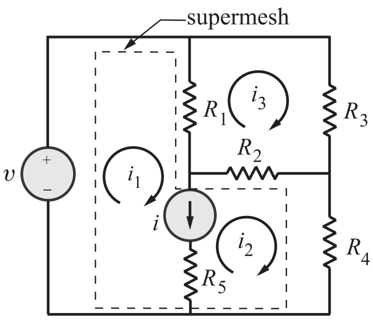


Figure 1.41 Circuit with a supermesh shown by the dashed line

EXAMPLE 1.21

Find the current i_o in the circuit shown in Fig. 1.42(a).

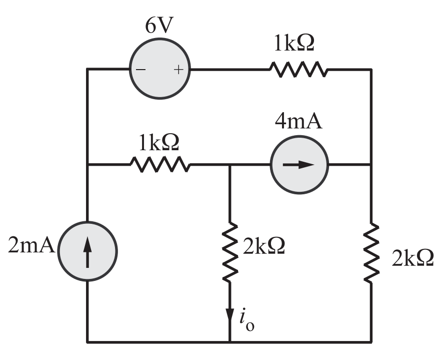


Figure 1.42(a)

SOLUTION

This problem is first solved by the techique explained in Section 1.10. Three mesh currents are specified as shown in Fig. 1.42(b). The mesh currents constrained by the current sources are

$$i = 2 \times 10^{-3} \text{ A}$$
$$i_2 - i_3 = 4 \times 10^{-3} \text{ A}$$

The KVL equations for meshes 2 and 3 respetively are

$$2 \times 10^3 i_2 + 2 \times 10^3 (i_2 - i_1) - v_{xy} = 0$$
$$-6 + 1 \times 10^3 i_3 + v_{xy} + 1 \times 10^3 (i_3 - i_1) = 0$$

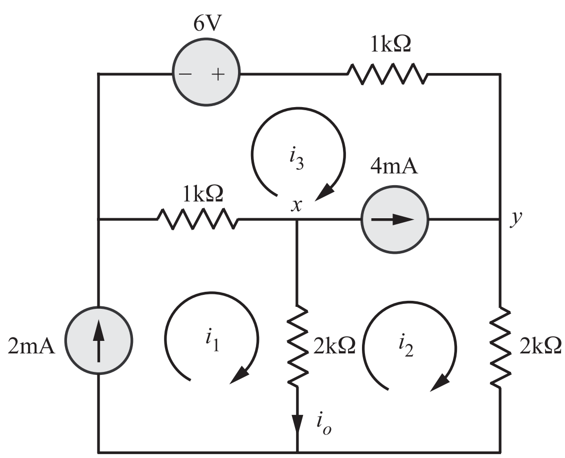


Figure 1.42(b)

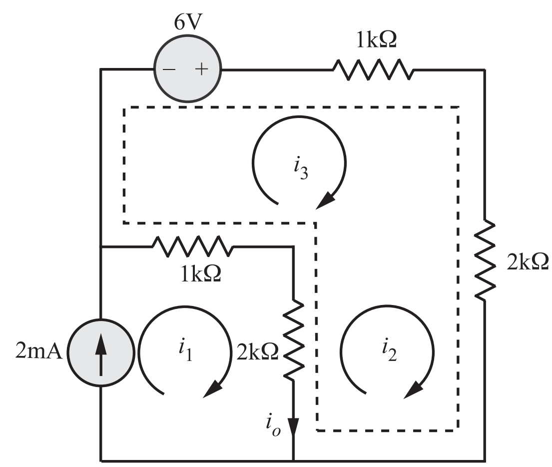


Figure 1.42(c)

Adding last two equations, we get

$$-6 + 1 \times 10^3 i_3 + 2 \times 10^3 i_2 + 2 \times 10^3 (i_2 - i_1) + 1 \times 10^3 (i_3 - i_1) = 0 \tag{1.39}$$

Substituting $i_1 = 2 \times 10^{-3}$A and $i_3 = i_2 - 4 \times 10^{-3}$A in the above equation, we get

$$-6 + 1 \times 10^3 \left[i_2 - 4 \times 10^{-3}\right] + 2 \times 10^3 i_2 + 2 \times 10^3 \left[i_2 - 2 \times 10^{-3}\right] + 1 \times 10^3 \left[i_2 - 4 \times 10^{-3} - 2 \times 10^{-3}\right] = 0$$

Solving we get

$$i_2 = \frac{10}{3}\ \text{mA}$$

Thus,

$$\begin{aligned} i_o &= i_1 - i_2 \\ &= 2 - \frac{10}{3} \\ &= \frac{-4}{3}\ \text{mA} \end{aligned}$$

The purpose of supermesh approach is to avoid introducing the unknown voltage v_{xy}. The supermesh is created by mentally removing the 4 mA current source as shown in Fig. 1.42(c). Then applying KVL equation around the dotted path, which defines the supermesh, using the orginal mesh currents as shown in Fig. 1.42(b), we get

$$-6 + 1 \times 10^3 i_3 + 2 \times 10^3 i_2 + 2 \times 10^3 (i_2 - i_1) + 1 \times 10^3 (i_3 - i_1) = 0$$

Note that the supermesh equation is same as equation 1.39 obtained earlier by introducing v_{xy}, the remaining procedure of finding i_o is same as before.

EXAMPLE 1.22

For the network shown in Fig. 1.43(a), find the mesh currents i_1, i_2 and i_3.

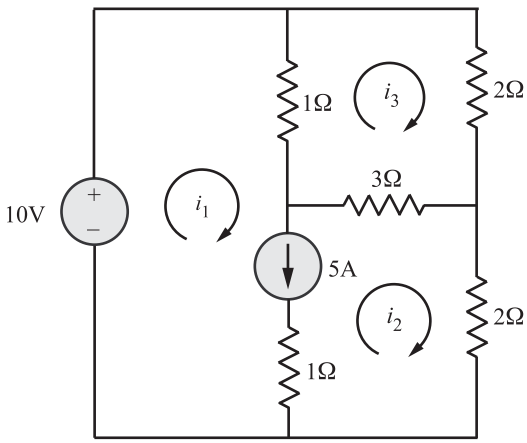


Figure 1.43(a)

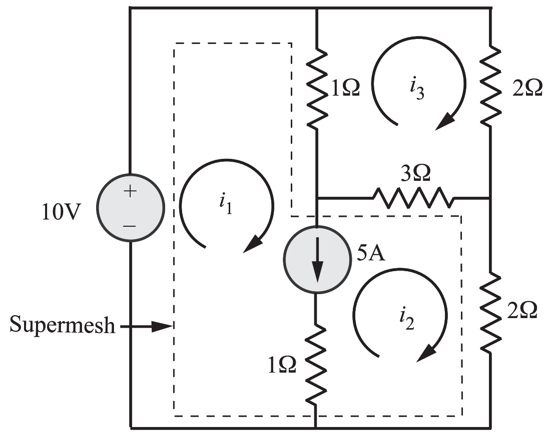


Figure 1.43(b)

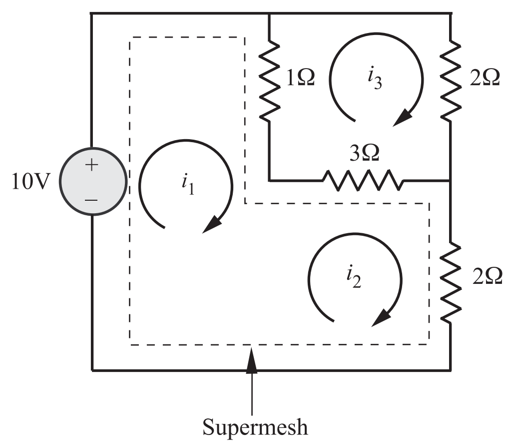


Figure 1.43(c)

SOLUTION

The 5A current source is in the common boundary of two meshes. The supermesh is shown as dotted lines in Figs.1.43(b) and 1.43(c), the branch having the 5A current source is removed from the circuit diagram. Then applying KVL around the dotted path, which defines the supermesh, using the original mesh currents as shown in Fig. 1.43(c), we find that

$$-10 + 1(i_1 - i_3) + 3(i_2 - i_3) + 2i_2 = 0$$

For mesh 3, we have

$$1(i_3 - i_1) + 2i_3 + 3(i_3 - i_2) = 0$$

Finally, the *constraint equation is*

$$i_1 - i_2 = 5$$

Then the above three eqations may be reduced to

$Supemesh$: $1i_1 + 5i_2 - 4i_3 = 10$

Mesh 3 : $-1i_1 - 3i_2 + 6i_3 = 0$

$current\ source$: $i_1 - i_2 = 5$

Solving the above simultaneous equations, we find that,

$$\boldsymbol{i_1 = 7.5\text{A}, i_2 = 2.5\text{A}, \text{ and } i_3 = 2.5\text{A}}$$

EXAMPLE 1.23

Find the mesh currents i_1, i_2 and i_3 for the network shown in Fig. 1.44.

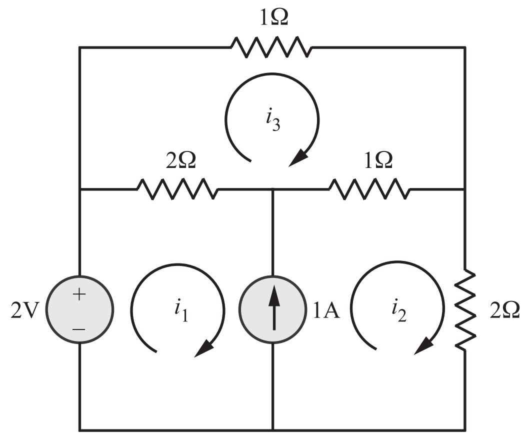


Figure 1.44

SOLUTION

Here we note that 1A independent current source is in the common boundary of two meshes. Mesh currents i_1, i_2 and i_3, are marked in the clockwise direction. The supermesh is shown as dotted lines in Figs. 1.45(a) and 1.45(b). In Fig. 1.45(b), the 1A current source is removed from the circuit diagram, then applying the KVL around the dotted path, which defines the supermesh, using original mesh currents as shown in Fig. 1.45(b), we find that

$$-2 + 2(i_1 - i_3) + 1(i_2 - i_3) + 2i_2 = 0$$

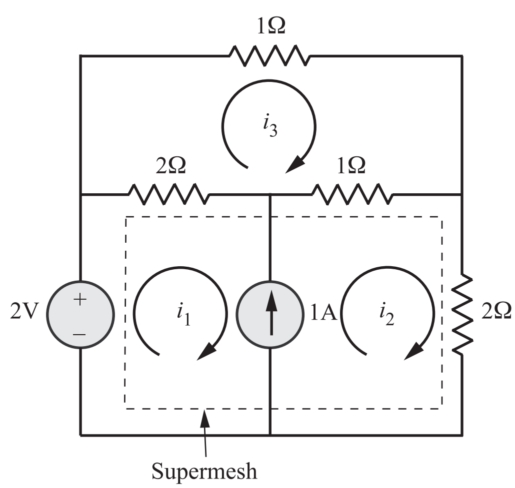


Figure 1.45(a)

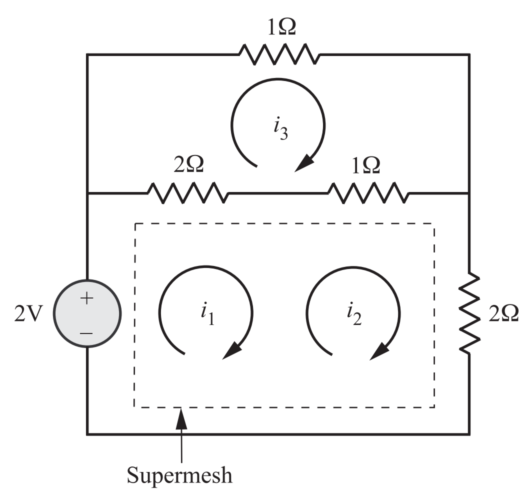


Figure 1.45(b).

For mesh 3, the KVL *equation is*

$$2(i_3 - i_1) + 1i_3 + 1(i_3 - i_2) = 0$$

Finally, the *constraint equation is*

$$i_1 - i_2 = 1$$

Then the above three equations may be reduced to
Supermesh : $2i_1 + 3i_2 - 3i_3 = 2$
Mesh 3 : $2i_1 + i_2 - 4i_3 = 0$
Current source: $i_1 - i_2 = 1$
Solving the above simultaneous equations, we find that
$\boldsymbol{i_1 = 1.55\text{A}, i_2 = 0.55\text{A}, i_3 = 0.91\text{A}}$

1.12 Mesh analysis for the circuits involving dependent sources

The persence of one or more dependent sources merely requires each of these source quantites and the variable on which it depends to be expressed in terms of assigned mesh currents. That is, to begin with, we treat the dependent source as though it were an independent source while writing the KVL equations. Then we write the *controlling equation* for the dependent source. The following examples illustrate the point.

EXAMPLE 1.24

(a) Use the mesh current method to solve for i_a in the circuit shown in Fig. 1.46.
(b) Find the power delivered by the independent current source.
(c) Find the power delivered by the dependent voltage source.

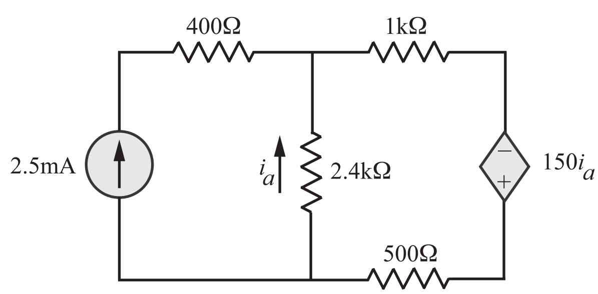


Figure 1.46

SOLUTION

(a) We mark two mesh currents i_1 and i_2 as shown in Fig. 1.47. We find that $i = 2.5$mA. *Applying KVL to mesh* 2, we find that

$$2400(i_2 - 0.0025) + 1500i_2 - 150(i_2 - 0.0025) = 0 \quad (\because i_a = i_2 - 2.5 \text{ mA})$$

$$\Rightarrow \qquad 3750i_2 = 6 - 0.375$$

$$= 5.625$$

$$\Rightarrow \qquad i_2 = \mathbf{1.5\ mA}$$

$$i_a = i_2 - 2.5 = \mathbf{-1.0mA}$$

(b) *Applying KVL to mesh* 1, we get

$-v_o + 2.5(0.4) - 2.4i_a = 0$

$\Rightarrow \quad v_o = 2.5(0.4) - 2.4(-1.0) = 3.4\text{V}$

$$P_{\text{ind.source}} = 3.4 \times 2.5 \times 10^{-3}$$
$$= \mathbf{8.5\ mW}(\text{delivered})$$

(c) $P_{\text{dep.source}} = 150i_a(i_2)$
$$= 150(-1.0 \times 10^{-3})(1.5 \times 10^{-3})$$
$$= \mathbf{-0.225\ mW}(\text{absorbed})$$

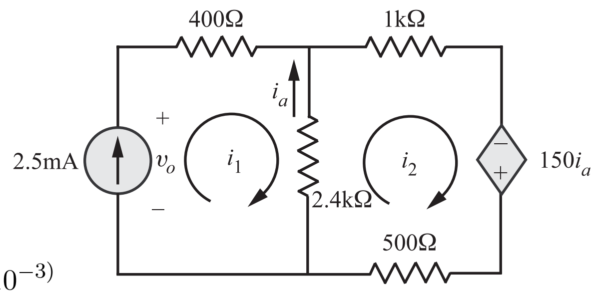


Figure 1.47

EXAMPLE 1.25

Find the total power delivered in the circuit using mesh-current method.

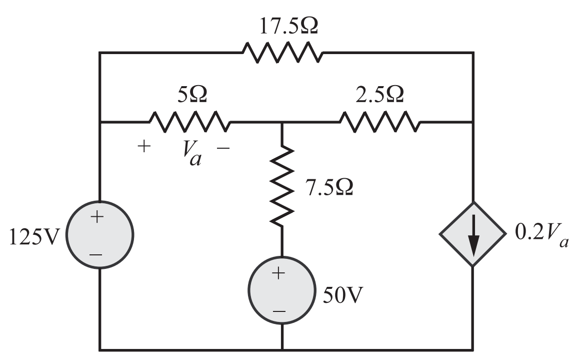


Figure 1.48

SOLUTION

Let us mark three mesh currents i_1, i_2 and i_3 as shown in Fig. 1.49.

KVL equations:

Mesh 1: $\quad 17.5i_1 + 2.5(i_1 - i_3) + 5(i_1 - i_2) = 0$

$\Rightarrow \quad 25i_1 - 5i_2 - 2.5i_3 = 0$

Mesh 2: $\quad -125 + 5(i_2 - i_1) + 7.5(i_2 - i_3) + 50 = 0$

$\Rightarrow \quad -5i_1 + 12.5i_2 - 7.5i_3 = 75$

Constraint equations:

$$i_3 = 0.2V_a$$
$$V_a = 5(i_2 - i_1)$$

Thus, $\quad i_3 = 0.2 \times 5(i_2 - i_1) = i_2 - i_1.$

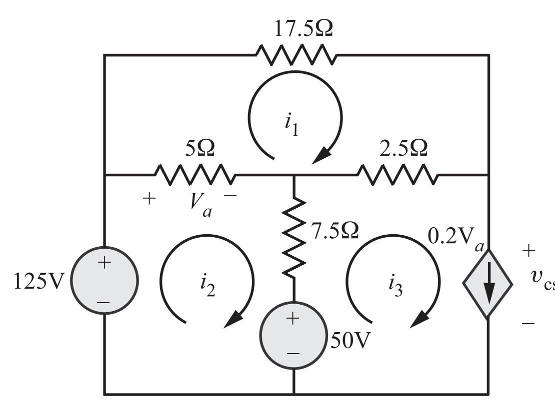


Figure 1.49

Making use of i_3 in the mesh equations, we get

$$\begin{aligned}
&Mesh\ 1: && 25i_1 - 5i_2 - 2.5(i_2 - i_1) = 0\\
&\Rightarrow && 27.5i_1 - 7.5i_2 = 0\\
&Mesh\ 2: && -5i_1 + 12.5i_2 - 7.5(i_2 - i_1) = 75\\
&\Rightarrow && 2.5i_1 + 5i_2 = 75
\end{aligned}$$

Solving the above two equations, we get

$$i_1 = 3.6 \text{ A}, i_2 = 13.2 \text{ A}$$

and

$$i_3 = i_2 - i_1 = 9.6 \text{ A}$$

Applying KVL through the path having $5\Omega \to 2.5\Omega \to v_{cs} \to 125\text{V}$ source, we get,

$$\begin{aligned}
5(i_2 - i_1) + 2.5(i_3 - i_1) + v_{cs} - 125 &= 0\\
\Rightarrow \quad v_{cs} &= 125 - 5(i_2 - i_1) - 2.5(i_3 - i_1)\\
&= 125 - 48 - 2.5(9.6 - 3.6) = 62 \text{ V}\\
P_{vcs} &= 62(9.6) = 595.2\text{W (absorbed)}\\
P_{50\text{V}} &= 50(i_2 - i_3) = 50(13.2 - 9.6) = 180\text{W (absorbed)}\\
P_{125\text{V}} &= 125i_2 = \mathbf{1650W\ (delivered)}
\end{aligned}$$

EXAMPLE 1.26

Use the mesh-current method to find the power delivered by the dependent voltage source in the circuit shown in Fig. 1.50.

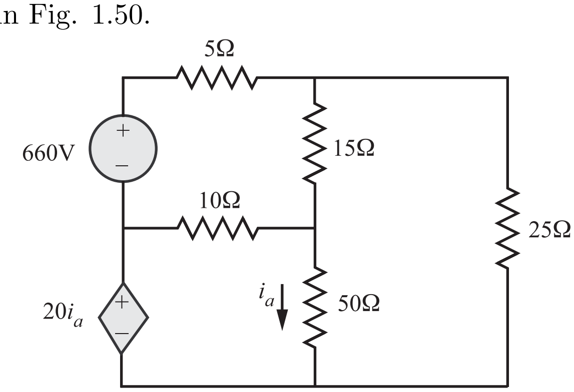


Figure 1.50

SOLUTION

Applying KVL to the meshes 1, 2 and 3 shown in Fig 1.51, we have

$$\begin{aligned}
&Mesh\ 1: && 5i_1 + 15(i_1 - i_3) + 10(i_1 - i_2) - 660 = 0\\
&\Rightarrow && 30i_1 - 10i_2 - 15i_3 = 660
\end{aligned}$$

$$
\begin{aligned}
&Mesh\ 2: && -20i_a + 10(i_2 - i_1) + 50(i_2 - i_3) = 0 \\
&\Rightarrow && 10(i_2 - i_1) + 50(i_2 - i_3) = 20i_a \\
&\Rightarrow && -10i_1 + 60i_2 - 50i_3 = 20i_a \\
&Mesh\ 3: && 15(i_3 - i_1) + 25i_3 + 50(i_3 - i_2) = 0 \\
&\Rightarrow && -15i_1 - 50i_2 + 90i_3 = 0
\end{aligned}
$$

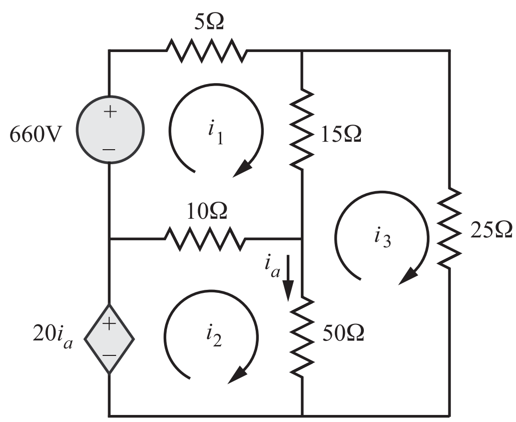


Figure 1.51

Also $\quad i_a = i_2 - i_3$

Solving, $\boldsymbol{i_1 = 42\text{A}, i_2 = 27\text{A}, i_3 = 22\text{A}, i_a = 5\text{A}}$.

Power delivered by the dependent voltage source $= P_{20i_a} = (20i_a)i_2$

$= \mathbf{2700W}$ (delivered)

1.13 Node voltage anlysis

In the nodal analysis, Kirchhoff's current law is used to write the equilibrium equations. A node is defined as a junction of two or more branches. If we define one node of the network as a reference node (a point of zero potential or ground), the remaining nodes of the network will have a fixed potential relative to this reference. Equations relating to all nodes except for the reference node can be written by applying KCL.

Refering to the circuit shown in Fig.1.52, we can arbitrarily choose any node as the reference node. However, it is convenient to choose the node with most connected branches. Hence, node 3 is chosen as the reference node here. It is seen from the network of Fig. 1.52 that there are three nodes.

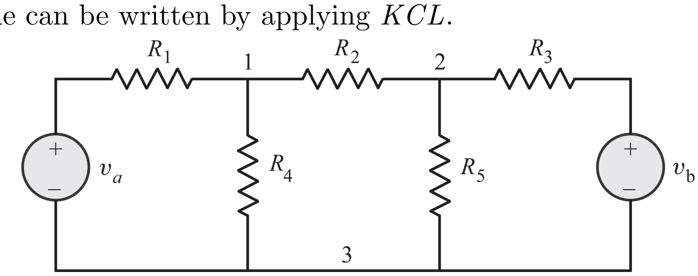


Figure 1.52 Circuit with three nodes where the lower node 3 is the reference node

Hence, number of equations based on KCL will be total number of nodes minus one. That is, in the present context, we will have only two KCL equations referred to as node equations. For applying KCL at node 1 and node 2, we assume that all the currents leave these nodes as shown in Figs. 1.53 and 1.54.

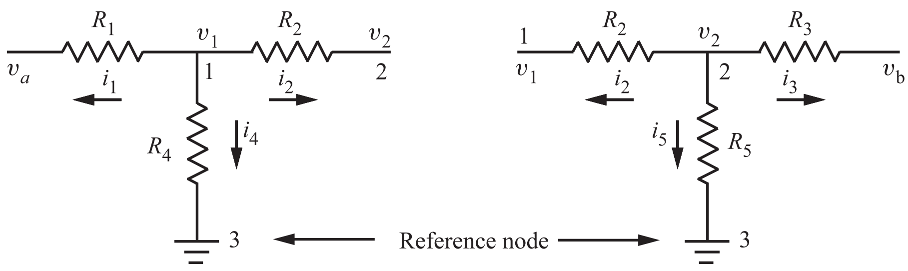


Figure 1.53 Simplified circuit for applying KCL at node 1

Figure 1.54 Simplified circuit for applying KCL at node 2

Applying KCL at node 1 *and* 2, we find that

(i) At node 1: $$i_1 + i_2 + i_4 = 0$$

$$\Rightarrow \quad \frac{v_1 - v_a}{R_1} + \frac{v_1 - v_2}{R_2} + \frac{v_1 - 0}{R_4} = 0$$

$$\Rightarrow \quad v_1\left[\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_4}\right] - v_2\frac{1}{R_2} = \frac{v_a}{R_1} \tag{1.40}$$

(ii) At node 2: $$i_2 + i_3 + i_5 = 0$$

$$\Rightarrow \quad \frac{v_2 - v_1}{R_2} + \frac{v_2 - v_b}{R_3} + \frac{v_2}{R_5} = 0$$

$$\Rightarrow \quad -v_1\left[\frac{1}{R_2}\right] + v_2\left[\frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_5}\right] = \frac{v_b}{R_3} \tag{1.41}$$

Putting equations (1.40) and (1.41) in matrix form, we get

$$\begin{bmatrix} \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_4} & -\frac{1}{R_2} \\ -\frac{1}{R_2} & \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_5} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{v_a}{R_1} \\ \frac{v_b}{R_3} \end{bmatrix}$$

The above matrix equation can be solved for node voltages v_1 and v_2 using Cramer's rule of determinants. Once v_1 and v_2 are obtainted, then by using Ohm's law, we can find all the branch currents and hence the solution of the network is obtained.

EXAMPLE 1.27

Refer the circuit shown in Fig. 1.55. Find the three node voltages v_a, v_b and v_c, when all the conductances are equal to 1S.

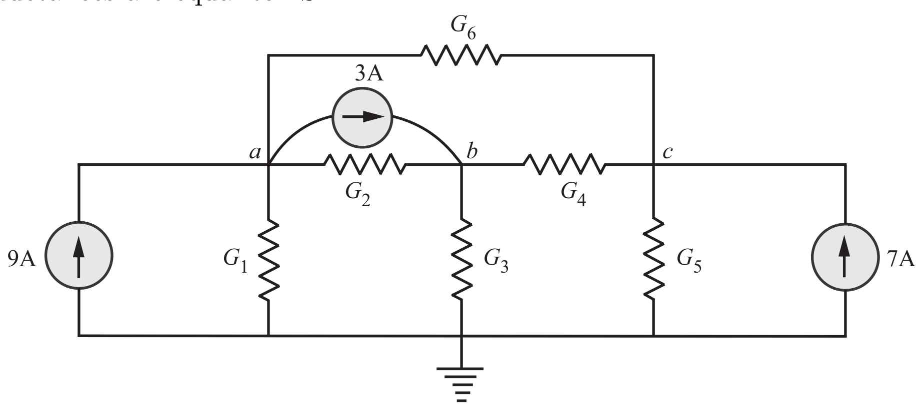


Figure 1.55

SOLUTION

At node **a**: $(G_1 + G_2 + G_6)v_a - G_2 v_b - G_6 v_c = 9 - 3$

At node **b**: $-G_2 v_a + (G_4 + G_2 + G_3)v_b - G_4 v_c = 3$

At node **c**: $-G_6 v_a - G_4 v_b + (G_4 + G_5 + G_6)v_c = 7$

Substituting the values of various conductances, we find that

$$3v_a - v_b - v_c = 6$$
$$-v_a + 3v_b - v_c = 3$$
$$-v_a - v_b + 3v_c = 7$$

Putting the above equations in matrix form, we see that

$$\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 7 \end{bmatrix}$$

Solving the matrix equation using cramer's rule, we get

$$v_a = 5.5\text{V}, \quad v_b = 4.75\text{V}, \quad v_c = 5.75\text{V}$$

The determinant Δ used for computing v_a, v_b and v_c in general form is given by

$$G = \begin{vmatrix} \sum_a G & -G_{ab} & -G_{ac} \\ -G_{ab} & \sum_b G & -G_{bc} \\ -G_{ac} & -G_{bc} & \sum_c G \end{vmatrix}$$

where $\sum_i G$ is the sum of the conductances at node i, and G_{ij} is the sum of conductances conecting nodes i and j.

The node voltage matrix equation for a circuit with k unknown node voltages is

$$\mathbf{G}\mathbf{v} = \mathbf{i_s},$$

where,

$$\mathbf{v} = \begin{bmatrix} v_a \\ v_b \\ \vdots \\ v_k \end{bmatrix}$$

is the vector consisting of k unknown node voltages.

The matrix

$$\mathbf{i_a} = \begin{bmatrix} i_{s1} \\ i_{s2} \\ \vdots \\ i_{sk} \end{bmatrix}$$

is the vector consisting of k current sources and i_{sk} is the sum of all the source currents entering the node k. If the k^{th} current source is not present, then $i_{sk} = 0$.

EXAMPLE 1.28

Use the node voltage method to find how much power the 2A source extracts from the circuit shown in Fig. 1.56.

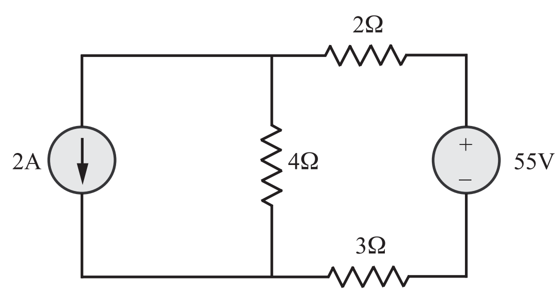


Figure 1.56

SOLUTION

Applying KCL at node a, we get

$$2 + \frac{v_a}{4} + \frac{v_a - 55}{5} = 0$$

$$\Rightarrow \quad v_a = 20\text{V}$$

$$P_{2\text{A source}} = 20(2) = 40\text{W (absorbing)}$$

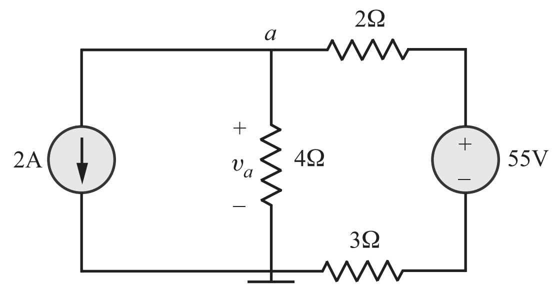


Figure 1.57

EXAMPLE 1.29

Refer the circuit shown in Fig. 1.58(a).

(a) Use the node voltage method to find the branch currents i_1 to i_6.

(b) Test your solution for the branch currents by showing the total power dissipated equals the power developed.

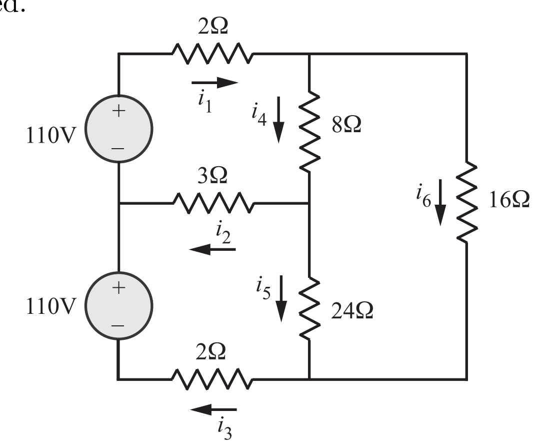


Figure 1.58(a)

SOLUTION

(a) At node v_1:

$$\frac{v_1 - 110}{2} + \frac{v_1 - v_2}{8} + \frac{v_1 - v_3}{16} = 0$$

$$\Rightarrow \qquad 11v_1 - 2v_2 - v_3 = 880$$

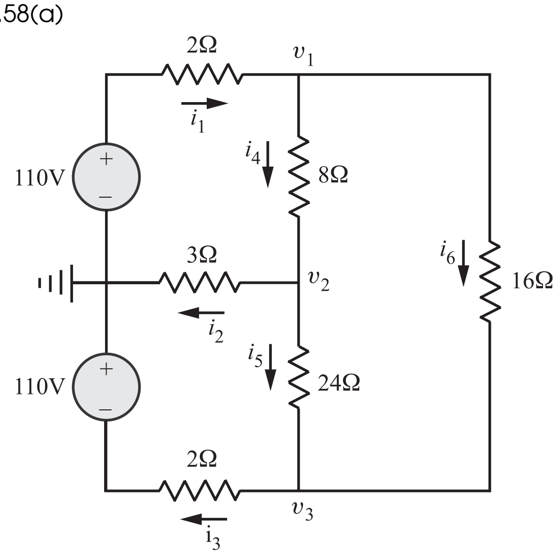


Figure 1.58(b)

At node v_2:

$$\frac{v_2 - v_1}{8} + \frac{v_2}{3} + \frac{v_2 - v_3}{24} = 0$$

$$\Rightarrow \qquad -3v_1 + 12v_2 - v_3 = 0$$

At node v_3:

$$\frac{v_3 + 110}{2} + \frac{v_3 - v_2}{24} + \frac{v_3 - v_1}{16} = 0$$

$$\Rightarrow \qquad -3v_1 - 2v_2 + 29v_3 = -2640$$

Solving the above nodal equations,we get

$$v_1 = 74.64\text{V}, \; v_2 = 11.79\text{V}, \; v_3 = -82.5\text{V}$$

Hence,

$$i_1 = \frac{110 - v_1}{2} = \mathbf{17.68A}$$

$$i_2 = \frac{v_2}{3} = \mathbf{3.93A}$$

$$i_3 = \frac{v_3 + 110}{2} = \mathbf{13.75A}$$
$$i_4 = \frac{v_1 - v_2}{8} = \mathbf{7.86A}$$
$$i_5 = \frac{v_2 - v_3}{24} = \mathbf{3.93A}$$
$$i_6 = \frac{v_1 - v_3}{16} = \mathbf{9.82A}$$

(b) Total power delivered $= 110i_1 + 110i_3 = \mathbf{3457.3W}$

Total power dissipated $= i_1^2 \times 2 + i_2^2 \times 3 + i_3^2 \times 2 + i_4^2 \times 8 + i_5^2 \times 24 + i_6^2 \times 16$

$= \mathbf{3457.3\ W}$

EXAMPLE 1.30

(a)Use the node voltage method to show that the output volatage v_o in the circuit of Fig 1.59(a) is equal to the average value of the source voltages.

(b) Find v_o if $v_1 = 150$V, $v_2 = 200$V and $v_3 = -50$V.

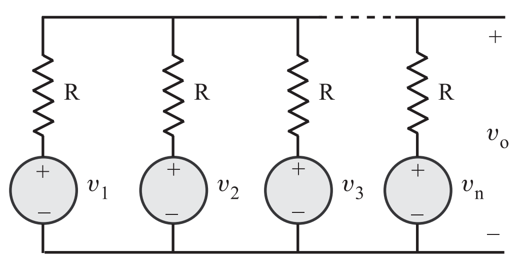


Figure 1.59(a)

SOLUTION

Applying KCL at node a, we get

$$\frac{v_o - v_1}{R} + \frac{v_o - v_2}{R} + \frac{v_o - v_3}{R} + \cdots + \frac{v_o - v_n}{R} = 0$$

$$\Rightarrow \; nv_o = v_1 + v_2 + \cdots + v_n$$

$$\text{Hence, } v_o = \frac{1}{n}\left[v_1 + v_2 + \cdots + v_n\right]$$

$$= \frac{1}{n}\sum_{k=1}^{n} v_k$$

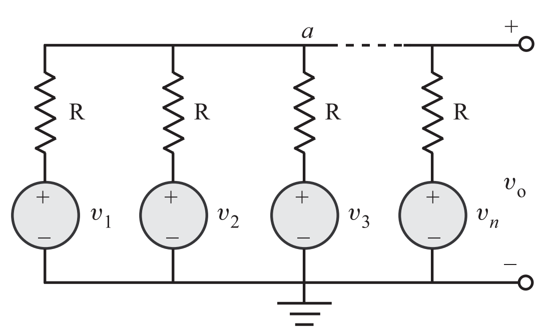


Figure 1.59(b)

(b) $$v_o = \frac{1}{3}(150 + 200 - 50) = \mathbf{100V}$$

EXAMPLE 1.31

Use nodal analysis to find v_o in the circuit of Fig. 1.60.

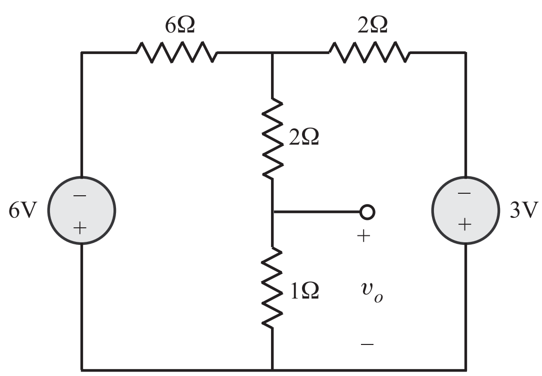


Figure 1.60

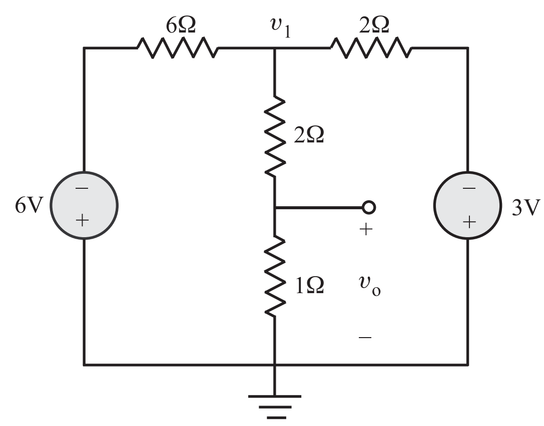


Figure 1.61

SOLUTION

Referring Fig 1.61, at node v_1:

$$\frac{v_1+6}{6}+\frac{v_1}{3}+\frac{v_1+3}{2}=0$$

$$\Rightarrow \quad \frac{v_1}{6}+\frac{v_1}{3}+\frac{v_1}{2}=-2.5$$

$$\Rightarrow \quad v_1=-2.5 \text{ V}$$

$$v_o=\left[\frac{v_1}{2+1}\right]\times 1$$

$$=\frac{-2.5}{3}\times 1$$

$$= \mathbf{-0.83volts}$$

EXAMPLE 1.32

Refer to the network shown in Fig. 1.62. Find the power delivered by 1A current source.

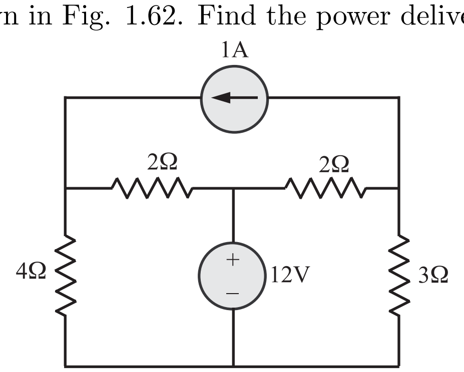


Figure 1.62

SOLUTION

Referring to Fig. 1.63, *applying KVL to the path* $v_a \rightarrow 4\Omega \rightarrow 3\Omega$, we get

$$v_a = v_1 - v_3$$

$$v_2 = 12\text{V}$$

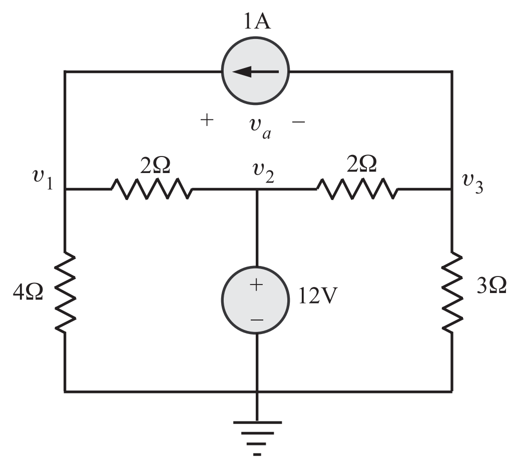


Figure 1.63

At node v_1 : $\dfrac{v_1}{4} + \dfrac{v_1 - v_2}{2} - 1 = 0$

$$\Rightarrow \quad \frac{v_1}{4} + \frac{v_1 - 12}{2} - 1 = 0$$

$$v_1 = 9.33 \text{ V}$$

At node v_2: $\dfrac{v_3}{3} + \dfrac{v_3 - v_2}{2} + 1 = 0$

$$\Rightarrow \quad \frac{v_3}{3} + \frac{v_3 - 12}{2} + 1 = 0$$

$$\Rightarrow \quad v_3 = 6\text{V}$$

Hence, $\quad v_a = 9.33 - 6 = 3.33$ volts

$$P_{1\text{A source}} = v_a \times 1$$

$$= 3.33 \times 1 = \mathbf{3.33W} \ (\textbf{delivering})$$

1.14 Supernode

Inorder to understand the concept of a supernode, let us consider an electrical circuit as shown in Fig. 1.64.

Applying KVL clockwise to the loop containing R_1, voltage source and R_2, we get

$$v_a = v_s + v_b$$

$$\Rightarrow \quad v_a - v_b = v_s (Constraint\ equation) \tag{1.42}$$

To account for the fact that the source voltage is known, we consider both v_a and v_b as part of one larger node represented by the dotted ellipse as shown in Fig. 1.64. We need a larger node because v_a and v_b are dependent (see equation 1.42). This larger node is called the supernode.

Applying KCL at nodes a and b, we get

$$\frac{v_a}{R_1} - i_a = 0$$

and $\quad \dfrac{v_b}{R_2} + i_a = i_s$

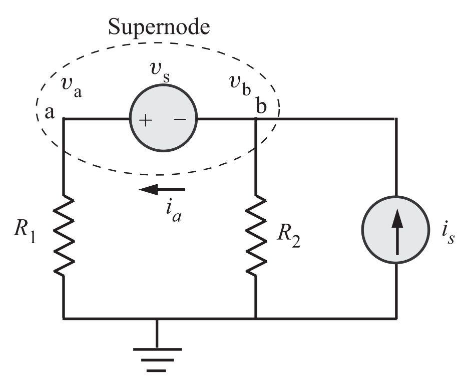


Figure 1.64 Circuit with a supernode incorporating v_a and v_b.

Adding the above two equations, we find that

$$\frac{v_a}{R_1} + \frac{v_b}{R_2} = i_s$$

$$\Rightarrow \qquad v_a G_1 + v_b G_2 = i_s \tag{1.43}$$

Solving equations (1.42) and (1.43), we can find the values of v_a and v_b.

When we apply KCL at the supernode, mentally imagine that the voltage source v_s is removed from the the circuit of Fig. 1.63, but the voltage at nodes a and b are held at v_a and v_b respectively. In other words, by applying KCL at supernode, we obtain

$$v_a G_1 + v_a G_2 = i_s$$

The equation is the same equation (1.43). As in supermesh, the KCL for supernode eliminates the problem of dealing with a current through a voltage source.

Procedure for using supernode:

1. Use it when a branch between non-reference nodes is connected by an independent or a dependent voltage source.
2. Enclose the voltage source and the two connecting nodes inside a dotted ellipse to form the supernode.
3. Write the constraint equation that defines the voltage relationship between the two non-reference node as a result of the presence of the voltage source.
4. Write the *KCL equation* at the supernode.
5. If the voltage source is dependent, then the *constraint equation* for the dependent source is also needed.

EXAMPLE 1.33

Refer the electrical circuit shown in Fig. 1.65 and find v_a.

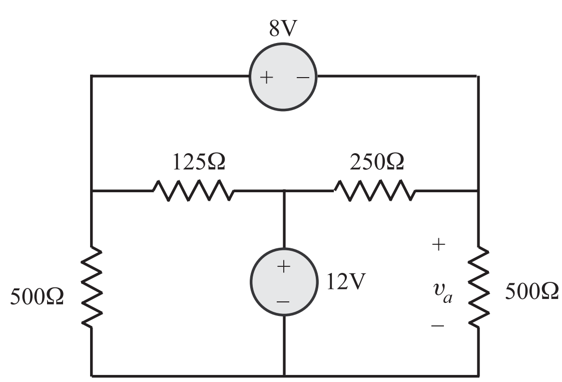


Figure 1.65

SOLUTION

The constraint equation is,

$$v_b - v_a = 8$$

$$\Rightarrow \quad v_b = v_a + 8$$

The KCL equation at the supernode is then,

$$\frac{v_a+8}{500} + \frac{(v_a+8)-12}{125} + \frac{v_a-12}{250} + \frac{v_a}{500} = 0$$

Therefore, $\quad \boldsymbol{v_a = 4\mathrm{V}}$

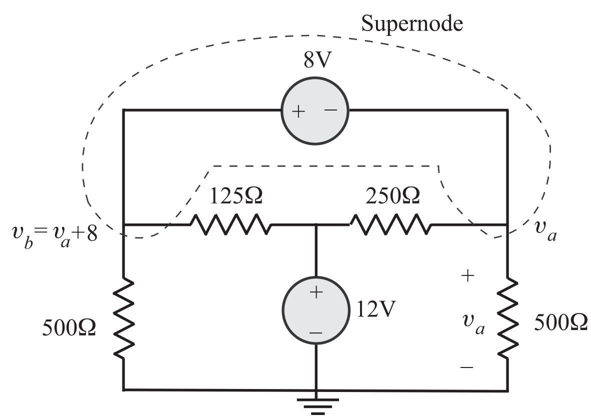


Figure 1.66

EXAMPLE 1.34

Use the nodal analysis to find v_o in the network of Fig. 1.67.

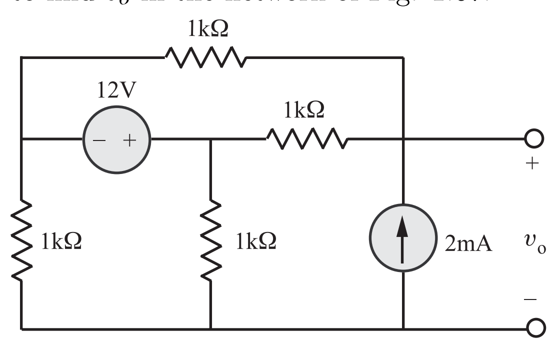


Figure 1.67

SOLUTION

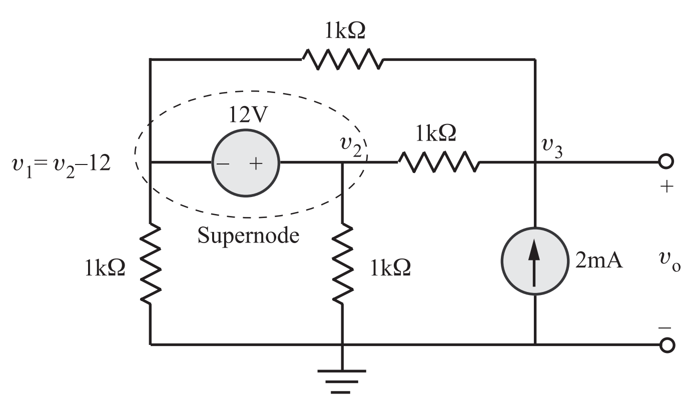


Figure 1.68

The constraint equation is,

$$v_2 - v_1 = 12$$
$$\Rightarrow \qquad v_1 = v_2 - 12$$

KCL at supernode:

$$\frac{v_2 - 12}{1 \times 10^3} + \frac{(v_2 - 12) - v_3}{1 \times 10^3} + \frac{v_2}{1 \times 10^3} + \frac{v_2 - v_3}{1 \times 10^3} = 0$$
$$\Rightarrow \qquad 4 \times 10^{-3} v_2 - 2 \times 10^{-3} v_3 = 24 \times 10^{-3}$$
$$\Rightarrow \qquad 4v_2 - 2v_3 = 24$$

At node v_3:

$$\frac{v_3 - v_2}{1 \times 10^3} + \frac{v_3 - (v_2 - 12)}{1 \times 10^3} = 2 \times 10^{-3}$$
$$\Rightarrow \qquad -2 \times 10^{-3} v_2 + 2 \times 10^{-3} v_3 = -10 \times 10^{-3}$$
$$-2v_2 + 2v_3 = -10$$

Solving we get $\qquad v_2 = 7\text{V}$

$$v_3 = 2\text{V}$$

Hence, $\qquad v_o = v_3 = \mathbf{2V}$

EXAMPLE 1.35

Refer the network shown in Fig. 1.69. Find the current I_o.

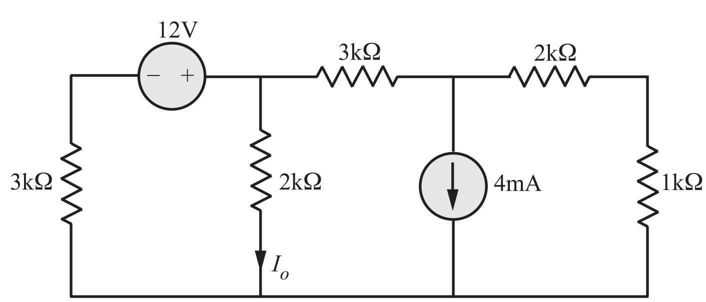


Figure 1.69

SOLUTION

Constriant equation:

$$v_3 = v_1 - 12$$

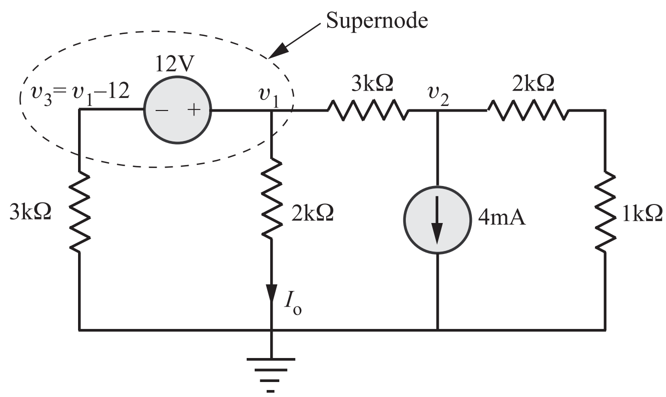


Figure 1.70

KCL at supernode:

$$\frac{v_1 - 12}{3 \times 10^3} + \frac{v_1}{2 \times 10^3} + \frac{v_1 - v_2}{3 \times 10^3} = 0$$

$$\Rightarrow \quad \frac{7}{6} \times 10^{-3} v_1 - \frac{1}{3} \times 10^{-3} v_2 = 4 \times 10^{-3}$$

$$\Rightarrow \quad \frac{7}{6} v_1 - \frac{1}{3} v_2 = 4$$

KCL at node 2:

$$\frac{v_2 - v_1}{3 \times 10^3} + \frac{v_2}{3 \times 10^3} + 4 \times 10^{-3} = 0$$

$$\Rightarrow \quad -\frac{1}{3} \times 10^{-3} v_1 + \frac{2}{3} \times 10^{-3} v_2 = -4 \times 10^{-3}$$

$$\Rightarrow \quad -\frac{1}{3} v_1 + \frac{2}{3} v_2 = -4$$

Putting the above two nodal equations in matrix form, we get

$$\begin{bmatrix} \frac{7}{6} & \frac{-1}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$$

Solving the above two matrix equations using Cramer's rule, we get

$$v_1 = 2\text{V}$$

$$\Rightarrow \quad I_o = \frac{v_1}{2 \times 10^3} = \frac{2}{2 \times 10^3} = \mathbf{1mA}$$

EXAMPLE 1.36

Refer the network shown in Fig. 1.71. Find the power delivered by the dependent voltage source in the network.

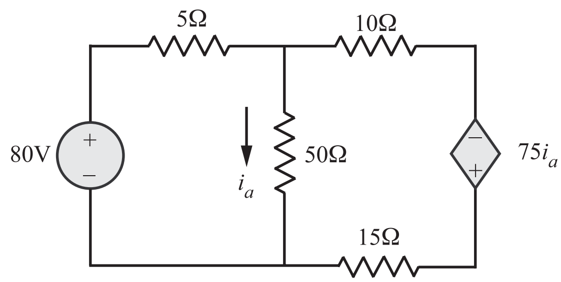


Figure 1.71

SOLUTION

Refer Fig. 1.72, *KCL at node* 1:

$$\frac{v_1 - 80}{5} + \frac{v_1}{50} + \frac{v_1 + 75i_a}{25} = 0$$

where $\quad i_a = \dfrac{v_1}{50}$

$$\Rightarrow \quad \frac{v_1 - 80}{5} + \frac{v_1}{50} + \frac{v_1 + 75\left(\frac{v_1}{50}\right)}{25} = 0$$

Solving we get $\qquad v_1 = \mathbf{50V}$

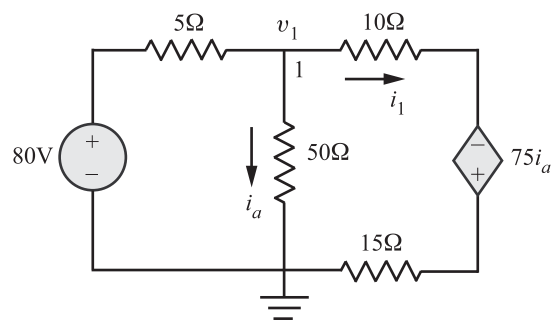


Figure 1.72

$$\Rightarrow \qquad i_a = \frac{v_1}{50} = \frac{50}{50} = 1\text{A}$$

Also,
$$\begin{aligned} i_1 &= \frac{v_1 - (-75i_a)}{(10+15)} \\ &= \frac{v_1 + 75i_a}{(10+15)} \\ &= \frac{50 + 75 \times 1}{(10+15)} = 5\text{A} \\ P_{75ia} &= (75i_a)i_1 \\ &= 75 \times 1 \times 5 \\ &= \mathbf{375W} \text{ (delivered)} \end{aligned}$$

EXAMPLE 1.37

Use the node-voltage method to find the power developed by the 20 V source in the circuit shown in Fig. 1.73.

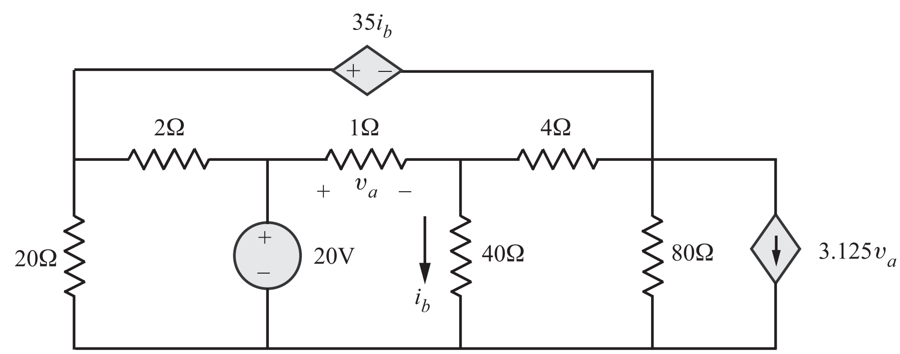


Figure 1.73

SOLUTION

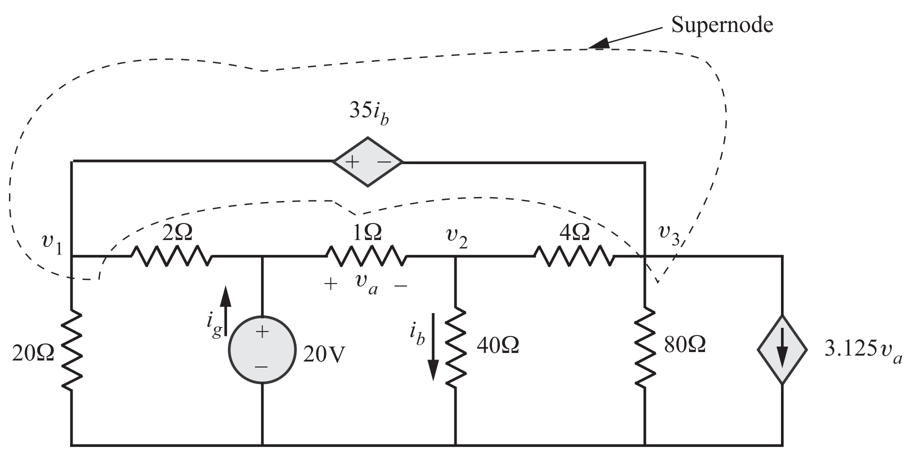


Figure 1.74

Constraint equations:

$$v_a = 20 - v_2$$
$$v_1 - 31i_b = v_3$$
$$i_b = \frac{v_2}{40}$$

Node equations:

(i) Supernode:

$$\frac{v_1}{20} + \frac{v_1 - 20}{2} + \frac{v_3 - v_2}{4} + \frac{v_3}{80} + 3.125v_a = 0$$

$$\Rightarrow \quad \frac{v_1}{20} + \frac{v_1 - 20}{2} + \frac{(v_1 - 35i_b) - v_2}{4} + \frac{(v_1 - 35i_b)}{80} + 3.125(20 - v_2) = 0$$

$$\Rightarrow \quad \frac{v_1}{20} + \frac{v_1 - 20}{2} + \frac{\left(v_1 - 35\frac{v_2}{40}\right) - v_2}{4} + \frac{\left(v_1 - 35\frac{v_2}{40}\right)}{80} + 3.125(20 - v_2) = 0$$

(ii) *At node* v_2:

$$\frac{v_2}{40} + \frac{v_2 - v_3}{4} + \frac{v_2 - 20}{1} = 0$$

$$\Rightarrow \qquad \frac{v_2}{40} + \frac{v_2 - (v_1 - 35i_b)}{4} + \frac{v_2 - 20}{1} = 0$$

$$\Rightarrow \qquad \frac{v_2}{40} + \frac{v_2 - \left(v_1 - 35\frac{v_2}{40}\right)}{4} + \frac{v_2 - 20}{1} = 0$$

Solving the above two nodal equations, we get

$$v_1 = -20.25\text{V}, \quad v_2 = 10\text{V}$$

Then

$$\begin{aligned} v_3 &= v_1 - 35i_b \\ &= v_1 - 35\frac{v_2}{40} \\ &= \mathbf{-29V} \end{aligned}$$

Also,

$$\begin{aligned} i_g &= \frac{20 - v_1}{2} + \frac{20 - v_2}{1} \\ &= \frac{20 + 20.25}{2} + \frac{(20 - 10)}{1} \\ &= \mathbf{30.125\ A} \end{aligned}$$

$$\begin{aligned} P_{20\text{V}} &= 20i_g = 20(30.125) \\ &= \mathbf{602.5\ W\ (delivered)} \end{aligned}$$

EXAMPLE 1.38

Refer the circuit shown in Fig. 1.75(a). Determine the current i_1.

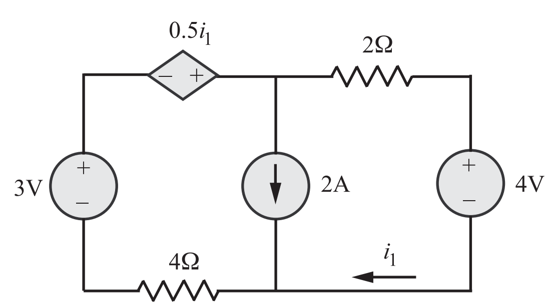


Figure 1.75(a)

SOLUTION

Constraint equation:

Applying KVL clockwise to the loop containing 3V source, dependent voltage source, 2A current source and 4Ω resitor, we get

$$-v_1 - 3 - 0.5i_1 + v_2 = 0$$
$$\Rightarrow \qquad v_1 - v_2 = -3 - 0.5i_1$$

Substituting $i_1 = \dfrac{v_2 - 4}{2}$, the above equation becomes

$$4v_1 - 3v_2 = -8$$

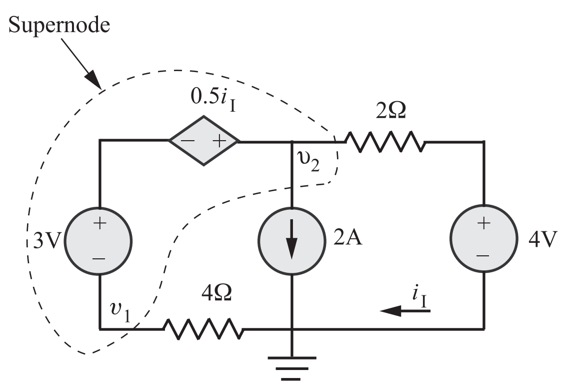


Figure 1.75(b)

KCL equation at supernode:

$$\frac{v_1}{4} + \frac{v_2 - 4}{2} = -2 \quad \Rightarrow \quad v_1 + 2v_2 = 0$$

Solving the constraint equation and the KCL equation at supernode simultaneously, we find that,

$$v_2 = 727.3 \text{ mV}$$
$$v_1 = -2v_2$$
$$= -1454.6 \text{ mV}$$

Then,
$$i_1 = \frac{v_2 - 4}{2}$$
$$= \mathbf{-1.636A}$$

EXAMPLE 1.39

Refer the network shown in Fig. 1.76(a). Find the node voltages v_d and v_c.

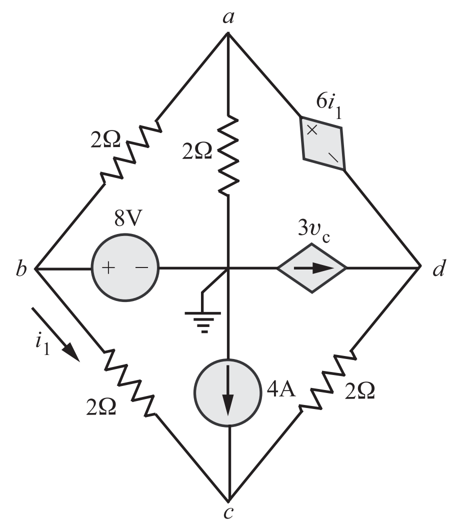


Figure 1.76(a)

SOLUTION

From the network, shown in Fig. 1.76 (b), by inspection, $v_b = 8$ V, $i_1 = \dfrac{v_b - v_c}{2}$

Constraint equation: $$v_a = 6i_1 + v_d$$

KCL at supernode: $$\frac{v_a - v_b}{2} + \frac{v_a}{2} + \frac{v_d - v_c}{2} = 3v_c$$

$$\Rightarrow \quad v_a \left[\frac{1}{2} + \frac{1}{2}\right] - \frac{1}{2} v_b + \frac{1}{2}\left[v_d - v_c\right] = 3v_c \tag{1.44}$$

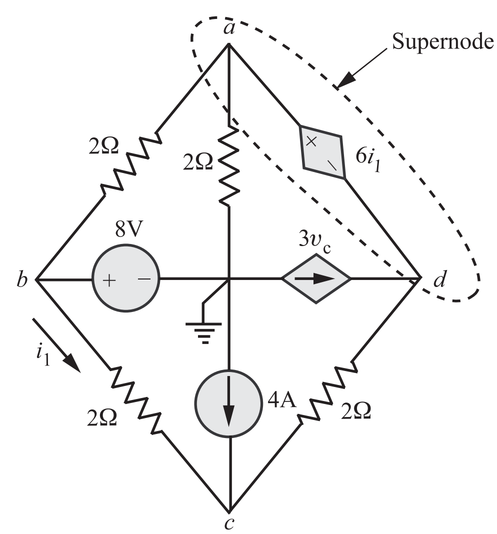


Figure 1.76(b)

Substituting $v_b = 8$ V in the constrained equation, we get

$$\begin{aligned} v_a &= 6\frac{(v_b - v_c)}{2} + v_d \\ &= 3(v_b - v_c) + v_d \\ &= 3(8 - v_c) + v_d \qquad (1.45) \end{aligned}$$

Substituting equation (1.45) into equation (1.44), we get

$$\begin{aligned} [3(8 - v_c) + v_d] - \frac{1}{2}(8) + \frac{1}{2}[v_d - v_c] &= 3v_c \\ \Rightarrow \quad 24 - 3v_c + v_d - 4 + \frac{1}{2}v_d - \frac{1}{2}v_c &= 3v_c \\ \Rightarrow \quad -6.5v_c + 1.5v_d &= -20 \qquad (1.46) \end{aligned}$$

KCL at node c:

$$\frac{v_c - v_b}{2} + \frac{v_c - v_d}{2} = 4$$

Substituting $v_b = 8$V, we have

$$\begin{aligned} \frac{v_c - 8}{2} + \frac{v_c - v_d}{2} &= 4 \\ \Rightarrow \quad v_c - 8 + v_c - v_d &= 8 \\ \Rightarrow \quad 2v_c - v_d &= 16 \\ \Rightarrow \quad v_c - 0.5v_d &= 8 \qquad (1.47) \end{aligned}$$

Solving equations (1.46) and (1.47), we get

$$v_c = \mathbf{-1.14V}$$
$$v_d = \mathbf{-18.3V}$$

EXAMPLE 1.40

For the circuit shown in Fig. 1.77(a), determine all the node voltages.

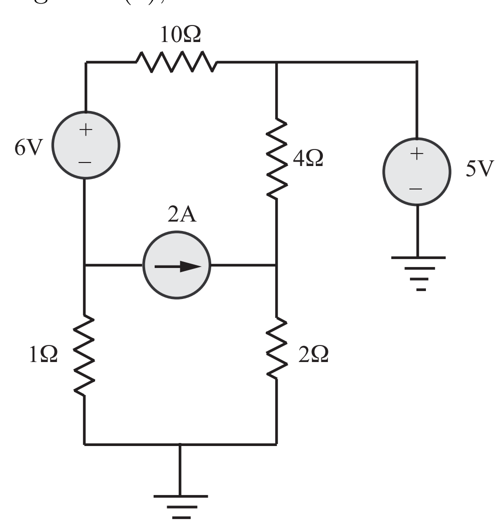


Figure 1.77(a)

SOLUTION

Refer Fig 1.77(b), by inspection, $v_2 = 5\text{V}$
Nodes 1 and 3 form a supernode.
Constraint equation:

$$v_1 - v_3 = 6$$

KCL at super node:

$$\frac{v_1 - v_2}{10} + \frac{v_3}{1} + 2 = 0$$

Substituting $v_2 = 5\text{V}$, we get

$$\frac{v_1 - 5}{10} + \frac{v_3}{1} = -2$$
$$\Rightarrow \quad v_1 - 5 + 10v_3 = -20$$
$$\Rightarrow \quad v_1 + 10v_3 = -15$$

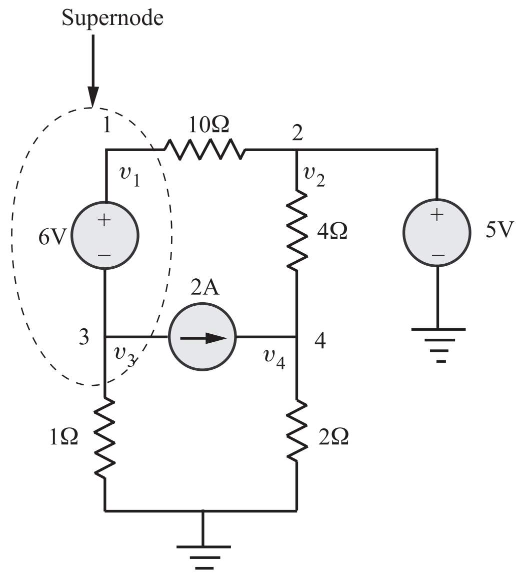


Figure 1.77(b)

Solving the constraint and the *KCL* equations at supernode simultaneously, we get

$$\boldsymbol{v_1 = 4.091\text{V}}$$
$$\boldsymbol{v_3 = -1.909\text{V}}$$

KCL at node 4 :

$$\frac{v_4}{2} + \frac{v_4 - v_2}{4} - 2 = 0$$

Substituting $v_2 = 5\text{V}$, we get

$$\frac{v_4}{2} + \frac{v_4 - 5}{4} - 2 = 0$$

Solving we get, $\quad v_4 = \mathbf{4.333V}$.

1.15 Brief review of impedance and admittance

Let us consider a general circuit with two accessible terminals, as shown in Fig. 1.78. If the time domain voltage and current at the terminals are given by

$$v = v_m \sin(\omega t + \phi_v)$$
$$i = i_m \sin(\omega t + \phi_i)$$

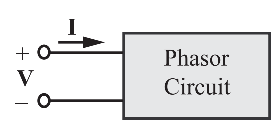


Figure 1.78 General phasor circuit

then the phasor quantities at the terminals are

$$\mathbf{V} = V_m \underline{/\phi_v}$$
$$\mathbf{I} = I_m \underline{/\phi_i}$$

We define the ratio of $\mathbf{V}$ to $\mathbf{I}$ as the impedence of the circuit, which is denoted as $\mathbf{Z}$. That is,

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}}$$

It is very important to note that impedance $\mathbf{Z}$ is a complex quantity, being the ratio of two complex quantities, but **it is not a phasor**. That is, it has no corresponding sinusoidal time-domain function, as current and voltage phasors do. Impedence is a complex constant that scales one phasor to produce another.

The impedence $\mathbf{Z}$ is written in rectangular form as

$$\mathbf{Z} = R + jX$$

where $R = \text{Real}[\mathbf{Z}]$ is the resistance and $X = \text{Im}[\mathbf{Z}]$ is the reactance. Both R and X, like $\mathbf{Z}$, are measured in ohms.

The magnitude of $\mathbf{Z}$ is written as $|\mathbf{Z}| = \sqrt{R^2 + X^2}$ and the angle of $\mathbf{Z}$ is denoted as $\phi_Z = \tan^{-1}\left[\dfrac{X}{R}\right]$.

The relationships are shown graphically in Fig. 1.79. The table below gives the various forms of $\mathbf{Z}$ for different combinations of R, L and C.

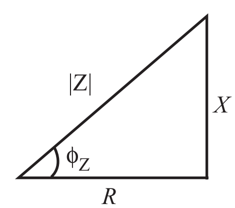


Figure 1.79 Graphical representation of impedance

	Type of the circuit	Impedance Z
1.	Purely resistive	$\mathbf{Z} = R$
2.	Purely inductive	$\mathbf{Z} = j\omega L = jX_L$
3.	Purely capactive	$\mathbf{Z} = \dfrac{-j}{\omega C} = -jX_C$
4.	RL	$\mathbf{Z} = R + j\omega L = R + jX_L$
5.	RC	$\mathbf{Z} = R - \dfrac{j}{\omega C} = R - jX_C$
6.	RLC	$\mathbf{Z} = R + j\omega L - \dfrac{j}{\omega C} = R + j(X_L - X_C)$

The reciprocal of impendance is denoted by

$$\mathbf{Y} = \frac{\mathbf{1}}{\mathbf{Z}}$$

is called admittance and is analogous to conductance in resistive circuits. Evidently, since $\mathbf{Z}$ is a complex number, so is $\mathbf{Y}$. The standard representation of admittance is

$$\mathbf{Y} = G + jB$$

The quantities $G = \text{Re}[\mathbf{Y}]$ and $B = \text{Im}[\mathbf{Y}]$ are respectively called conductance and suspectence. The units of $\mathbf{Y}$, G and B are all siemens.

1.16 Kirchhoff's Laws: Applied to alternating circuits

If a complex excitation, say $v_m e^{j(\omega t+\theta)}$, is applied to a circuit, then complex voltages, such as $v_1 e^{j(\omega t+\theta_1)}, v_2 e^{j(\omega t+\theta_2)}$ and so on, appear across the elements in the circuit. Kirchhoff's voltage law applied around a typical loop results in an equation such as

$$v_1\ e^{j(\omega t+\theta_1)} + v_2\ e^{j(\omega t+\theta_2)} + \ldots + v_N e^{j(\omega t+\theta_N)} = 0$$

Dividing by $e^{j\omega t}$, we get

$$v_1 e^{j\theta_1} + v_2 e^{j\theta_2} + \ldots + v_N e^{j\theta_N} = 0$$

$$\Rightarrow \qquad \mathbf{V}_1 + \mathbf{V}_2 + \ldots + \mathbf{V}_N = 0$$

where $\qquad \mathbf{V}_i = V_i \underline{/\theta_i}\ , i = 1, 2, \cdots N$

are the phasor voltage around the loop.

Thus KVL holds good for phasors also. A similar approach will establish KCL also. At any node having N connected branches,

$$\mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_N = 0$$

where $\qquad \mathbf{I}_i = I_i \underline{/\theta_i}\ , i = 1, 2 \cdots N$

Thus, KCL holds good for phasors also.

EXAMPLE 1.41

Determine $\mathbf{V}_1$ and $\mathbf{V}_2$, the node voltage phasors using nodal technique for the circuit shown in Fig. 1.80.

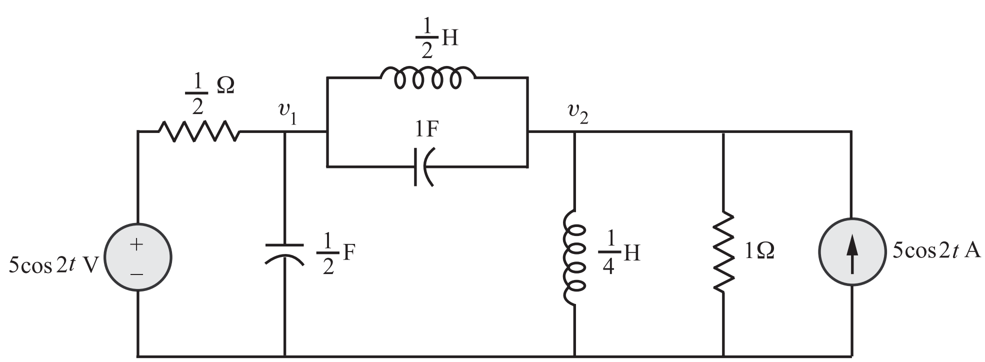


Figure 1.80

SOLUTION

First step in the analysis is to convert the circuit of Fig. 1.80 into its phasor version (frequency domain representation).

$$5\cos 2t \quad \Rightarrow \quad 5\underline{/0^\circ}\ , \omega = 2 \text{ rad/s}$$

$$\frac{1}{4}\text{ H} \quad \Rightarrow \quad j\omega L = j\frac{1}{2}\Omega$$

$$\frac{1}{2}\text{ F} \quad \Rightarrow \quad \frac{-j}{\omega C} = -j1\Omega, \qquad 1\text{F} \quad \Rightarrow \quad \frac{-j}{\omega C} = -j\frac{1}{2}\Omega$$

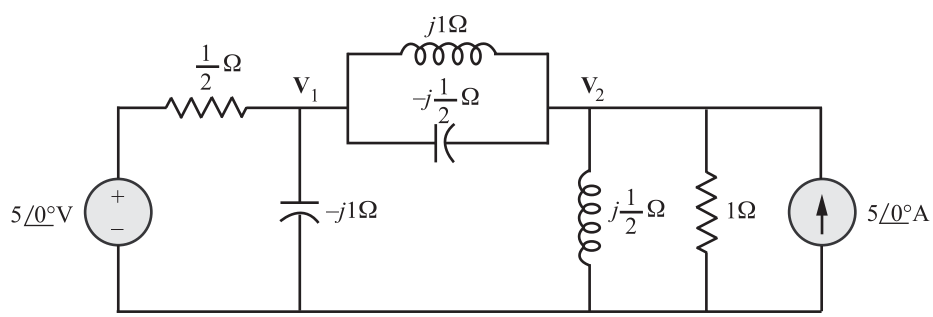


Figure 1.80(a)

$\Rightarrow$

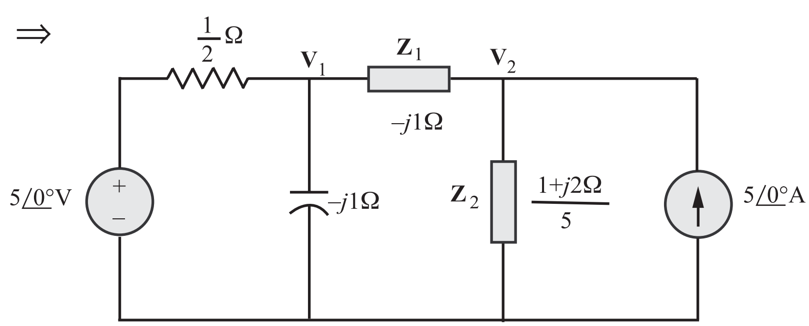


Figure 1.80(b)

Fig. 1.80(a) and (b) are the two versions of the phasor circuit of Fig. 1.80.

$$\mathbf{Z}_1 = j1\Omega || \left(-j\frac{1}{2}\Omega\right)$$

$$= \frac{j1\left(-j\frac{1}{2}\right)}{j1 - j\frac{1}{2}} = -j1\Omega$$

$$\mathbf{Z}_2 = j\frac{1}{2}\Omega || 1\Omega$$
$$= \frac{\left(j\frac{1}{2}\right)(1)}{\left(j\frac{1}{2}+1\right)} = \frac{1+j2}{5}\Omega$$

KCL at node $\mathbf{V}_1$:

$$2\left(\mathbf{V}_1 - 5\,\underline{/0^\circ}\right) + \frac{\mathbf{V}_1}{-j1} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{-j1} = 0$$
$$\Rightarrow \qquad (2+j2)\mathbf{V}_1 - j1\mathbf{V}_2 = 10$$

KCL at node $\mathbf{V}_2$:

$$\frac{\mathbf{V}_2 - \mathbf{V}_1}{-j1} + \frac{\mathbf{V}_2}{\frac{1+j2}{5}} = 5\angle 0^\circ$$
$$\Rightarrow \qquad j\mathbf{V}_2 - j\mathbf{V}_1 + \mathbf{V}_2 - 2j\mathbf{V}_2 = 5$$
$$\Rightarrow \qquad -j1\mathbf{V}_{1+}(1-j1)\mathbf{V}_2 = 5$$

Putting the above equations in a matrix form, we get

$$\begin{bmatrix} 2+j2 & -j1 \\ -j1 & 1-j1 \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

Solving $\mathbf{V}_1$ and $\mathbf{V}_2$ by Cramer's rule, we get

$$\mathbf{V}_1 = 2 - j1 \text{ V}$$
$$\mathbf{V}_2 = 2 + j4 \text{ V}$$

In polar form,

$$\mathbf{V}_1 = \sqrt{5}\,\underline{/-26.6^\circ} \text{ V}$$
$$\mathbf{V}_2 = 2\sqrt{5}\,\underline{/63.4^\circ} \text{ V}$$

In time domain,

$$\boldsymbol{v_1 = \sqrt{5}\cos(2t - 26.6^\circ)} \textbf{ V}$$
$$\boldsymbol{v_2 = 2\sqrt{5}\cos(2t + 63.4^\circ)} \textbf{ V}$$

EXAMPLE 1.42

Find the source voltage $\mathbf{V}_s$ shown in Fig. 1.81 using nodal technique. Take $\mathbf{I} = 3\,\underline{/45^\circ}$ A.

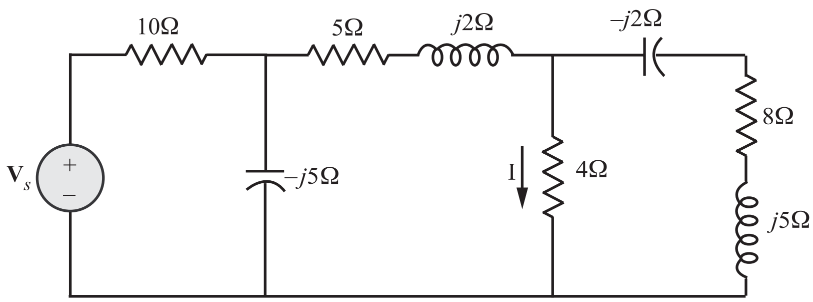


Figure 1.81

SOLUTION

Refer to Fig. 1.81(a).

KCL at node 1:

$$\frac{\mathbf{V}_1 - \mathbf{V}_s}{10} + \frac{\mathbf{V}_1}{-j5} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{5 + j2} = 0$$

$$\Rightarrow \qquad (11 + j12)\mathbf{V}_1 - (5 + j2)\mathbf{V}_s = 10\mathbf{V}_2 \tag{1.48}$$

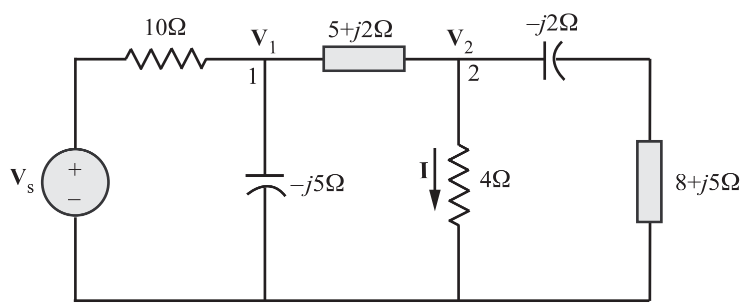


Figure 1.81(a)

KCL at node 2:

$$\frac{\mathbf{V}_2 - \mathbf{V}_1}{5 + j2} + \mathbf{I} + \frac{\mathbf{V}_2}{8 + j3} = 0$$

$$\Rightarrow \qquad (8 + j3)\mathbf{V}_1 = (13 + j5)\mathbf{V}_2 + (34 + j31)\mathbf{I} \tag{1.49}$$

Also,

$$\mathbf{V}_2 = 4\mathbf{I} = 4(3\,\underline{/45^\circ}) = 12\,\underline{/45^\circ}$$
$$= 6\sqrt{2} + j6\sqrt{2} \tag{1.50}$$

Substituting equations (1.48) and (1.50) in equation (1.49), we get

$$(8 + j3)\mathbf{V}_1 = 74.24 + j290.62$$

$$\Rightarrow \qquad \mathbf{V}_1 = \frac{300\,\underline{/75.7^\circ}}{8.54\,\underline{/20.6^\circ}}$$

$$= 35.1\,\underline{/55.1^\circ}$$

$$= 20.1 + j28.8 \text{ V}$$

Substituting $\mathbf{V}_1$ and $\mathbf{V}_2$ in equation (1.48) yields

$$(5 + j2)\mathbf{V}_s = -209.4 + j473.1$$

Therefore, $$\mathbf{V}_s = \frac{517.4\,\underline{/113.9^\circ}}{5.38\,\underline{/21.8^\circ}} = \mathbf{96.1\,\underline{/92.1^\circ}\ V}$$

EXAMPLE 1.43

Find the voltage $v(t)$ in the network shown in Fig. 1.82 using nodal technique.

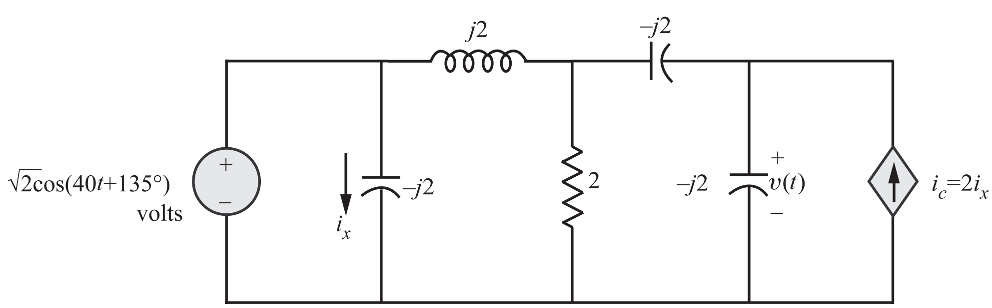


Figure 1.82

SOLUTION

Converting the circuit diagram shown in Fig. 1.82 into a phasor circuit diagram, we get

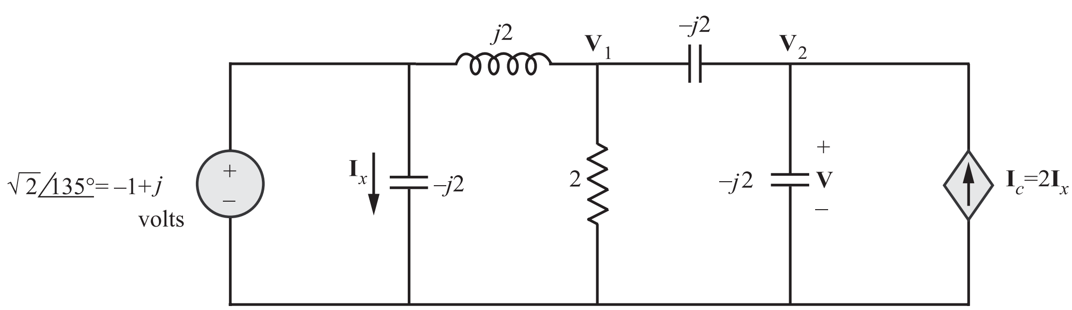


Figure 1.83

At node $\mathbf{V}_1$:

$$\frac{\mathbf{V}_1-(-1+j)}{j2}+\frac{\mathbf{V}_1}{2}+\frac{\mathbf{V}_1-\mathbf{V}_2}{-j2}=0$$

$$\Rightarrow \qquad \mathbf{V}_1-j\mathbf{V}_2=1+j \tag{1.51}$$

At node$\mathbf{V}_2$: $$\frac{\mathbf{V}_2-\mathbf{V}_1}{-j2}+\frac{\mathbf{V}_2}{-j2}-\mathbf{I}_c=0$$

Also $$\mathbf{I}_c=2\mathbf{I}_x=\frac{2(-1+j)}{-j2}=-1-j$$

Hence, $$\frac{\mathbf{V}_2-\mathbf{V}_1}{-j2}+\frac{\mathbf{V}_2}{-j2}=-1-j$$

$$\Rightarrow \qquad -j\mathbf{V}_1+j2\mathbf{V}_2=-2-j2 \tag{1.52}$$

Solving equations (1.51) and (1.52) using Cramer's rule we get

$$\mathbf{V}_2=\sqrt{2}\,\underline{/135^\circ}\ \text{V}$$

Therefore, $$\boldsymbol{v(t)=v_2(t)=\sqrt{2}\cos(4t+135^\circ)\ \mathrm{V}}$$

EXAMPLE 1.44

Refer to the circuit of Fig. 1.84. Using nodal technique, find the current i.

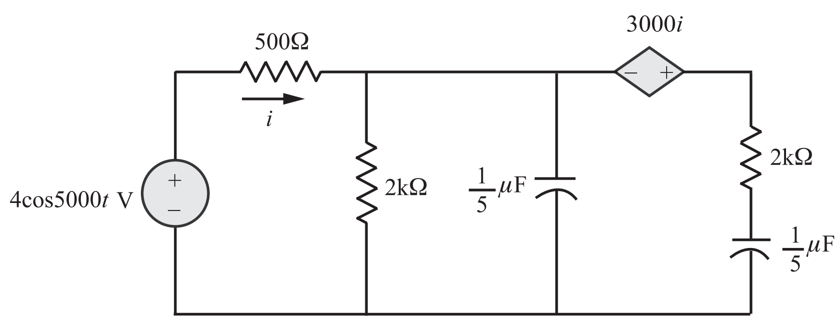


Figure 1.84

SOLUTION

Reactance of $\frac{1}{5}\mu$F capacitor $=\dfrac{1}{j\omega C}=\dfrac{1}{j5000\times\frac{1}{5}\times 10^{-6}}=-j1\text{k}\Omega$

The parallel combinations of 2kΩ and $-j1$kΩ is

$$\mathbf{Z}_p=\frac{2\times10^3(-j10^3)}{2\times10^3-j10^3}=\frac{2}{5}(1-j2)\text{k}\Omega$$

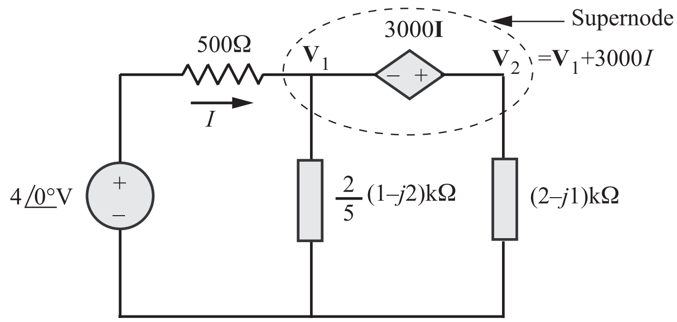


Figure 1.85

The phasor circuit of Fig. 1.84 is as shown in Fig. 1.85.
Constraint equation :

$$\mathbf{V_2 = V_1 + 3000I}$$

KCL at supernode :

$$\frac{\mathbf{V}_1 - 4\,\underline{/0^\circ}}{500} + \frac{\mathbf{V}_1}{\frac{2}{5}(1-j2)\times 10^3} + \frac{\mathbf{V}_2}{(2-j1)\times 10^3} = 0$$

Substituting $\mathbf{V}_2 = \mathbf{V}_1 + 3000\mathbf{I}$ in the above equation, we get

$$\frac{\mathbf{V}_1 - 4\,\underline{/0^\circ}}{500} + \frac{\mathbf{V}_1}{\frac{2}{5}(1-j2)\times 10^3} + \frac{\mathbf{V}_1 + 3000\mathbf{I}}{(2-j1)\times 10^3} = 0$$

Also,

$$\mathbf{I} = \frac{4\,\underline{/0^\circ} - V_1}{500} \tag{1.53}$$

Hence,

$$\frac{\mathbf{V}_1 - 4\angle 0^\circ}{500} + \frac{\mathbf{V}_1}{\frac{2}{5}(1-j2)\times 10^3} + \frac{\mathbf{V}_1 + 3000\left(\frac{4-\mathbf{V}_1}{500}\right)}{(2-j1)\times 10^3} = 0$$

Solving for $\mathbf{V}_1$ and substituting the same in equation (1.53), we get $\mathbf{I} = 24\,\underline{/53.1^\circ}$ mA
Hence, in time-domain, we have

$$\boldsymbol{i = 24\cos(5000t + 53.1^\circ)\mathrm{mA}}$$

EXAMPLE 1.45

Use nodal analysis to find $\mathbf{V}_o$ in the circuit shown in Fig. 1.86.

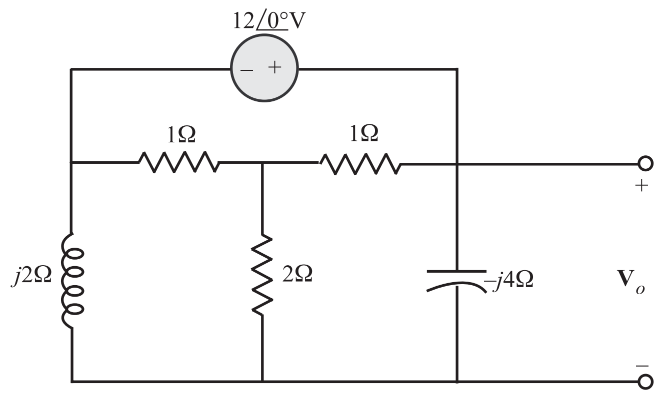


Figure 1.86

SOLUTION

The voltage source and its two connecting nodes form the supernode as shown in Fig. 1.87.

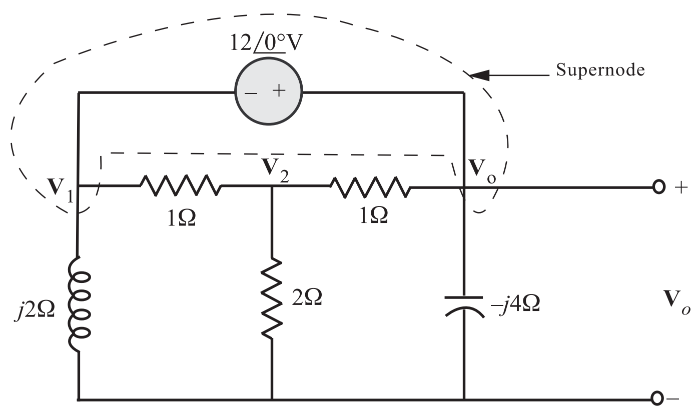


Figure 1.87

Constraint equation:

Applying *KVL* clockwise to the loop formed by $12\underline{/0^\circ}$ source, $j2\Omega$ and $-j4\Omega$ we get

$$-12\underline{/0^\circ} + \mathbf{V}_o - \mathbf{V}_1 = 0$$

$$\Rightarrow \qquad \mathbf{V}_1 = \mathbf{V}_o - 12\underline{/0^\circ}$$

KCL at supernode:

$$\frac{\mathbf{V}_1}{j2} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{1} + \frac{\mathbf{V}_o - \mathbf{V}_2}{1} + \frac{\mathbf{V}_o}{-j4} = 0$$

Substituting $\mathbf{V}_1 = \mathbf{V}_o - 12$ in the above equation

we get, $$\frac{-j}{2}(\mathbf{V}_o - 12) + (\mathbf{V}_o - 12 - \mathbf{V}_2) + \mathbf{V}_o - \mathbf{V}_2 + \frac{j}{4}\mathbf{V}_o = 0$$

$$\Rightarrow \quad \mathbf{V}_o\left(\frac{-j}{2} + 1 + 1 + \frac{j}{4}\right) + \mathbf{V}_2(-1 - 1) = 12 - j6$$

$$\Rightarrow \quad \mathbf{V}_o\left(2 - \frac{1}{4}j\right) - 2\mathbf{V}_2 = 12 - j6$$

KCL at $\mathbf{V}_2$: $$\frac{\mathbf{V}_2 - \mathbf{V}_1}{1} + \frac{\mathbf{V}_2}{2} + \frac{\mathbf{V}_2 - \mathbf{V}_o}{1} = 0$$

Substituting $\mathbf{V}_1 = \mathbf{V}_o - 12\angle 0^\circ$ in the above equation

we get, $$\mathbf{V}_2 - (\mathbf{V}_o - 12\angle 0^\circ) + \frac{1}{2}\mathbf{V}_2 + \mathbf{V}_2 - \mathbf{V}_o = 0$$

$$\Rightarrow \quad -2\mathbf{V}_o + \frac{5}{2}\mathbf{V}_2 = -12\angle 0^\circ$$

Solving the two nodal equations,we get

$$\mathbf{V_o = 11.056 - \mathit{j}8.09 = 13.7\angle -36.2^\circ \ V}$$

EXAMPLE 1.46

Find i_1 in the circuit of Fig. 1.88 using nodal analysis.

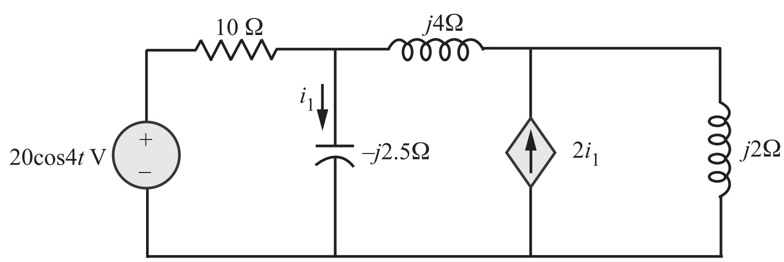


Figure 1.88

SOLUTION

The phasor equivalent circuit is as shown in Fig. 1.88(a).

KCL at node $\mathbf{V}_1$:

$$\frac{\mathbf{V}_1 - 20\angle 0^\circ}{10} + \frac{\mathbf{V}_1}{-j2.5} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{j4} = 0$$

$$\Rightarrow \quad (1 + j1.5)\mathbf{V}_1 + j2.5\mathbf{V}_2 = 20$$

KCL at node $\mathbf{V}_2$:

$$\frac{\mathbf{V}_2 - \mathbf{V}_1}{j4} + \frac{\mathbf{V}_2}{j2} = 2\mathbf{I}_1$$

But $$\mathbf{I}_1 = \frac{\mathbf{V}_1}{-j2.5}$$

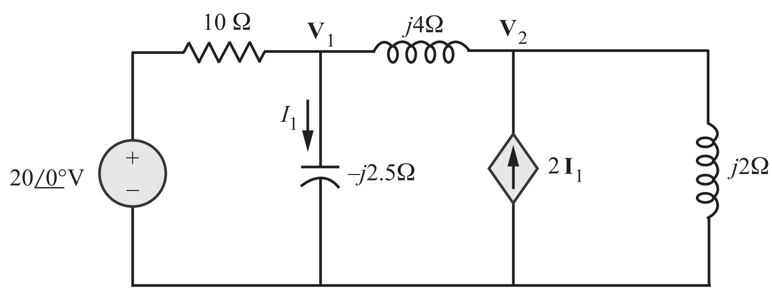


Figure 1.88(a)

Hence,
$$\frac{\mathbf{V}_2 - \mathbf{V}_1}{j4} + \frac{\mathbf{V}_2}{j2} = \frac{2\mathbf{V}_1}{-j2.5}$$
$$\Rightarrow \quad -j0.55\mathbf{V}_1 - j0.75\mathbf{V}_2 = 0$$

Multiplying throughout by $j20$, we get

$$11\mathbf{V}_1 + 15\mathbf{V}_2 = 0$$

Putting the two nodal equations in matrix form, we get

$$\begin{bmatrix} 1 + j1.5 & j2.5 \\ 11 & 15 \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 0 \end{bmatrix}$$

Solving the matrix equation, we get

$$\mathbf{V}_1 = 18.97 \underline{/18.43^\circ} \text{ V}$$
$$\mathbf{V}_2 = 13.91 \underline{/-161.56^\circ} \text{ V}$$

The current
$$\mathbf{I}_1 = \frac{\mathbf{V}_1}{-j2.5} = 7.59 \underline{/108.4^\circ} \text{ A}$$

Transforming this to the time-domain, we get

$$\boldsymbol{i_1 = 7.59\cos(4t + 108.4^\circ)\,\mathrm{A}}$$

EXAMPLE 1.47

Use the node-voltage method to find the steady-state expression for $v_o(t)$ in the circuit shown in Fig. 1.89 if

$$v_{g1} = 10\cos(5000t + 53.13^\circ)\text{V}$$
$$v_{g2} = 8\sin 5000t \text{ V}$$

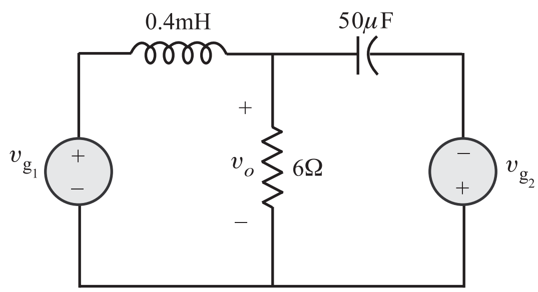


Figure 1.89

SOLUTION

The first step is to convert the circuit of Fig. 1.89 into a phasor circuit.

$$10\cos(5000t + 53.13^\circ)\text{V},\ \omega = 5000\text{rad/sec} \quad\Rightarrow\quad 10\,\underline{/53.13^\circ} = 6 + j8\text{V}$$
$$8\sin 5000t = 8\cos(5000t - 90^\circ)\text{V} \quad\Rightarrow\quad 8\,\underline{/-90^\circ} = -j8\text{V}$$
$$L = 0.4\text{ mH} \quad\Rightarrow\quad j\omega L = j2\Omega$$
$$C = 50\mu F \quad\Rightarrow\quad \frac{1}{j\omega C} = -j4\Omega$$

The phasor circuit is shown in Fig. 1.89(a).

KCL at node 1:

$$\frac{\mathbf{V}_o - (6 + j8)}{j2} + \frac{\mathbf{V}_o}{6} + \frac{\mathbf{V}_o - (-j8)}{-j4} = 0$$

Solving we get $\mathbf{V}_o = 12\,\underline{/0^\circ}$ V

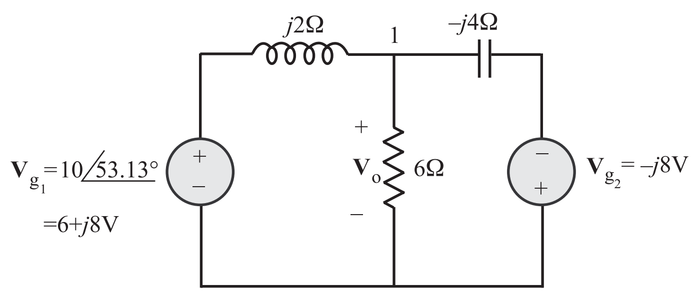


Figure 1.89(a)

Hence, the steady-state expression is

$$v_o(t) = 12\cos 5000\text{t}$$

EXAMPLE 1.48

Solve the example (1.47) using mesh-current method.

SOLUTION

Refer Fig. 1.90.

$$KVL\ to\ mesh\ 1: \quad [6 + j2]\mathbf{I}_1 - 6\mathbf{I}_2 = 10\,\underline{/53.13^\circ}$$
$$KVL\ to\ mesh\ 2: \quad -6\mathbf{I}_1 + (6 - j4)\mathbf{I}_2 = 8\,\underline{/-90^\circ}$$

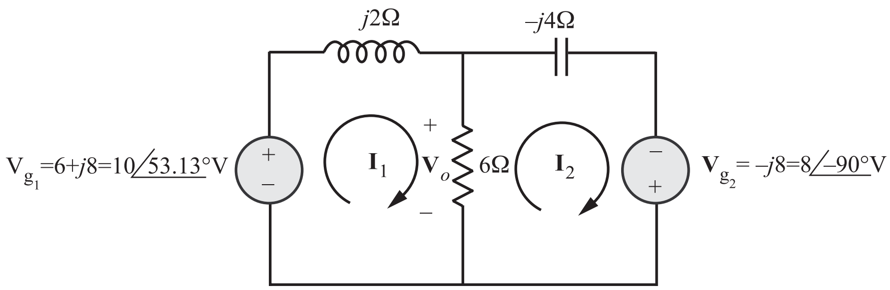


Figure 1.90

Putting the above equations in matrix form, we get

$$\begin{bmatrix} 6+j2 & -6 \\ -6 & 6-j4 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 10\,\underline{/53.13^\circ} \\ 8\,\underline{/-90^\circ} \end{bmatrix}$$

Solving for $\mathbf{I}_1$ and $\mathbf{I}_2$, we get

$$\mathbf{I}_1 = 4 + j3$$
$$\mathbf{I}_2 = 2 + j3$$

Now, $$\mathbf{V}_o = (\mathbf{I}_1 - \mathbf{I}_2)6 = 12$$
$$= 12\,\underline{/0^\circ} \ \ \text{V}$$

Hence in time domain, $$\boldsymbol{v_o} = \mathbf{12\cos 5000}\boldsymbol{t}\ \mathbf{Volts}$$

EXAMPLE 1.49

Determine the current $\mathbf{I}_o$ in the circuit of Fig. 1.91 using mesh analysis.

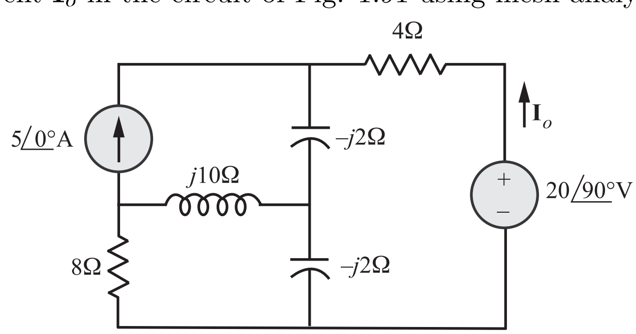


Figure 1.91

SOLUTION

Refer Fig 1.92

KVL for mesh 1 :

$$(8 + j10 - j2)\mathbf{I}_1 - (-j2)\mathbf{I}_2 - j10\mathbf{I}_3 = 0$$

$$\Rightarrow \qquad (8 + j8)\mathbf{I}_1 + j2\mathbf{I}_2 = j10\mathbf{I}_3 \tag{1.54}$$

KVL for mesh 2 :

$$(4 - j2 - j2)\mathbf{I}_2 - (-j2)\mathbf{I}_1 - (-j2)\mathbf{I}_3 + 20\,\underline{/90^\circ} = 0$$

$$\Rightarrow \qquad j2\mathbf{I}_1 + (4 - j4)\mathbf{I}_2 + j2\mathbf{I}_3 = -j20 \tag{1.55}$$

For mesh 3,

$$\mathbf{I}_3 = 5 \tag{1.56}$$

Sustituting the value of $\mathbf{I}_3$ in the equations (1.54) and (1.55), we get

$$(8 + j8)\mathbf{I}_1 + j2\mathbf{I}_2 = j50$$

$$j2\mathbf{I}_1 + (4 - j4)\mathbf{I}_2 = -j20 - j10$$

$$= -j30$$

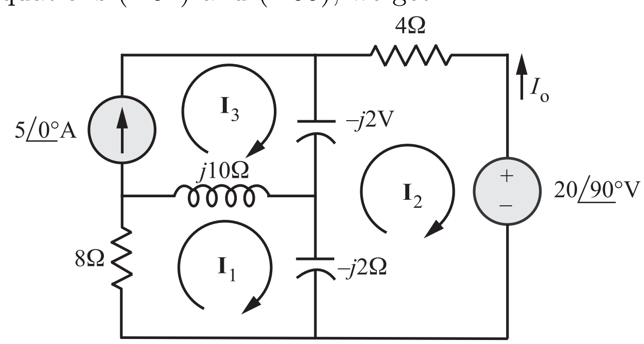


Figure 1.92

Putting the above equations in matrix form,we get

$$\begin{bmatrix} 8 + j8 & j2 \\ j2 & 4 - j4 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} j50 \\ -j30 \end{bmatrix}$$

Using Cramer's rule,we get

$$\mathbf{I}_2 = 6.12\,\underline{/-35.22^\circ}\text{ A}$$

The required current:

$$\mathbf{I}_o = -\mathbf{I}_2$$

$$= \mathbf{6.12\,\underline{/144.78^\circ}\text{ A}}$$

EXAMPLE 1.50

Find $\mathbf{V}_{oc}$ using mesh technique.

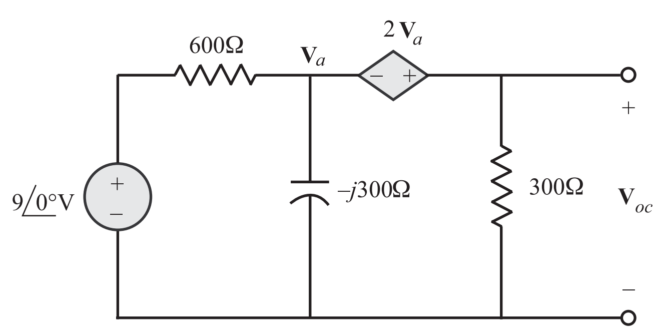


Figure 1.93

SOLUTION

Applying KVL clockwise for mesh 1 :

$$600\mathbf{I}_1 - j300(\mathbf{I}_1 - \mathbf{I}_2) - 9 = 0$$

$$\Rightarrow \qquad (600 - j300)\mathbf{I}_1 + j300\mathbf{I}_2 = 9$$

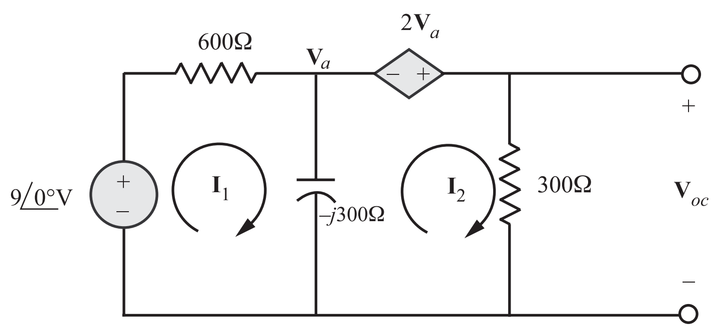


Figure 1.94

Applying KVL clockwise for mesh 2 :

$$-2\mathbf{V}_a + 300\mathbf{I}_2 - j300(\mathbf{I}_2 - \mathbf{I}_1) = 0$$

Also, $$\mathbf{V}_a = -j300(\mathbf{I}_1 - \mathbf{I}_{2)}$$

Hence, $$-2(-j300(\mathbf{I}_1 - \mathbf{I}_2)) + 300\mathbf{I}_2 - j300\,(\mathbf{I}_2 - \mathbf{I}_1) = 0$$

$$\Rightarrow \quad j3\mathbf{I}_1 + (1 - j3)\mathbf{I}_2 = 0$$

Putting the above two mesh equations in matrix form, we get

$$\begin{bmatrix} 600 - j300 & j300 \\ j3 & 1 - j3 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$

Using Cramer's rule, we find that

$$\mathbf{I}_2 = 0.0124\,\underline{/-16°}\ \text{A}$$

Hence, $$\mathbf{V_{oc}} = \mathbf{300I_2} = \mathbf{3.72}\,\underline{\mathbf{/-16°}}\ \mathbf{V}$$

EXAMPLE 1.51

Find the steady current i_1 when the source voltage is $v_s = 10\sqrt{2}\cos(\omega t + 45°)$ V and the current source is $i_s = 3\cos\omega t$ A for the circuit of Fig. 1.95. The circuit provides the impedence in ohms for each element at the specified ω.

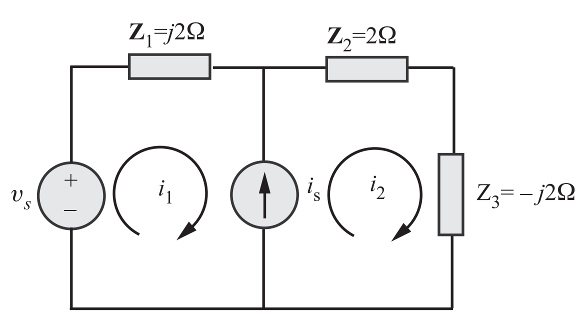


Figure 1.95

SOLUTION

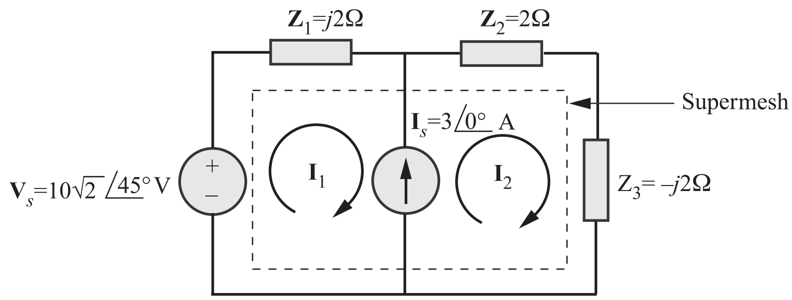


Figure 1.96

The first step is to convert the circuit of Fig. 1.95 into a phasor circuit. The phasor circuit is shown in Fig. 1.96.

$$v_s = 10\sqrt{2}\cos(\omega t + 45^\circ) \quad \Rightarrow \quad \mathbf{V}_s = 10\sqrt{2}\,\underline{/45^\circ} = 10(1+j)$$

$$i_s = 3\cos\omega t \quad \Rightarrow \quad \mathbf{I}_s = 3\,\underline{/0^\circ}$$

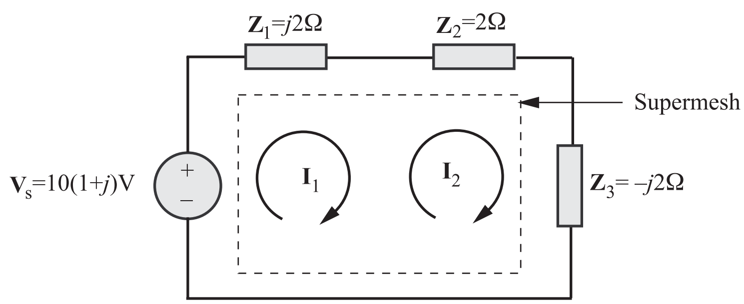


Figure 1.96(a)

Constraint equation:

$$\mathbf{I}_2 - \mathbf{I}_1 = \mathbf{I}_s = 3\,\underline{/0^\circ}$$

Applying KVL clockwise around the supermesh we get

$$\mathbf{I}_1\mathbf{Z}_1 + \mathbf{I}_2(\mathbf{Z}_2 + \mathbf{Z}_3) - \mathbf{V}_s = 0$$

Substituting $\mathbf{I}_2 = \mathbf{I}_1 + \mathbf{I}_s$ (from the constraint equation)

we get, $\mathbf{I}_1\mathbf{Z}_1 + (\mathbf{I}_1 + \mathbf{I}_s)(\mathbf{Z}_2 + \mathbf{Z}_3) = \mathbf{V}_s$

$$\Rightarrow \quad (\mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3)\mathbf{I}_1 = \mathbf{V}_s - (\mathbf{Z}_2 + \mathbf{Z}_3)\mathbf{I}_s$$

$$\Rightarrow \quad \mathbf{I}_1 = \frac{\mathbf{V}_s - (\mathbf{Z}_2 + \mathbf{Z}_3)\mathbf{I}_s}{\mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3} = \frac{(10 + j10) - (2 - j2)3}{2}$$

$$= 2 + j8 = 8.25\,\underline{/76^\circ}\ \text{A}$$

Hence in time domain,

$$\boldsymbol{i_1 = 8.25\cos(\omega t + 76^\circ)\ \mathrm{A}}$$

EXAMPLE 1.52

Find the steady-state sinusoidal current i_1 for the circuit of Fig. 1.97, when $v_s = 10\sqrt{2}\cos(100t + 45°)$ V.

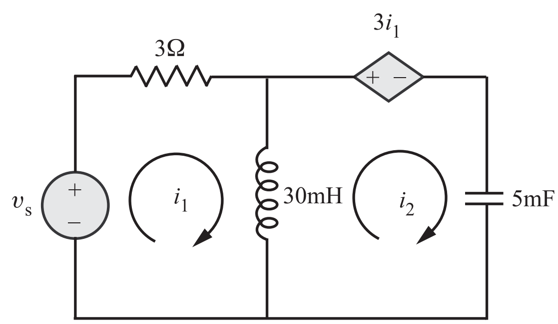


Figure 1.97

SOLUTION

The first step is to convert the circuit of Fig. 1.97 int to a phasor circuit. The phasor circuit is shown in Fig. 1.98.

$$v_s = 10\sqrt{2}\cos(100t + 45°)$$

$$\begin{aligned}
& \Rightarrow & \mathbf{V}_s &= 10\sqrt{2}\,\underline{/45°}\,, \qquad \omega = 100 \text{ rad/ sec} \\
L = 30 \text{ mH} & \Rightarrow & X_L &= j\omega L \\
& & &= j100 \times 30 \times 10^{-3} = j3\Omega \\
C = 5 \text{ mF} & \Rightarrow & X_C &= \frac{1}{j\omega C} \\
& & &= \frac{1}{j100 \times 5 \times 10^{-3}} = -j2\Omega
\end{aligned}$$

KVL for mesh 1 :

$$(3 + j3)\mathbf{I}_1 - j3\mathbf{I}_2 = 10 + j10$$

KVL for mesh 2 :

$$(3 - j3)\mathbf{I}_1 + (j3 - j2)\mathbf{I}_2 = 0$$

Putting the above two mesh equations in matrix form, we get

$$\begin{bmatrix} 3 + j3 & -j3 \\ 3 - j3 & j1 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 10 + j10 \\ 0 \end{bmatrix}$$

Using Cramer's rule,we get

$$\mathbf{I}_1 = 1.05\,\underline{/71.6^\circ}\ \text{A}$$

Thus the steady state time response is,

$$\boldsymbol{i_1 = 1.05\cos(100t + 71.6^\circ)\mathrm{A}}$$

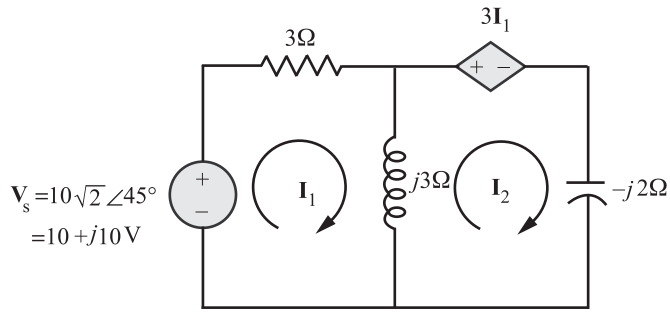


Figure 1.98

EXAMPLE 1.53

Determine $\mathbf{V}_o$ using mesh analysis.

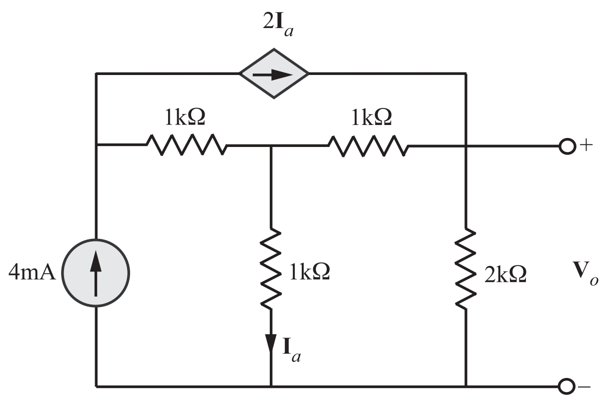


Figure 1.99

SOLUTION

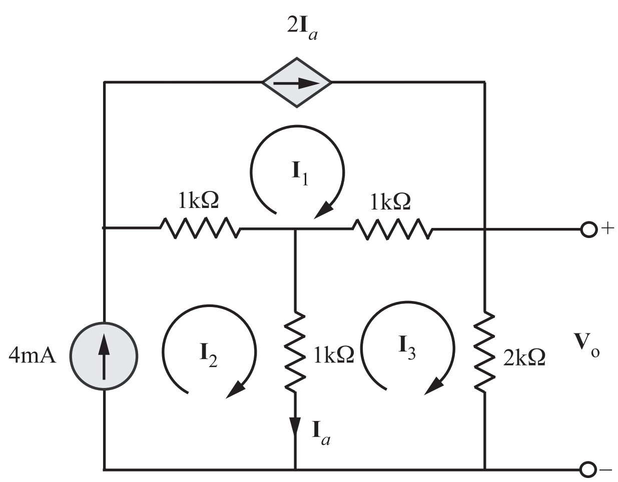


Figure 1.100

From Fig. 1.100, we find by inspection that,

$$\mathbf{I}_1 = 2\mathbf{I}_a = 2(\mathbf{I}_2 - \mathbf{I}_3)$$
$$\mathbf{I}_2 = 4 \text{ mA}$$

Applying KVL clockwise to mesh 3, we get

$$1 \times 10^3(\mathbf{I}_3 - \mathbf{I}_2) + 1 \times 10^3(\mathbf{I}_3 - \mathbf{I}_1) + 2 \times 10^3\mathbf{I}_3 = 0$$

Substituting $\mathbf{I}_1 = 2(\mathbf{I}_2 - \mathbf{I}_3)$ and $\mathbf{I}_2 = 4$ mA in the above equation and solving for $\mathbf{I}_3$, we get,

$$\mathbf{I}_3 = 2 \text{ mA}$$

Then,

$$\mathbf{V}_o = 2 \times 10^3\mathbf{I}_3 = 4\text{V}$$

EXAMPLE 1.54

Find $\mathbf{V}_o$ in the network shown in Fig. 1.101 using mesh analysis.

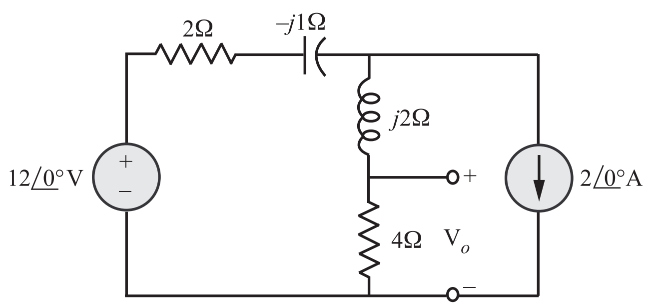


Figure 1.101

SOLUTION

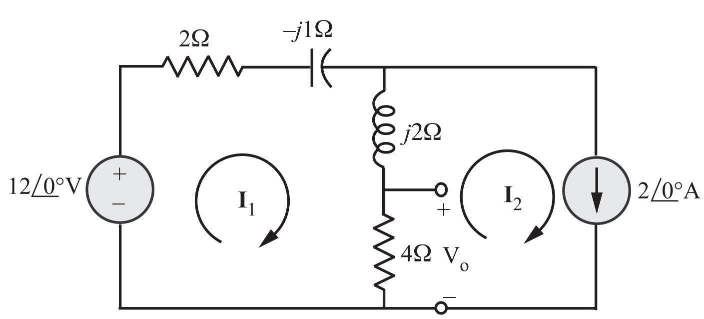


Figure 1.102

By inspection, we find that $\mathbf{I}_2 = 2\,\underline{/0^\circ}$ A.

Applying KVL clockwise to mesh 1, we get

$$-12 + \mathbf{I}_1(2 - j1) + (\mathbf{I}_1 - \mathbf{I}_2)(4 + j2) = 0$$

Substituting $\mathbf{I}_2 = 2\,\underline{/0^\circ}$ in the above equation yields,

$$-12 + \mathbf{I}_1(2 - j1 + 4 + j2) - 2(4 + j2) = 0$$

$$\Rightarrow \qquad \mathbf{I}_1 = \frac{20 + j4}{6 + j1} = 3.35\,\underline{/1.85^\circ}\ \text{A}$$

Hence

$$\mathbf{V}_o = 4(\mathbf{I}_1 - \mathbf{I}_2)$$
$$= 5.42\,\underline{/4.57^\circ}\ \text{V}$$

Wye $\rightleftharpoons$ Delta transformation

For reducing a complex network to a single impedance between any two terminals, the reduction formulas for impedances in series and parallel are used. However, for certain configurations of network, we cannot reduce the interconnected impedances to a single equivalent impedance between any two terminals by using series and parallel impedance reduction techniques. That is the reason for this topic.

Consider the networks shown in Fig. 1.103 and 1.104.

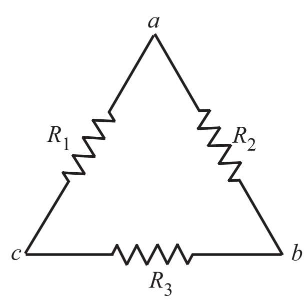


Figure 1.103 Delta resistance network

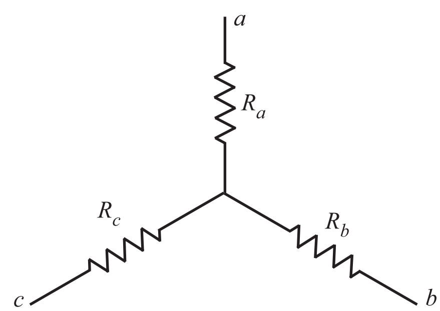


Figure 1.104 Wye resistance network

It may be noted that resistors in Fig. 1.103 form a Δ (delta), and resistors in Fig. 1.104. form a Υ (Wye). If both these configurations are connected at only the three terminals a, b and c, it would be very advantageous if an equivalence is established between them. It is possible to relate the resistances of one network to those of the other such that their terminal characteristics are the same. The relationship between the two configurations is called $\Upsilon - \Delta$ transformation.

We are interested in the relationship between the resistances R_1, R_2 and R_3 and the resitances R_a, R_b and R_c. For deriving the relationship, we assume that for the two networks to be equivalent at each corresponding pair of terminals, it is necessary that the resistance at the corresponding terminals be equal. That is, for example, resistance at terminals b and c with a open-circuited must be same for both networks. Hence, by equating the resistances for each corresponding set of terminals, we get the following set of equations :

(i)
$$R_{ab}(\Upsilon) = R_{ab}(\Delta)$$

$$\Rightarrow \quad R_a + R_b = \frac{R_2(R_1 + R_3)}{R_2 + R_1 + R_3} \tag{1.57}$$

(ii)
$$R_{bc}(\Upsilon) = R_{bc}(\Delta)$$

$$\Rightarrow \quad R_b + R_c = \frac{R_3(R_1 + R_2)}{R_3 + R_1 + R_2} \tag{1.58}$$

(iii)
$$R_{ca}(\Upsilon) = R_{ca}(\Delta)$$

$$\Rightarrow \quad R_c + R_a = \frac{R_1(R_2 + R_3)}{R_1 + R_2 + R_3} \tag{1.59}$$

Solving equations (1.57), (1.58) and (1.59) gives

$$R_a = \frac{R_1 R_2}{R_1 + R_2 + R_3} \tag{1.60}$$

$$R_b = \frac{R_2 R_3}{R_1 + R_2 + R_3} \tag{1.61}$$

$$R_c = \frac{R_1 R_3}{R_1 + R_2 + R_3} \tag{1.62}$$

Hence, each resistor in the Υ network is the product of the resistors in the two adjacent Δ branches, divided by the sum of the three Δ resistors.

To obtain the conversion formulas for transforming a *wye* network to an equivalent *delta* network, we note from equations (1.60) to (1.62) that

$$R_a R_b + R_b R_c + R_c R_a = \frac{R_1 R_2 R_3 (R_1 + R_2 + R_3)}{(R_1 + R_2 + R_3)^2} = \frac{R_1 R_2 R_3}{R_1 + R_2 + R_3} \tag{1.63}$$

Dividing equation (1.63) by each of the equations (1.60) to (1.62) leads to the following relationships :

$$R_1 = \frac{R_a R_b + R_b R_c + R_a R_c}{R_b} \tag{1.64}$$

$$R_2 = \frac{R_a R_b + R_b R_c + R_a R_c}{R_c} \tag{1.65}$$

$$R_3 = \frac{R_a R_b + R_b R_c + R_a R_c}{R_a} \tag{1.66}$$

Hence each resistor in the Δ network is the sum of all possible products of Υ resistors taken two at a time, divided by the opposite Υ resistor.

Then Υ and Δ are said to be balanced when

$$R_{1=} R_2 = R_3 = R_\Delta \text{ and } R_a = R_b = R_c = R_\Upsilon$$

Under these conditions the conversions formula become

$$R_{\Upsilon} = \frac{1}{3} R_{\Delta}$$

and
$$R_{\Delta} = 3R_{\Upsilon}$$

EXAMPLE 1.55

Find the value of resistance between the terminals $a - b$ of the network shown in Fig. 1.105.

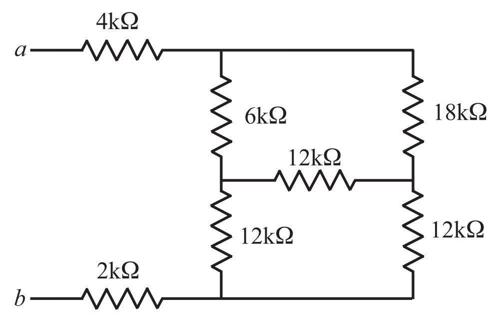


Figure 1.105

SOLUTION

Let us convert the upper Δ to Υ

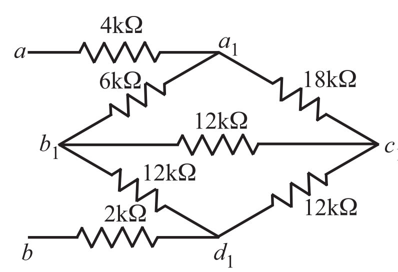
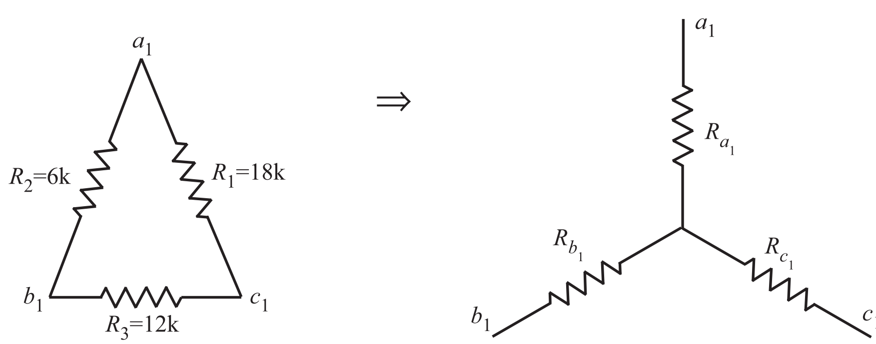


Figure 1.106

$$R_{a_1} = \frac{(6\text{k})(18\text{k})}{6\text{k} + 12\text{k} + 18\text{k}} = 3 \text{ k}\Omega$$

$$R_{b_1} = \frac{(6\text{k})(12\text{k})}{6\text{k} + 12\text{k} + 18\text{k}} = 2 \text{ k}\Omega$$

$$R_{c_1} = \frac{(12\text{k})(18\text{k})}{6\text{k} + 12\text{k} + 18\text{k}} = 6 \text{ k}\Omega$$



The network shown in Fig. 1.106 is now reduced to that shown in Fig. 1.106(a)

Hence, $$R_{ab} = 4 + 3 + 7.875 + 2$$
$$= \mathbf{16.875k\Omega}$$

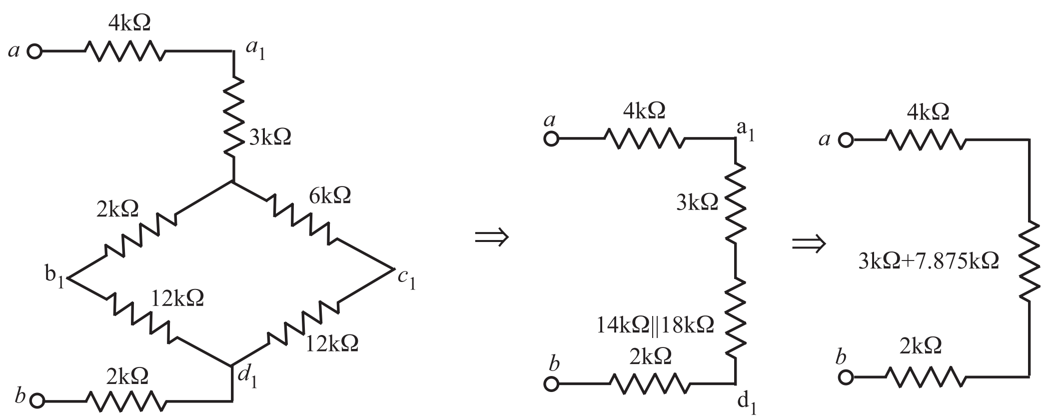


Figure 1.106(a)

EXAMPLE 1.56

Find the resistance R_{ab} using $\Upsilon - \Delta$ transformation.

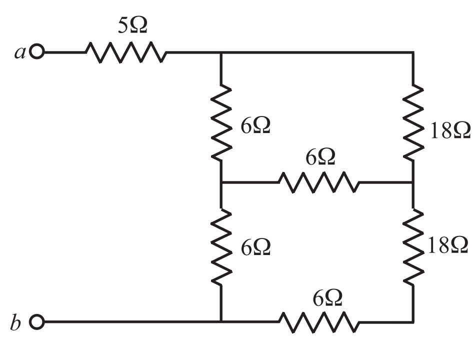


Figure 1.107

SOLUTION

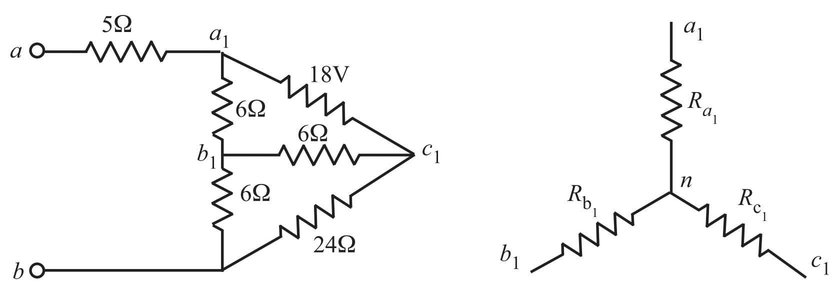
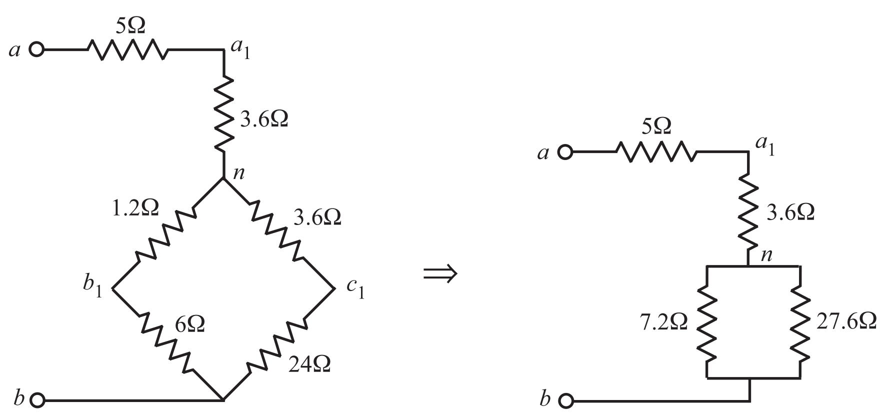


Figure 1.108

Let us convert the upper Δ between the points a_1, b_1 and c_1 into an equivalent Υ.

$$R_{a_1} = \frac{6 \times 18}{6 + 18 + 6} = 3.6\Omega$$
$$R_{b_1} = \frac{6 \times 6}{6 + 18 + 6} = 1.2\Omega$$
$$R_{c_1} = \frac{6 \times 18}{6 + 18 + 6} = 3.6\Omega$$

Figure 1.108 now becomes



$$R_{ab} = 5 + 3.6 + 7.2||27.6$$
$$= 8.6 + \frac{7.2 \times 27.6}{7.2 + 27.6}$$
$$= \mathbf{14.31\Omega}$$

EXAMPLE 1.57

Obtain the equivalent resistance R_{ab} for the circuit of Fig. 1.109 and hence find i.

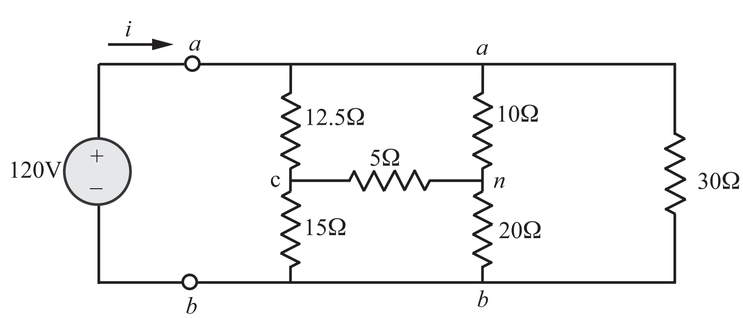
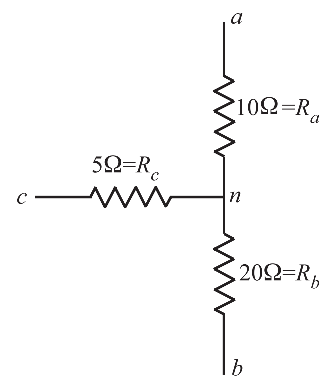


Figure 1.109

SOLUTION

Let us convert Υ between the terminals a, b and c into an equivalent Δ.

$$\begin{aligned}
R_{ab} &= \frac{R_aR_b + R_bR_c + R_cR_a}{R_c} \\
&= \frac{10\times 20 + 20\times 5 + 5\times 10}{5} = 70\Omega \\
R_{bc} &= \frac{R_aR_b + R_bR_c + R_cR_a}{R_a} \\
&= \frac{10\times 20 + 20\times 5 + 5\times 10}{10} = 35\Omega \\
R_{ca} &= \frac{R_aR_b + R_bR_c + R_cR_a}{R_b} \\
&= \frac{10\times 20 + 20\times 5 + 5\times 10}{20} = 17.5\Omega
\end{aligned}$$



The circuit diagram of Fig. 1.109 now becomes the circuit diagram shown in Fig. 1.109(a). Combining three pairs of resistors in parallel, we obtain the circuit diagram of Fig. 1.109(b).

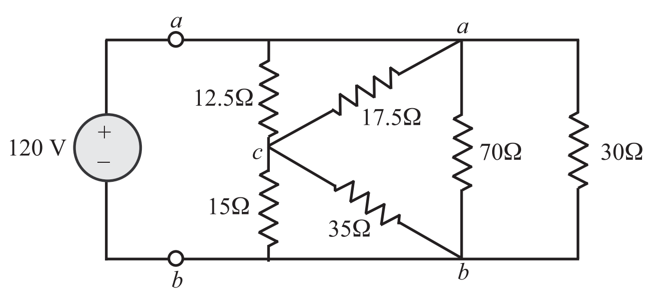


Figure 1.109(a)

$$\begin{aligned}
70||30 &= \frac{70\times 30}{70+30} = 21\Omega \\
12.5||17.5 &= \frac{12.5\times 17.5}{12.5+17.5} = 7.292\Omega \\
15||35 &= \frac{15\times 35}{15+35} = 10.5\Omega \\
R_{ab} &= (7.292 + 10.5)||21 = 9.632\Omega
\end{aligned}$$

Thus, $i = \dfrac{v_s}{R_{ab}} = \mathbf{12.458\ A}$

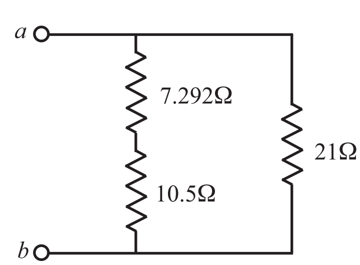


Figure 1.109(b)

Nodal versus mesh analysis

The analysis of a complex circuit can usually be accomplished by either the node voltage or mesh current method. One may ask : Given a network to be analyzed, how do we know which method is better or more efficient? The choice is dictated by two factors.

When a circuit contains only voltage sources, it is probably easier to use the mesh current method. Conversely, when the circuit contains only current sources, it will be easier to use the node voltage method. Also, a circuit with fewer nodes than meshes is better analyzed using nodal analysis, while a circuit with fewer meshes than nodes is better analyzed using mesh analysis. In other words, the best technique is one which gives smaller number of equations.

Another point to consider while choosing between the two methods is, what information is required. If node voltages are required, it may be advantageous to apply nodal analysis. On the other hand, if you need to know several currents, it may be wise to proceed directly with mesh current analysis.

It is often advantageous if we know both the techniques. The first advantage lies in the fact that the second method can verify the results of the first method. Also, both the methods have limitations. For example, while analysing a transistor circuit, only mesh method is suited and while analysing an Op-amp circuit, nodal method is only applicable. Mesh technique is applicable for planar[1] networks. However, nodal method suits to both planar and nonplanar [2] networks.

Reinforcement Problems

R.P 1.1

Find the power dissipated in the 80Ω resistor using mesh analysis.

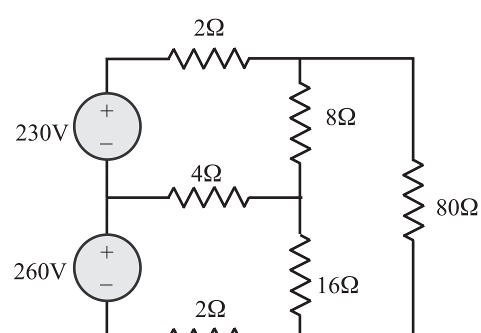


Figure R.P.1.1

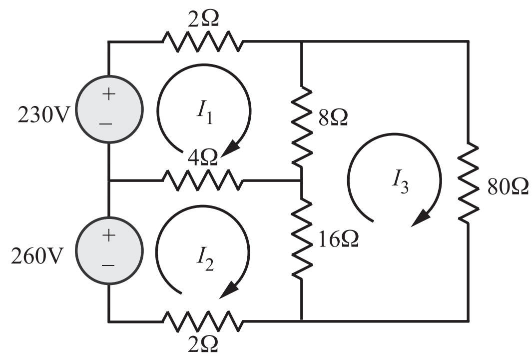
[1]A planar network can be drawn on a plane without branches crossing each other.

[2]A nonplanar network is one in which crossover is identified and cannot be eliminated by redrawing the branches.

SOLUTION

KVL *clockwise to mesh* 1 :

$$14I_1 - 4I_2 - 8I_3 = 230$$



KVL *clockwise to mesh* 2 :

$$-4I_1 + 22I_2 - 16I_3 = 260$$

KVL *clockwise to mesh* 3 :

$$-8I_1 - 16I_2 + 104I_3 = 0$$

Putting the above mesh equations in matrix form, we get

$$\begin{bmatrix} 14 & -4 & -8 \\ -4 & 22 & -16 \\ -8 & -16 & 104 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 230 \\ 260 \\ 0 \end{bmatrix}$$

The current I_3 is found from the above matrix equation by using Cramer's rule.

$$I_3 = 5\text{A}$$

Thus, $$P_{80} = I_3^2 R_{80} = 5^2 \times 80 = \mathbf{2000W}(\mathbf{dissipated})$$

R.P 1.2

Refer the circuit shown in Fig. R.P. 1.2. The current $i_o = 4$A. Find the power dissipated in the 70 Ω resistor.

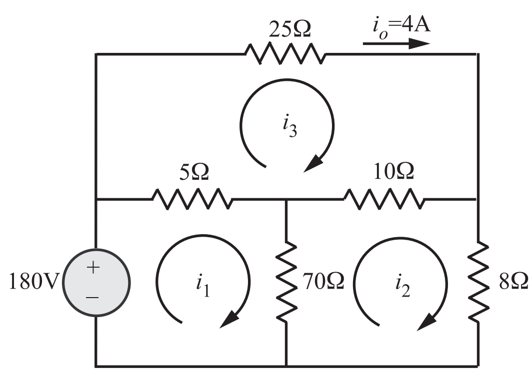


Figure R.P.1.2

SOLUTION

By inspection, we find that the mesh current $i_3 = i_o = 4$A

KVL *clockwise to mesh* 1 : $\quad 75i_1 - 70i_2 - 5i_3 = 180$

Substituting $i_3 = 4\text{A}$, we get $\quad 75i_1 - 70i_2 = 200$
KVL clockwise to mesh 2 : $\quad -70i_1 + 88i_2 - 10i_3 = 0$
Substituting the value $i_3 = 4\text{A}$, we get $\quad -70i_1 + 88i_2 = 40$
Puting the two mesh equations in matrix from, we get

$$\begin{bmatrix} 75 & -70 \\ -70 & 88 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 200 \\ 40 \end{bmatrix}$$

Using Cramer's rule, we get

$$i_1 = 12\text{A}, i_2 = 10\text{A}$$
$$P_{70} = (i_1 - i_2)^2 70 = 4 \times 70$$
$$= \mathbf{280\ W\ (dissipated)}$$

R.P 1.3

Solve for current **I** in the circuit of Fig. R.P. 1.3 using nodal analysis.

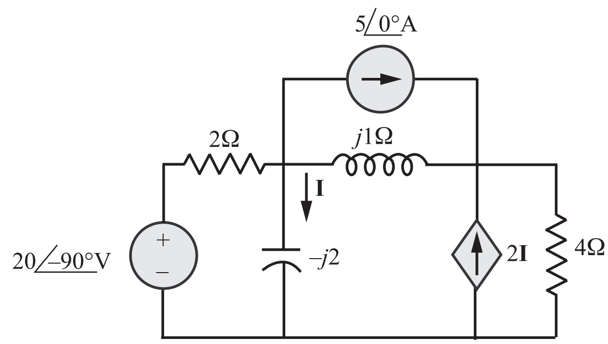


Figure R.P.1.3

SOLUTION

KCL at node $\mathbf{V}_1$:

$$\frac{\mathbf{V}_1 - 20\,\underline{/-90^\circ}}{2} + \frac{\mathbf{V}_1}{-j2} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{j1} + 5\,\underline{/0^\circ} = 0$$

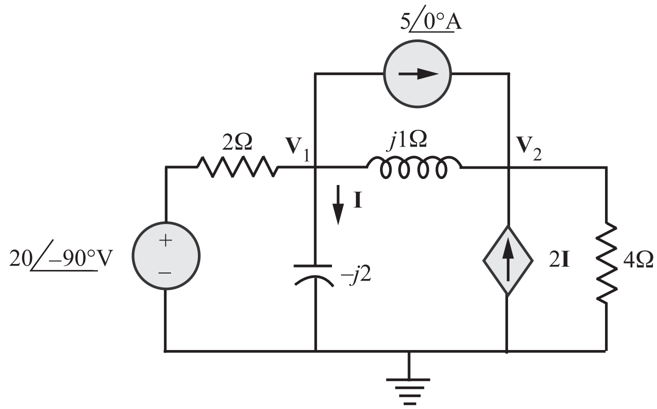
$$\Rightarrow \qquad (0.5 - j0.5)\mathbf{V}_1 + j\mathbf{V}_2 = -5 - j10$$

KCL at node $\mathbf{V}_2$:

$$\frac{\mathbf{V}_2 - \mathbf{V}_1}{j1} + \frac{\mathbf{V}_2}{4} - 2\mathbf{I} - 5\,\underline{/0^\circ} = 0$$

Also, $$\mathbf{I} = \frac{\mathbf{V}_1}{-j2}$$

Hence, $$\frac{\mathbf{V}_2 - \mathbf{V}_1}{j1} + \frac{\mathbf{V}_2}{4} + \frac{2}{j2}\mathbf{V}_1 - 5\,\underline{/0^\circ} = 0$$



$$\Rightarrow \qquad (0.25 - j)\mathbf{V}_2 = 5$$

$$\Rightarrow \qquad \mathbf{V}_2 = \frac{5}{0.25 - j}$$

Making use of $\mathbf{V}_2$ in the nodal equation at node $\mathbf{V}_1$, we get

$$-5 - j10 - \frac{j5}{0.25 - j} = 0.5(1 - j)\mathbf{V}_1$$

$$\Rightarrow \qquad (1 - j)\mathbf{V}_1 = -10 - j20 - \left(\frac{j40}{1 - j4}\right)$$

$$\Rightarrow \qquad \mathbf{V}_1 = 15.81 \angle -46.5^\circ \text{ V}$$

Hence,

$$\mathbf{I} = \frac{\mathbf{V}_1}{-j2} = \frac{15.81 \angle -46.5^\circ}{2 \angle -90^\circ}$$

$$= \mathbf{7.906 \angle 43.5^\circ \text{ A}}$$

R.P 1.4

Find $\mathbf{V}_o$ shown in the Fig. R.P. 1.4 using Nodal technique.

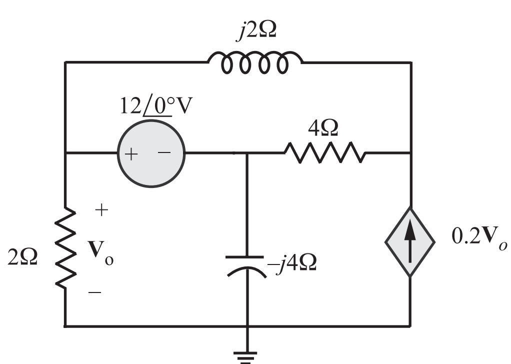


Figure R.P.1.4

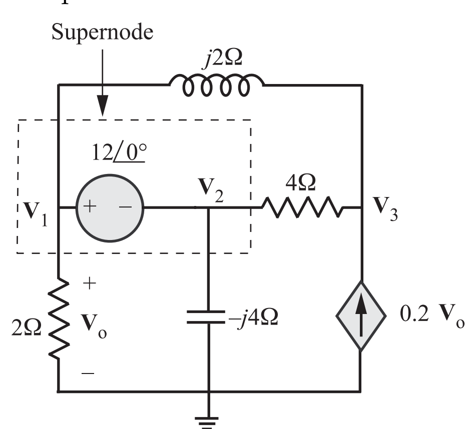


Figure R.P.1.4(a)

SOLUTION

We find from Fig RP 1.4(a) that,

$$\mathbf{V}_1 = \mathbf{V}_o$$

Constraint equation:

Applying KVL clockwise along the path consisting of voltage source, capacitor, and 2Ω resistor, we find that

$$12\,\underline{/0^\circ} + \mathbf{V}_2 - \mathbf{V}_1 = 0$$

$$\Rightarrow \qquad \mathbf{V}_1 = \mathbf{V}_2 + 12\,\underline{/0^\circ}$$

or

$$\mathbf{V}_2 = \mathbf{V}_1 - 12$$

KCL *at Supernode* :

$$\frac{\mathbf{V}_1 - \mathbf{V}_3}{j2} + \frac{\mathbf{V}_1}{2} + \frac{\mathbf{V}_2}{-j4} + \frac{\mathbf{V}_2 - \mathbf{V}_3}{4} = 0$$

$$\Rightarrow \quad (2 - j2)\mathbf{V}_1 + (1 + j)\mathbf{V}_2 + (-1 + j2)\mathbf{V}_3 = 0$$

KCL *at node* 3 :

$$\frac{\mathbf{V}_3 - \mathbf{V}_1}{j2} + \frac{\mathbf{V}_3 - \mathbf{V}_2}{4} - 0.2\mathbf{V}_o = 0 \tag{1.67}$$

Substituting $\mathbf{V}_o = \mathbf{V}_1$, we get

$$(0.8 - j2)\mathbf{V}_1 + \mathbf{V}_2 + (-1 + j2)\mathbf{V}_3 = 0 \tag{1.68}$$

Subtracting equation (1.68) from (1.67), we get

$$1.2\mathbf{V}_1 + j\mathbf{V}_2 = 0 \tag{1.69}$$

Substituting $\mathbf{V}_2 = \mathbf{V}_1 - 12$ (from the constraint equation), we get

$$1.2\mathbf{V}_1 + j(\mathbf{V}_1 - 12) = 0$$

$$\Rightarrow \qquad \mathbf{V}_1 = \frac{j12}{1.2 + j} = \mathbf{V}_o$$

Hence

$$\mathbf{V}_o = \mathbf{7.68}\,\underline{/\mathbf{50.2^\circ}}\ \mathbf{V}$$

R.P 1.5

Solve for i_o using mesh analysis.

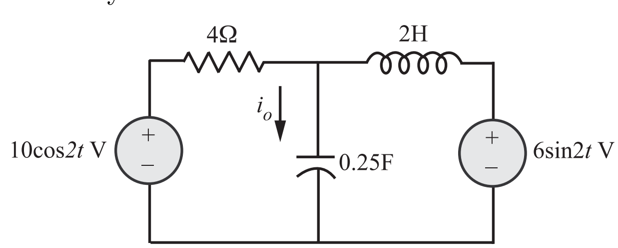


Figure R.P. 1.5

SOLUTION

The first step in the analysis is to draw the phasor circuit equivalent of Fig. R.P.1.5.

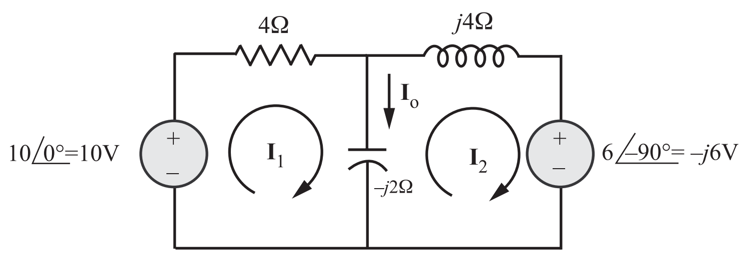


Figure R.P. 1.5(a)

$$\omega = 2$$

$$\begin{aligned}
10\cos 2t &\Rightarrow 10\,\underline{/0^\circ}\ \mathbf{V}\\
6\sin 2t = 6\cos(2t-90) &\Rightarrow 6\,\underline{/-90^\circ} = -j6\mathbf{V}\\
L = 2H &\Rightarrow X_L = j\omega L = j4\Omega\\
C = 0.25F &\Rightarrow X_C = \frac{1}{j\omega C} = \frac{1}{j2\left(\frac{1}{4}\right)} = -j2\Omega
\end{aligned}$$

$Applying\ KVL\ clockwise\ to\ mesh\ 1:$

$$-10 + (4-j2)\mathbf{I}_1 + j2\mathbf{I}_2 = 0$$

$$\Rightarrow \qquad (2-j1)\mathbf{I}_1 + j\mathbf{I}_2 = 5$$

$Applying\ KVL\ clockwise\ to\ mesh\ 2:$

$$j2\mathbf{I}_1 + (j4-j2)\mathbf{I}_2 + (-j6) = 0$$

$$\mathbf{I}_1 + \mathbf{I}_2 = 3$$

Putting the above mesh equations in a matrix form, we get

$$\begin{bmatrix} 2-j & j \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Using Cramer's rule, we get

$$\begin{aligned}
\mathbf{I}_1 &= 2 + j0.5,\\
\mathbf{I}_2 &= 1 - j0.5,\\
\mathbf{I}_o &= \mathbf{I}_1 - \mathbf{I}_2 = 1 + j = 1.414\,\underline{/45^\circ}
\end{aligned}$$

Hence $\quad i_o(t) = \mathbf{1.414}\cos\left(\mathbf{2t + 45^\circ}\right)\ \mathbf{A}$

R.P 1.6

Refer the circuit shown in Fig. R.P. 1.6. Find **I** using mesh analysis.

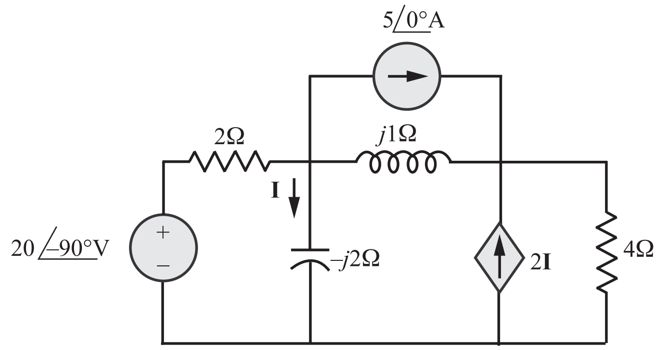


Figure R.P.1.6

SOLUTION

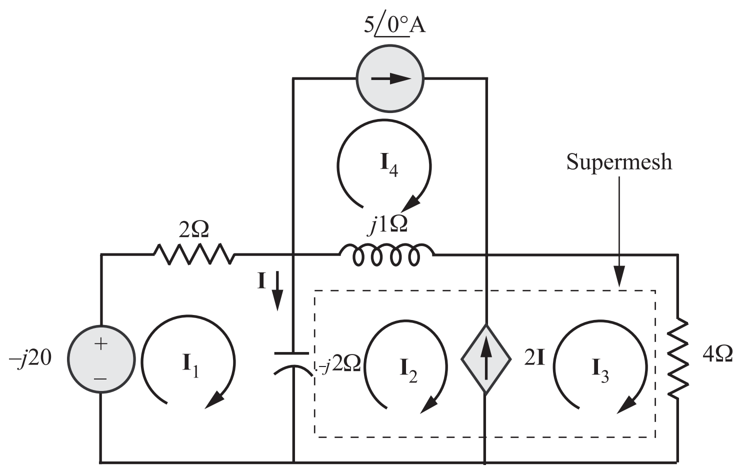


Figure R.P.1.6(a)

Constraint equation:

$$\mathbf{I}_3 - \mathbf{I}_2 = 2\mathbf{I}$$

$$\Rightarrow \quad \mathbf{I}_3 - \mathbf{I}_2 = 2(\mathbf{I}_1 - \mathbf{I}_2)$$

$$\Rightarrow \quad \mathbf{I}_3 = 2\mathbf{I}_1 - \mathbf{I}_2$$

Also, for mesh 4, $\quad \mathbf{I}_4 = 5 \text{ A}$

Applying KVL clockwise for mesh 1 :

$$-(-j20) + (2 - j2)\mathbf{I}_1 + j2\mathbf{I}_2 = 0$$

$$\Rightarrow \quad (1 - j)\mathbf{I}_1 + j\mathbf{I}_2 = -j10 \tag{1.70}$$

Applying KVL clockwise for the supermesh :

$$(j - j2)\mathbf{I}_2 + j2\mathbf{I}_1 + 4\mathbf{I}_3 - j\mathbf{I}_4 = 0$$

Substituting $\mathbf{I}_3 = 2\mathbf{I}_1 - \mathbf{I}_2$ and $\mathbf{I}_4 = 5\text{A}$

we get $$(8 + j2)\mathbf{I}_1 - (4 + j)\mathbf{I}_2 = j5 \tag{1.71}$$

Putting equations (1.70) and (1.71) in matrix form, we get

$$\begin{bmatrix} 1 - j & j \\ 8 + j2 & -(4 + j) \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} -j10 \\ j5 \end{bmatrix}$$

Solving for $\mathbf{I}_1$ and $\mathbf{I}_2$, we get

$$\begin{aligned} \mathbf{I}_1 &= -(5.44 + j4.26) \text{ A} \\ \mathbf{I}_2 &= -(11.18 + j9.7) \text{ A} \\ \mathbf{I} &= \mathbf{I}_1 - \mathbf{I}_2 \\ &= 5.735 + j5.44 \\ &= \mathbf{7.9\, \underline{/43.49^\circ}\ \ A} \end{aligned}$$

R.P 1.7

Calculate $\mathbf{V}_o$ in the circuit of Fig. R.P. 1.7 using the method of source transformation.

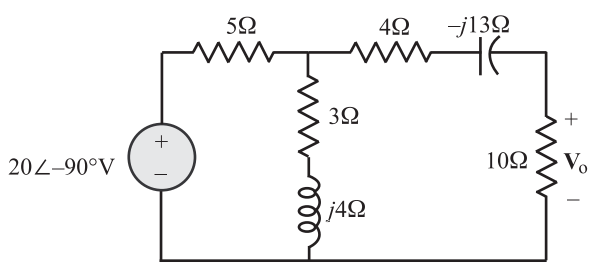


Figure R.P. 1.7

SOLUTION

Transform the voltage source to a current source and obtain the circuit shown in Fig. R.P.1.7(a).

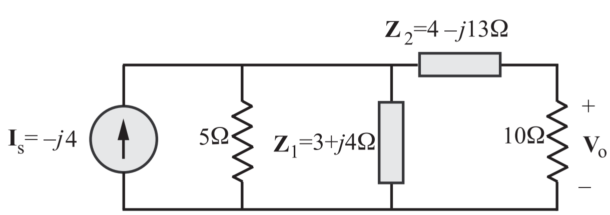


Figure R.P.1.7(a)

$$\mathbf{I}_s = \frac{20\,\underline{/-90^\circ}}{5} = 4\,\underline{/-90^\circ}\text{ A}$$

$$\mathbf{Z}_p = 5\Omega || 3 + j4 = \frac{5 \times (3 + j4)}{5 + (3 + j4)} = 2.5 + j1.25\Omega$$

Converting the current source in Fig. R.P. 1.7(b) to a voltage source gives the circuit as shown in Fig. R.P. 1.7(c).

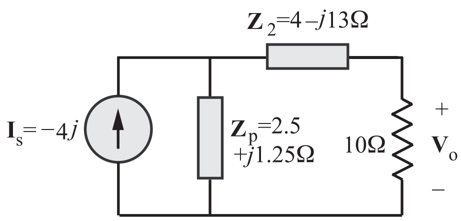


Figure R.P.1.7(b)

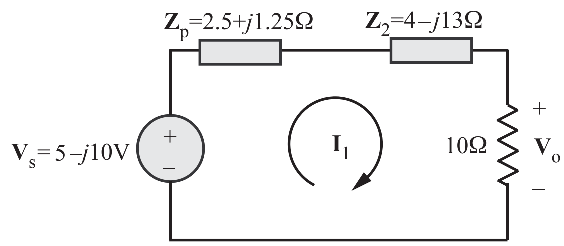


Figure R.P.1.7(c)

$$\begin{aligned}\mathbf{V}_s = \mathbf{I}_s\mathbf{Z}_p &= -4j(2.5+j1.25)\\ &= 5-j10\mathbf{V}\\ \mathbf{V}_o &= 10\mathbf{I}\\ &= \left[\frac{\mathbf{V}_s}{\mathbf{Z}_p+\mathbf{Z}_2+10}\right]10\\ &= \frac{5-j10}{[2.5+j1.25+4-j13+10]}\times 10\\ &= \mathbf{5.519\,\underline{/-28^\circ}\ V}\end{aligned}$$

R.P 1.8

Find v_x and i_x in the circuit shown in Fig. R.P. 1.8.

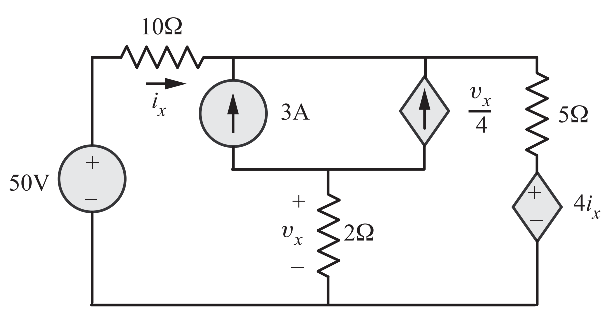


Figure R.P. 1.8

SOLUTION

Constraint equation: $$i_2 - i_1 = 3 + \frac{v_x}{4}$$

$$\Rightarrow \qquad i_2 = i_1 + 3 + \frac{v_3}{4}$$

The above equation becomes very clear if one writes *KCL equation* at node B of Fig. R.P. 1.8(a).

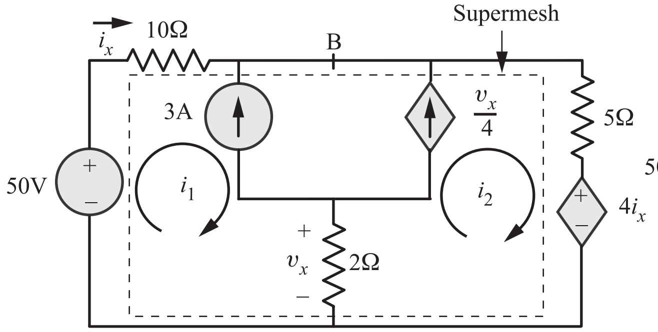


Figure R.P. 1.8(a)

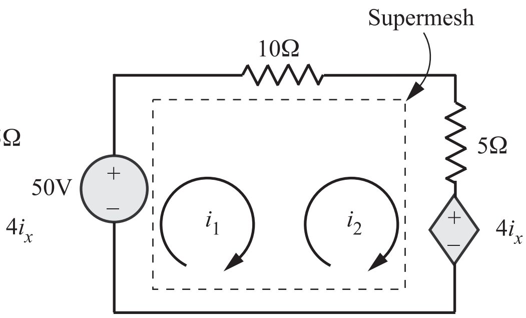


Figure R.P. 1.8(b)

Applying KVL clockwise to the supermesh in Fig. R.P. 1.8(b), we get

$$-50 + 10i_1 + 5i_2 + 4i_x = 0$$

But $i_x = i_1$. Hence, $-50 + 10i_1 + 5i_2 + 4i_1 = 0$

$$\Rightarrow \quad 14i_1 + 5i_2 = 50 \tag{1.72}$$

Making use of $v_x = (i_1 - i_2) \times 2$ in the constraint equation, we get

$$i_2 = i_1 + 3 + \frac{(i_1 - i_2) \times 2}{4}$$

$$\Rightarrow \quad i_2 = i_1 + 3 + \frac{i_1 - i_2}{2}$$

$$\Rightarrow \quad 2i_2 = 2i_1 + 6 + i_1 - i_2$$

$$\Rightarrow \quad 3i_1 - 3i_2 + 6 = 0$$

$$\Rightarrow \quad i_1 - i_2 = -2 \tag{1.73}$$

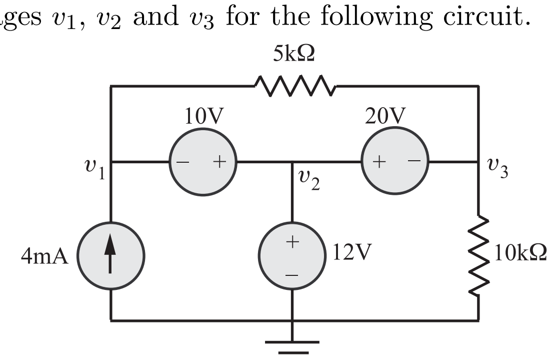
Solving equations (1.72) and (1.73) gives $i_1 = 2.105$ A, $i_2 = 4.105$ A

Thus, $$v_x = 2(i_1 - i_2) = \mathbf{-4\ V}$$

and $$i_x = i_1 = \mathbf{2.105\ A}$$

R.P 1.9

Obtain the node voltages v_1, v_2 and v_3 for the following circuit.



SOLUTION

We have a supernode as shown in Fig. R.P. 1.9(a). By inspection, we find that $\mathbf{V}_2 = 12\text{V}$.
Refer Fig. R.P. 1.9(b) for further analysis.

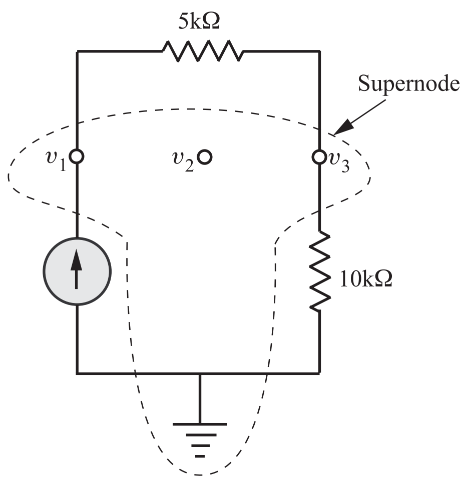


Figure R.P.1.9(a)

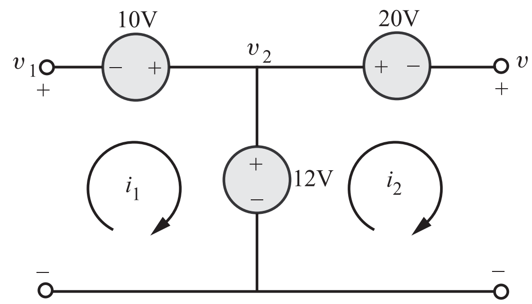


Figure R.P.1.9(b).

KVL clockwise to mesh 1 :

$$-v_1 - 10 + 12 = 0 \quad \Rightarrow \quad v_1 = 2$$

KVL clockwise to mesh 2 :

$$-12 + 20 + v_3 = 0$$

$$\Rightarrow \qquad v_3 = -8 \text{ V}$$

Hence, $\boldsymbol{v_1 = 2}$ **V**, $\boldsymbol{v_2 = 12}$ **V**, $\boldsymbol{v_3 = -8}$ **V**

R.P 1.10

Find the equivalent resistance R_{ab} for the circuit shown in Fig. R.P.1.10.

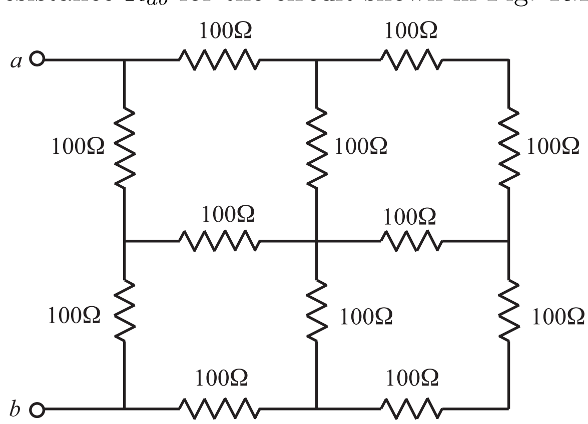


Figure R.P. 1.10

SOLUTION

The circuit is redrawn marking the nodes c to j in Fig. R.P. 1.10(a). It can be seen that the network consists of four identical stars :

(i) ae, ef, cb
(ii) ac, cf, cd
(iii) dg, gf, gj
(iv) bh, fh, hj

Converting each stars in to its equivalent delta, the network is redrawn as shown in Fig. R.P. 1.10(b), noting that each resistance in delta is $100 \times 3 = 300\Omega$, eliminating nodes c, e, g, h.

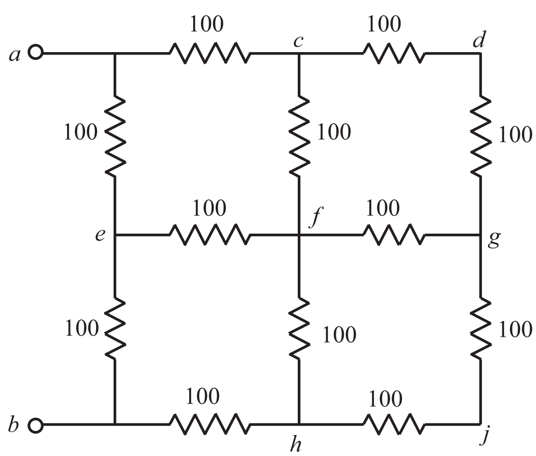


Figure R.P.1.10(a)

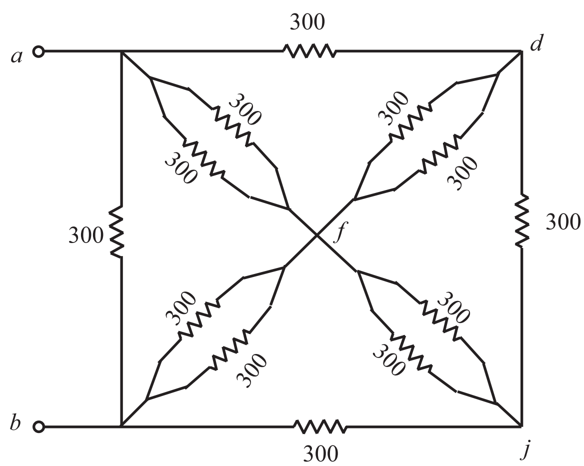


Figure R.P.1.10(b)

Reducing the parallel resistors, we get the circuit as in Fig. R.P. 1.10(c).

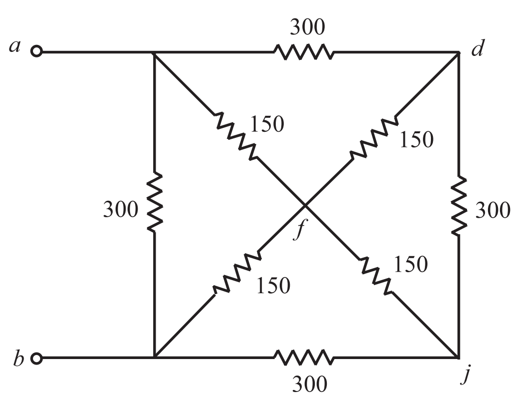


Figure R.P.1.10(c)

Hence, there are two identical deltas afd and bfj. Converting them to their equivalent stars, we get the circuit as shown in Fig. R.P.1.10(d).

$$R_{ak} = R_{bl} = R_{kd} = R_{lj} = \frac{300 \times 150}{600} = 75\ \Omega$$

$$R_{kf} = R_{fl} = \frac{150^2}{600} = 37.5\ \Omega$$

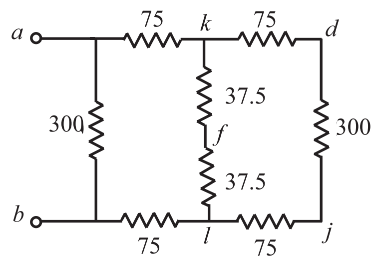


Figure R.P.1.10(d)

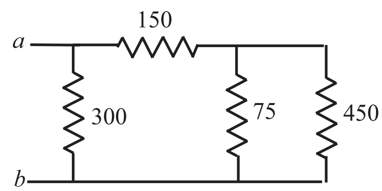


Figure R.P.1.10(e)

The circuit is further reduced to Fig. R.P. 1.10(e) and then to Fig. R.P. 1.10(f) and (g). Then the equivalent resistance is

$$R_{ab} = \frac{214.286 \times 300}{514.286} = 125\ \Omega$$

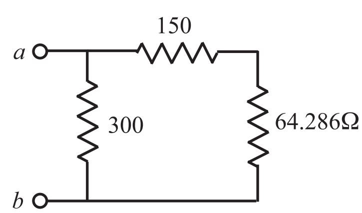


Figure R.P.1.10(f)

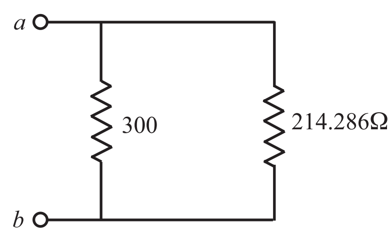


Figure R.P.1.10(g)

R.P 1.11

Obtain the equivalent resistance R_{ad} for the circuit shown in Fig. R.P.1.11.

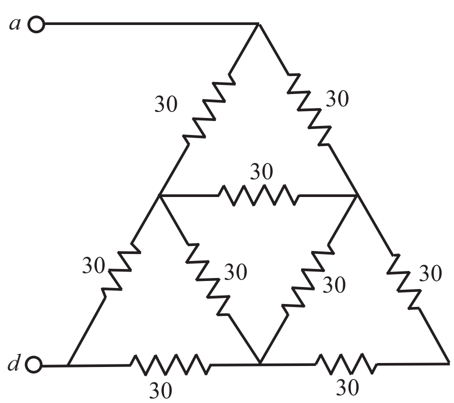


Figure R.P.1.11

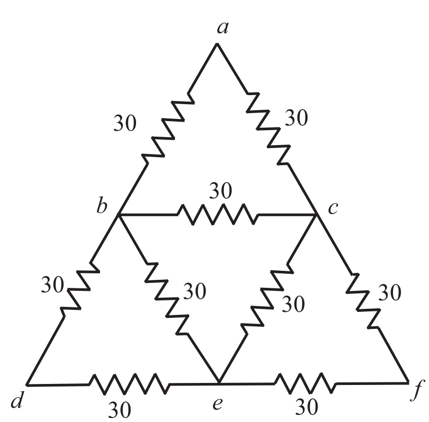


Figure R.P.1.11(a)

SOLUTION

The circuit is redrawn as shown Fig. 1.11(a), marking the nodes a to f to identify the deltas in it. It contains 3 deltas abc, bde and def with 3 equal resistors of 30 Ω each. For each delta, their equivalent star contains 3 resistors each of value $\frac{30}{3} = 10\Omega$. Then the circuit becomes as shown in Fig. R.P. 1.11(b) where f is isolated.

On simplification, we get the circuit as shown in Fig. R.P.1.11(c) and further reduced to Fig. R.P.1.11(d).

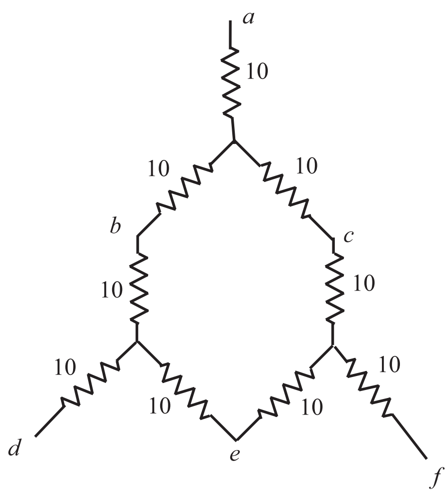


Figure R.P.1.11(b)

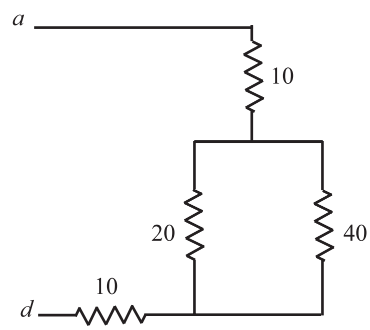


Figure R.P.1.11(c)

Figure R.P.1.11(d)

Then the equivalent ressitance,

$$R_{ad} = 10 + 13.33 + 10 = 33.33\ \Omega$$

R.P 1.12

Draw a network for the following mesh equations in matrix form :

$$\begin{bmatrix} 5+j5 & -j5 & 0 \\ -j5 & 8+j8 & -6 \\ 0 & -6 & 10 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \mathbf{I}_3 \end{bmatrix} = \begin{bmatrix} 30\underline{/-0^\circ} \\ 0 \\ -20\underline{/-0^\circ} \end{bmatrix}$$

SOLUTION

The general form of the mesh equations in matrix form for a network having three mashes is given by

$$\begin{bmatrix} \mathbf{Z}_{11} & -\mathbf{Z}_{12} & -\mathbf{Z}_{13} \\ -\mathbf{Z}_{21} & \mathbf{Z}_{22} & -\mathbf{Z}_{23} \\ -\mathbf{Z}_{31} & -\mathbf{Z}_{32} & \mathbf{Z}_{33} \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \mathbf{I}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{V}_1 \underline{/\theta_1} \\ \mathbf{V}_2 \underline{/\theta_2} \\ \mathbf{V}_3 \underline{/\theta_3} \end{bmatrix}$$

and, $$\mathbf{Z}_{11} = \mathbf{Z}_{10} + \mathbf{Z}_{12} + \mathbf{Z}_{13}$$

where $\mathbf{Z}_{10} =$ Sum of the impedances confined to mesh 1 alone

$\mathbf{Z}_{12} =$ Sum of the impedances common to meshes 1 and 2

$\mathbf{Z}_{13} =$ Sum of the impedances common to meshes 1 and 3

Similiar difenitions hold good for $\mathbf{Z}_{22}$ and $\mathbf{Z}_{33}$. Also, $\mathbf{Z}_{ij} = \mathbf{Z}_{ji}$

For the present problem,

$$\begin{aligned} \mathbf{Z}_{11} &= 5 + j5\Omega \\ \mathbf{Z}_{12} = \mathbf{Z}_{21} &= j5\Omega \\ \mathbf{Z}_{13} = \mathbf{Z}_{31} &= 0\Omega \\ \mathbf{Z}_{23} = \mathbf{Z}_{32} &= 6\Omega \end{aligned}$$

We know that, $$\mathbf{Z}_{11} = \mathbf{Z}_{10} + \mathbf{Z}_{12} + \mathbf{Z}_{13}$$

$$\Rightarrow \quad 5 + j5 = \mathbf{Z}_{10} + j5 + 0$$

$$\Rightarrow \quad \mathbf{Z}_{10} = 5\Omega$$

Similarly, $$\mathbf{Z}_{22} = \mathbf{Z}_{20} + \mathbf{Z}_{21} + \mathbf{Z}_{23}$$

$$\Rightarrow \quad 8 + j8 = \mathbf{Z}_{20} + j5 + 6$$

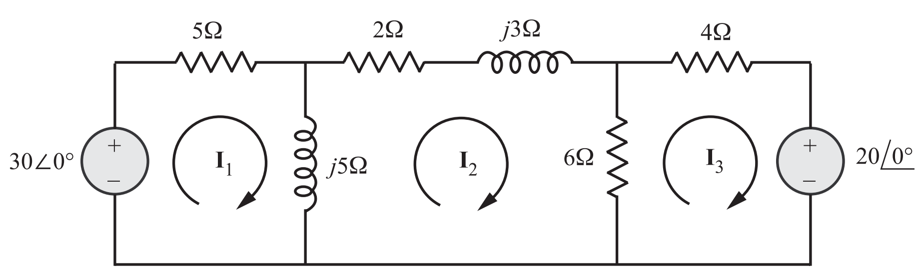
$$\Rightarrow \quad \mathbf{Z}_{20} = 2 + j3\Omega$$

Finally, $$\mathbf{Z}_{33} = \mathbf{Z}_{30} + \mathbf{Z}_{31} + \mathbf{Z}_{32}$$

$$\Rightarrow \quad 10 = \mathbf{Z}_{30} + 0 + 6$$

$$\Rightarrow \quad \mathbf{Z}_{30} = 4\Omega$$

Making use of the above impedances, we can configure a network as shown below :



R.P 1.13

Draw a network for the following nodal equations in matrix form.

$$\begin{bmatrix} \left(\dfrac{1}{-j10}+\dfrac{1}{10}\right) & -\dfrac{1}{10} \\ -\dfrac{1}{10} & \left(\dfrac{1}{5}(1-j)+\dfrac{1}{10}\right) \end{bmatrix} \begin{bmatrix} \mathbf{V}_a \\ \mathbf{V}_b \end{bmatrix} = \begin{bmatrix} 10\,\underline{/0^\circ} \\ 0 \end{bmatrix}$$

SOLUTION

The general form of the nodal equations in matrix form for a network having two nodes is given by

$$\begin{bmatrix} \mathbf{Y}_{11} & -\mathbf{Y}_{12} \\ -\mathbf{Y}_{21} & \mathbf{Y}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{I}_1\,\underline{/\theta_1} \\ \mathbf{I}_2\,\underline{/\theta_2} \end{bmatrix}$$

where $\mathbf{Y}_{11} = \mathbf{Y}_{10} + \mathbf{Y}_{12}$ and $\mathbf{Y}_{22} = \mathbf{Y}_{20} + \mathbf{Y}_{21}$.

$\mathbf{Y}_{10}$ = sum of admittances connected at node 1 alone.

$\mathbf{Y}_{12} = \mathbf{Y}_{21}$ = sum of admittances common to nodes 1 and 2.

$\mathbf{Y}_{20}$ = sum of admittances connected at node 2 alone.

For the present problem,

$$\mathbf{Y}_{11} = \frac{1}{-j10} + \frac{1}{10}\ \text{S}$$

$$\mathbf{Y}_{12} = \mathbf{Y}_{21} = \frac{1}{10}\ \text{S}$$

$$\mathbf{Y}_{22} = \frac{1}{5}(1-j) + 10\ \text{S}$$

We know that, $\mathbf{Y}_{11} = \mathbf{Y}_{10} + \mathbf{Y}_{12}$

$$\Rightarrow \quad \frac{1}{-j10} + \frac{1}{10} = \mathbf{Y}_{10} + \frac{1}{10}$$

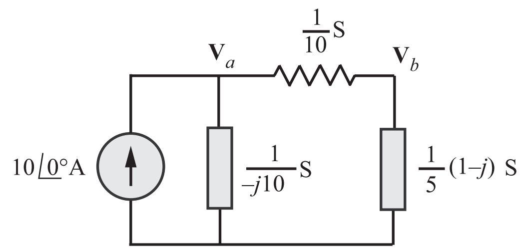
$$\Rightarrow \quad \mathbf{Y}_{10} = \frac{-1}{j10}\ \text{S}$$

Similarly, $\mathbf{Y}_{22} = \mathbf{Y}_{20} + \mathbf{Y}_{21}$

$$\Rightarrow \quad \frac{1}{5}(1-j) + \frac{1}{10} = \mathbf{Y}_{20} + \frac{1}{10}$$

$$\Rightarrow \quad \mathbf{Y}_{20} = \frac{1}{5}(1-j)\ \text{S}$$

Making use of the above admittances, we can configure a network as shown below :



Exercise problems

E.P 1.1

Refer the circuit shown in Fig. E.P.1.1. Using mesh analysis, find the current delivered by the source. Verify the result using nodal technique.

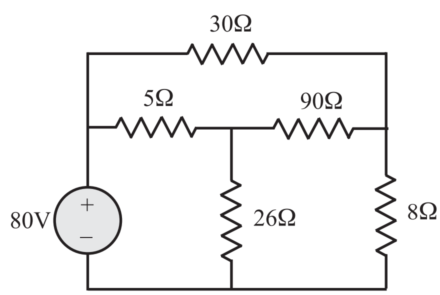


Figure E.P. 1.1

Ans : 5A

E.P 1.2

For the resistive circuit shown in Fig. E.P. 1.2. by using source transformation and mesh analysis, find the current supplied by the 20 V source.

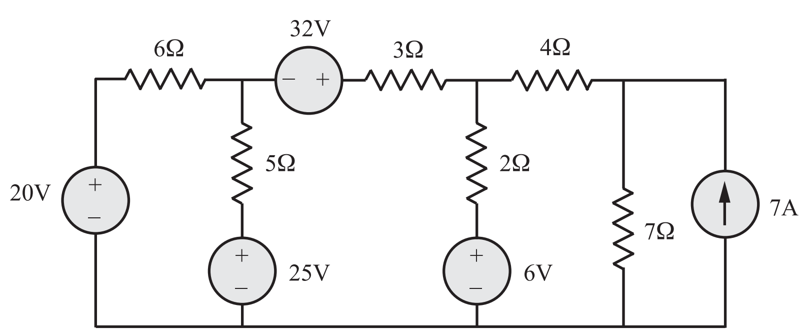


Figure E.P. 1.2

Ans : 2.125A

E.P 1.3

Find the voltage v using nodal technique for the circuit shown in Fig. E.P. 1.3.

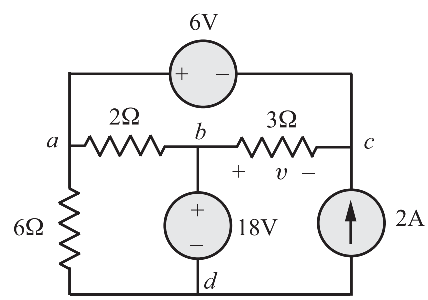


Figure E.P. 1.3

Ans : $\boldsymbol{v = 5\text{V}}$

E.P 1.4

Refer the network shown in Fig. E.P. 1.4. Find the currents i_1 and i_2 using nodal analysis.

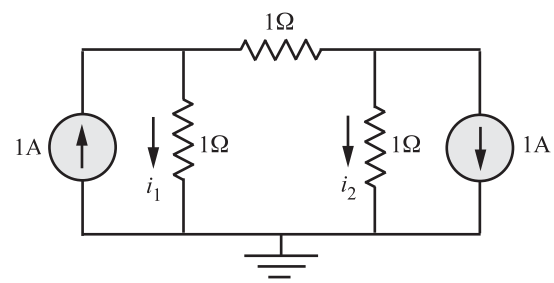


Figure E.P. 1.4

Ans : $\boldsymbol{i_1 = 1\text{ A}, i_2 = -1\text{ A}}$

E.P 1.5

For the network shown in Fig. E.P. 1.5, find the currents through the resistors R_1 and R_2 using nodal technique.

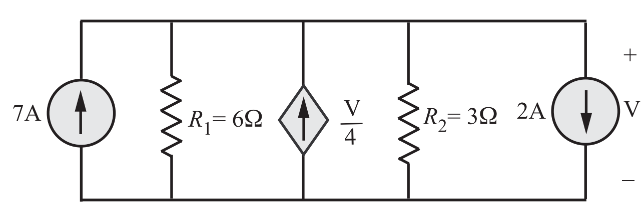


Figure E.P. 1.5

Ans : **3.33A, 6.67A**

E.P 1.6

Use the mesh-current method to find the branch currents i_1, i_2 and i_3 in the circuit of Fig. E.P. 1.6.

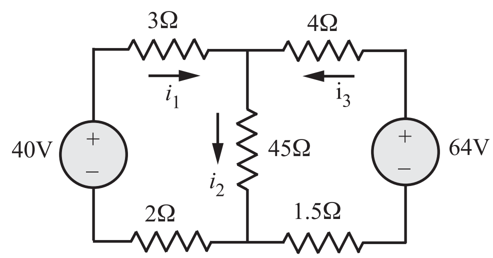


Figure E.P. 1.6

Ans : $\boldsymbol{i_1 = -1.72\text{A},\ i_2 = 1.08\text{A}\ ,\ i_3 = 2.8\text{A}}$

E.P 1.7

Refer the network shown in Fig. E.P. 1.7. Find the power delivered by the dependent voltage source in the network.

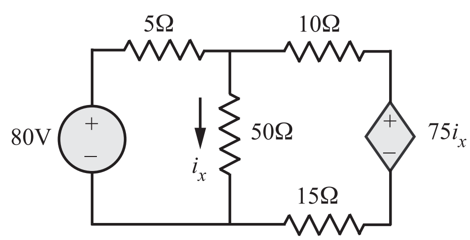


Figure E.P. 1.7

Ans : **−375 Watts**

E.P 1.8

Find the current I_x using (i) nodal analysis and (ii) mesh analysis.

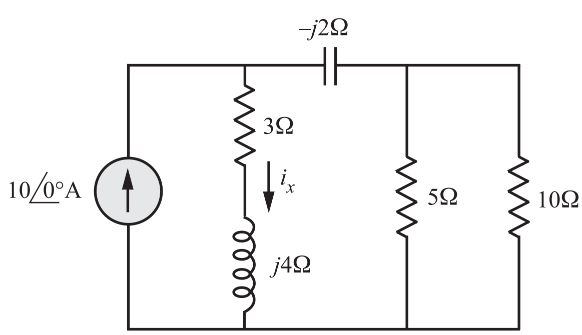


Figure E.P. 1.8

Ans : $\boldsymbol{I_x = \dfrac{150(3 + j4)}{95 + j30}\text{A}}$

E.P 1.9

Determine the current i_x in the circuit shown in Fig. E.P. 1.9

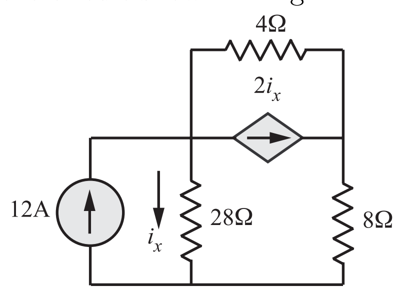


Figure E.P. 1.9

Ans : $\boldsymbol{i_x = 3\text{A}}$

E.P 1.10

Determine the resistance between the terminals $a - b$ of the network shown in Fig. E.P. 1.10.

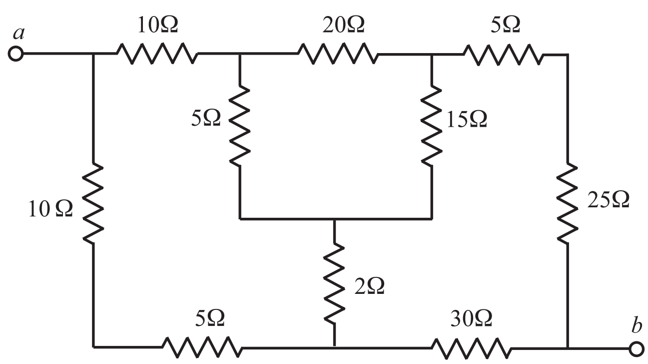


Figure E.P. 1.10

Ans : **23.6 Ω**

E.P 1.11

Determine the resistance between the points A and B in the network shown in Fig. E.P. 1.11.

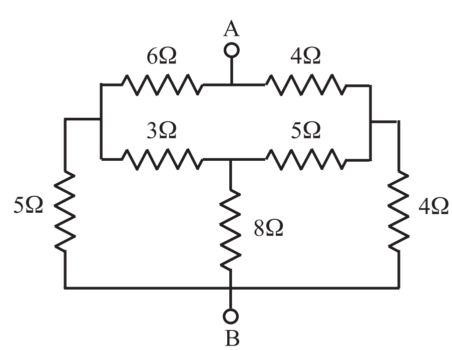


Figure E.P. 1.11

Ans : **4.23 Ω**

E.P 1.12

Determine the current in the galvanometer branch of the bridge network shown in Fig. E.P. 1.12.

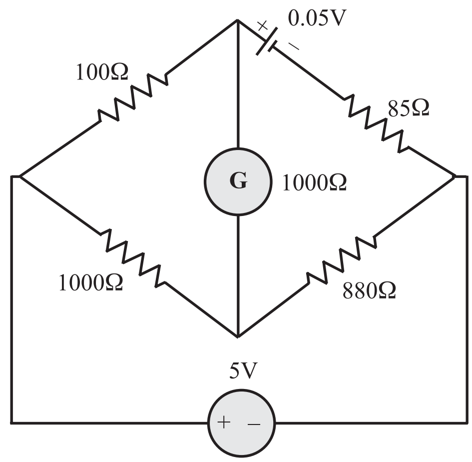


Figure E.P. 1.12

Ans : $\mathbf{10.62\mu A}$

Chapter 3 Circuit Theorems

Many electric circuits are complex, but it is an engineer's goal to reduce their complexity to analyze them easily. In the previous chapters, we have mastered the ability to solve networks containing independent and dependent sources making use of either mesh or nodal analysis. In this chapter, we will introduce new techniques to strengthen our armoury to solve complicated networks. Also, these new techniques in many cases do provide insight into the circuit's operation that cannot be obtained from mesh or nodal analysis. Most often, we are interested only in the detailed performance of an isolated portion of a complex circuit. If we can model the remainder of the circuit with a simple equivalent network, then our task of analysis gets greatly reduced and simplified. For example, the function of many circuits is to deliver maximum power to load such as an audio speaker in a stereo system. Here, we develop the required relationship betweeen a load resistor and a fixed series resistor which can represent the remaining portion of the circuit. Two of the theorems that we present in this chapter will permit us to do just that.

3.1 Superposition theorem

The principle of superposition is applicable only for linear systems. The concept of superposition can be explained mathematically by the following response and excitation principle :

$$i_1 \rightarrow v_1$$
$$i_2 \rightarrow v_2$$
$$\text{then,} \qquad i_1 + i_2 \rightarrow v_1 + v_2$$

The quantity to the left of the arrow indicates the excitation and to the right, the system response. Thus, we can state that a device, if excited by a current i_1 will produce a response v_1. Similarly, an excitation i_2 will cause a response v_2. Then if we use an excitation $i_1 + i_2$, we will find a response $v_1 + v_2$.

The principle of superposition has the ability to reduce a complicated problem to several easier problems each containing only a single independent source.

Superposition theorem states that,

In any linear circuit containing multiple independent sources, the current or voltage at any point in the network may be calculated as algebraic sum of the individual contributions of each source acting alone.

When determining the contribution due to a particular independent source, we disable all the remaining independent sources. That is, all the remaining voltage sources are made zero by replacing them with short circuits, and all remaining current sources are made zero by replacing them with open circuits. Also, it is important to note that if a dependent source is present, it must remain active (unaltered) during the process of superposition.

Action Plan:

(i) In a circuit comprising of many independent sources, only one source is allowed to be active in the circuit, the rest are deactivated (turned off).

(ii) To deactivate a voltage source, replace it with a short circuit, and to deactivate a current source, replace it with an open circuit.

(iii) The response obtained by applying each source, one at a time, are then added algebraically to obtain a solution.

Limitations: Superposition is a fundamental property of linear equations and, therefore, can be applied to any effect that is linearly related to the cause. That is, we want to point out that, superposition principle applies only to the current and voltage in a linear circuit but it cannot be used to determine power because power is a non-linear function.

EXAMPLE 3.1

Find the current in the 6 Ω resistor using the principle of superposition for the circuit of Fig. 3.1.

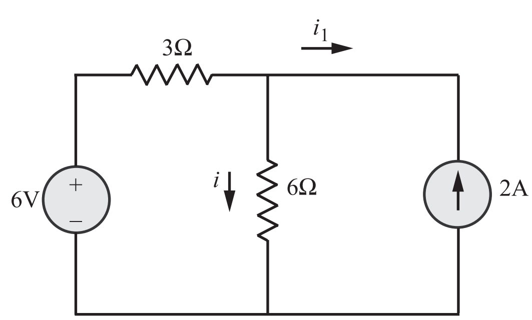


Figure 3.1

SOLUTION

As a first step, set the current source to zero. That is, the current source appears as an open circuit as shown in Fig. 3.2.

$$i_1 = \frac{6}{3+6} = \frac{6}{9}\text{A}$$

As a next step, set the voltage to zero by replacing it with a short circuit as shown in Fig. 3.3.

$$i_2 = \frac{2 \times 3}{3+6} = \frac{6}{9}\text{A}$$

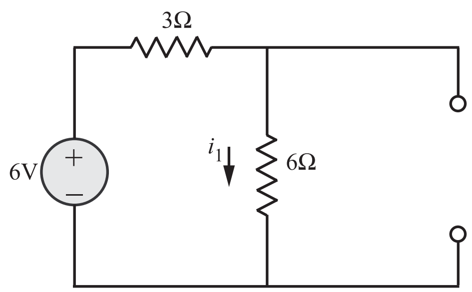


Figure 3.2

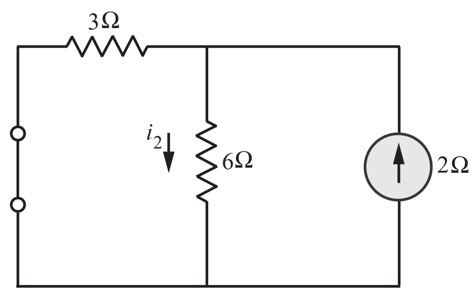


Figure 3.3

The total current i is then the sum of i_1 and i_2

$$i = i_1 + i_2 = \mathbf{\frac{12}{9}A}$$

EXAMPLE 3.2

Find i_o in the network shown in Fig. 3.4 using superposition.

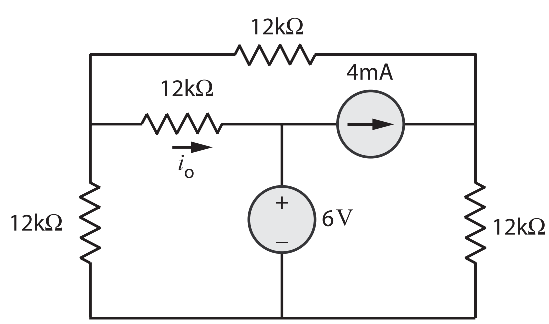


Figure 3.4

SOLUTION

As a first step, set the current source to zero. That is, the current source appears as an open circuit as shown in Fig. 3.5.

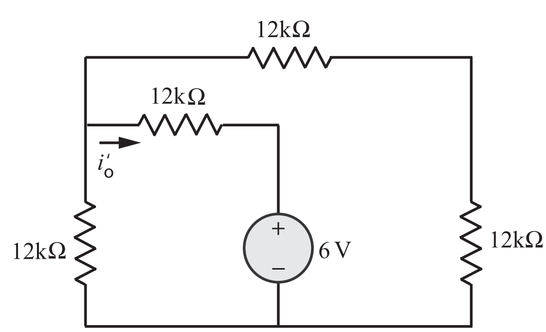
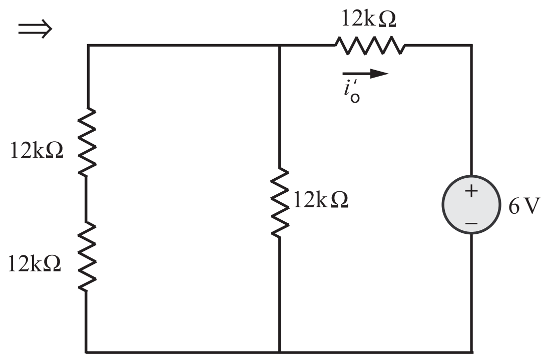
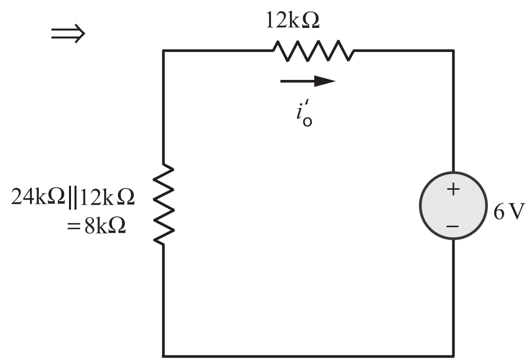


Figure 3.5

$\Rightarrow$



$\Rightarrow$



$$i_o' = \frac{-6}{(8+12)\times 10^3} = -0.3 \text{ mA}$$

As a second step, set the voltage source to zero. This means the voltage source in Fig. 3.4 is replaced by a short circuit as shown in Figs. 3.6 and 3.6(a). Using current division principle,

$$i_A = \frac{iR_2}{R_1 + R_2}$$

where $R_1 = (12 \text{ k}\Omega || 12 \text{ k}\Omega) + 12 \text{ k}\Omega$

$= 6 \text{ k}\Omega + 12 \text{ k}\Omega$

$= 18 \text{ k}\Omega$

and $R_2 = 12 \text{ k}\Omega$

$\Rightarrow$ $$i_A = \frac{4\times 10^{-3} \times 12 \times 10^3}{(12+18)\times 10^3}$$

$= 1.6 \text{ mA}$

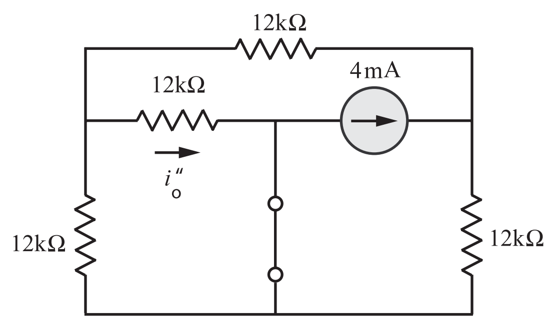


Figure 3.6

Again applying the current division principle,

$$i_o'' = \frac{i_A \times 12}{12+12} = 0.8 \text{ mA}$$

Thus, $i_o = i_o' + i_o'' = -0.3 + 0.8 = \mathbf{0.5\ mA}$

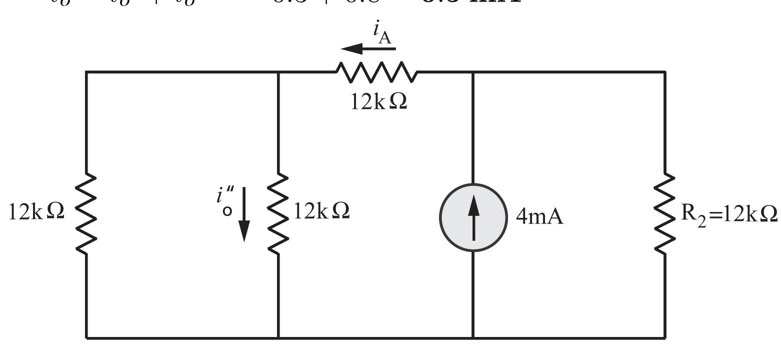


Figure 3.6(a)

EXAMPLE 3.3

Use superposition to find i_o in the circuit shown in Fig. 3.7.

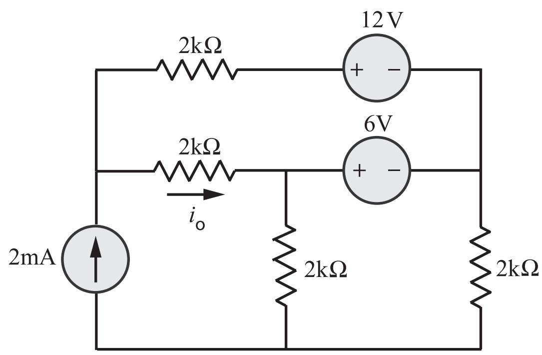


Figure 3.7

SOLUTION

As a first step, keep only the 12 V source active and rest of the sources are deactivated. That is, 2 mA current source is opened and 6 V voltage source is shorted as shown in Fig. 3.8.

$$i'_o = \frac{12}{(2+2)\times 10^3}$$
$$= 3 \text{ mA}$$

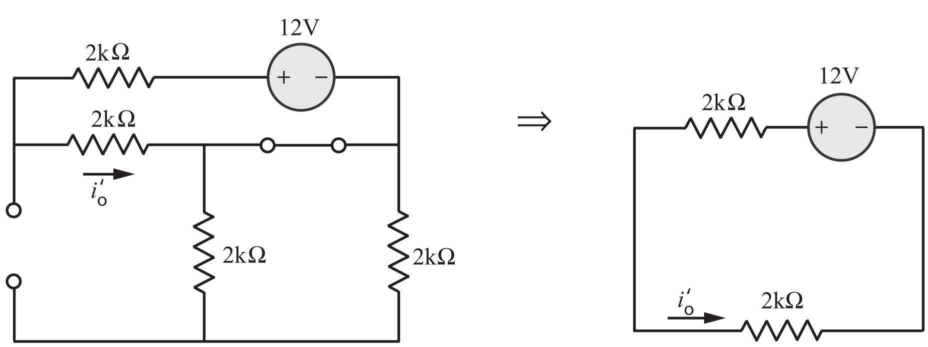


Figure 3.8

As a second step, keep only 6 V source active. Deactivate rest of the sources, resulting in a circuit diagram as shown in Fig. 3.9.

Applying KVL clockwise to the upper loop, we get

$$-2\times 10^3 i_o'' - 2\times 10^3 i_o'' - 6 = 0$$

$$\Rightarrow \qquad i_o'' = \frac{-6}{4\times 10^3} = -1.5\ \text{mA}$$

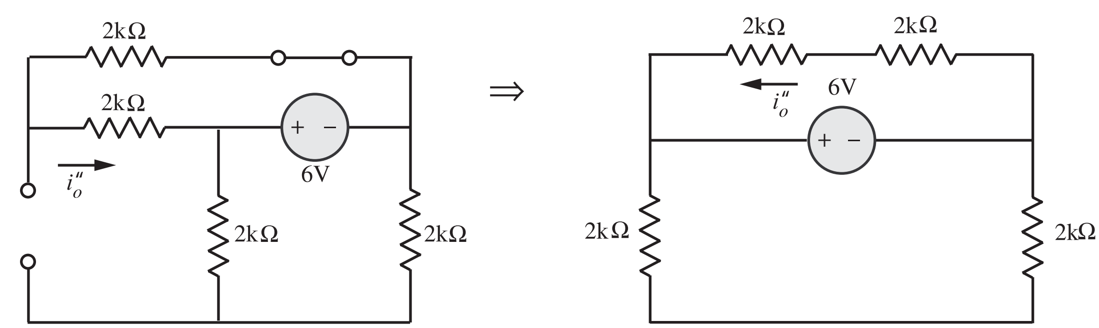


Figure 3.9

As a final step, deactivate all the independent voltage sources and keep only 2 mA current source active as shown in Fig. 3.10.

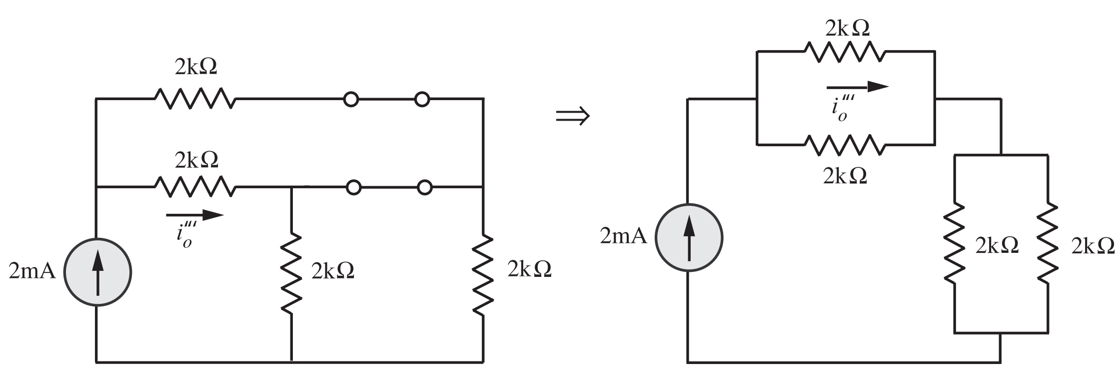


Figure 3.10

Current of 2 mA splits equally.

Hence, $\qquad i_o''' = 1\text{mA}$

Applying the superposition principle, we find that

$$\begin{aligned} i_o &= i_o' + i_o'' + i_o''' \\ &= 3 - 1.5 + 1 \\ &= \mathbf{2.5\ mA} \end{aligned}$$

EXAMPLE 3.4

Find the current i for the circuit of Fig. 3.11.

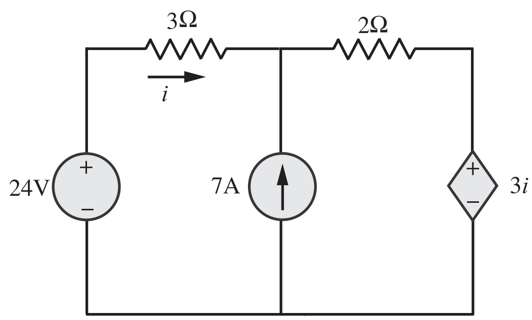


Figure 3.11

SOLUTION

We need to find the current i due to the two independent sources.

As a first step in the analysis, we will find the current resulting from the independent voltage source. The current source is deactivated and we have the circuit as shown as Fig. 3.12.

Applying *KVL* clockwise around loop shown in Fig. 3.12, we find that

$$5i_1 + 3i_1 - 24 = 0$$

$$\Rightarrow \qquad i_1 = \frac{24}{8} = 3\text{A}$$

As a second step, we set the voltage source to zero and determine the current i_2 due to the current source. For this condition, refer to Fig. 3.13 for analysis.

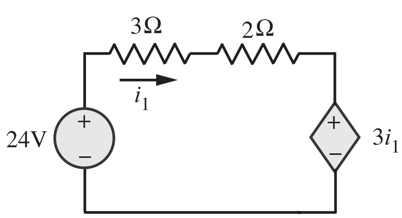


Figure 3.12

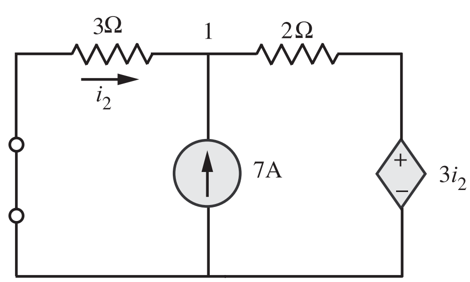


Figure 3.13

Applying KCL at node 1, we get

$$i_2 + 7 = \frac{v_1 - 3i_2}{2} \tag{3.1}$$

Noting that

$$-\,i_2 = \frac{v_1 - 0}{3}$$

we get,

$$v_1 = -3i_2 \tag{3.2}$$

Making use of equation (3.2) in equation (3.1) leads to

$$i_2 + 7 = \frac{-3i_2 - 3i_2}{2}$$

$$\Rightarrow \qquad i_2 = -\frac{7}{4}\text{A}$$

Thus, the total current

$$\begin{aligned} i &= i_1 + i_2 \\ &= 3 - \frac{7}{4}\text{ A} = \mathbf{\frac{5}{4}A} \end{aligned}$$

EXAMPLE 3.5

For the circuit shown in Fig. 3.14, find the terminal voltage V_{ab} using superposition principle.

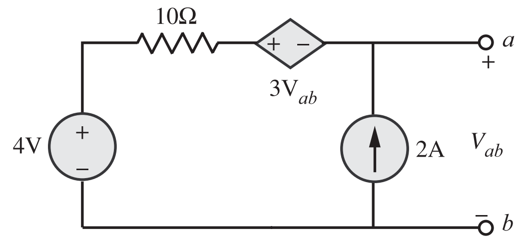


Figure 3.14

SOLUTION

As a first step in the analysis, deactivate the independent current source. This results in a circuit diagram as shown in Fig. 3.15.

Applying KVL clockwise gives

$$-4 + 10 \times 0 + 3V_{ab_1} + V_{ab_1} = 0$$

$$\Rightarrow \qquad 4V_{ab_1} = 4$$

$$\Rightarrow \qquad V_{ab_1} = 1\text{V}$$

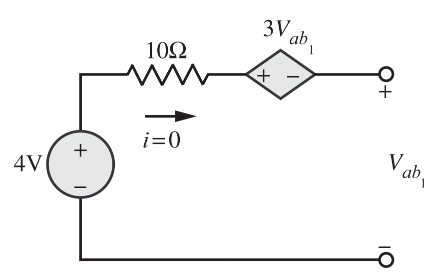


Figure 3.15

Next step in the analysis is to deactivate the independent voltage source, resulting in a circuit diagram as shown in Fig. 3.16.

Applying KVL gives

$$-10 \times 2 + 3V_{ab_2} + V_{ab_2} = 0$$

$$\Rightarrow \qquad 4V_{ab_2} = 20$$

$$\Rightarrow \qquad V_{ab_2} = 5\text{V}$$

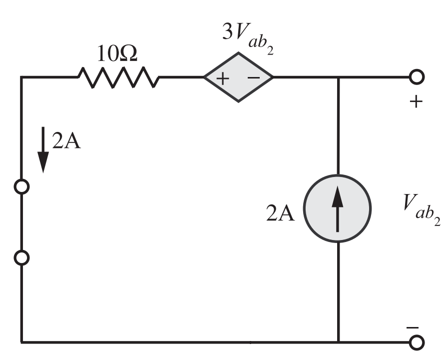


Figure 3.16

According to superposition principle,

$$
\begin{aligned}
V_{ab} &= V_{ab_1} + V_{ab_2} \\
&= 1 + 5 = \mathbf{6V}
\end{aligned}
$$

EXAMPLE 3.6

Use the principle of superposition to solve for v_x in the circuit of Fig. 3.17.

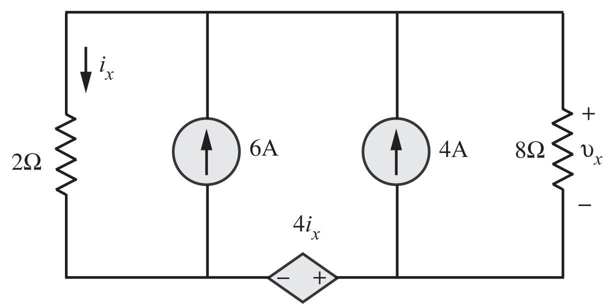


Figure 3.17

SOLUTION

According to the principle of superposition,

$$v_x = v_{x_1} + v_{x_2}$$

where v_{x_1} is produced by 6A source alone in the circuit and v_{x_2} is produced solely by 4A current source.

To find v_{x_1}, deactivate the 4A current source. This results in a circuit diagram as shown in Fig. 3.18.

KCL at node x_1 :

$$\frac{v_{x_1}}{2} + \frac{v_{x_1} - 4i_{x_1}}{8} = 6$$

But $$i_{x_1} = \frac{v_{x_1}}{2}$$

Hence, $$\frac{v_{x_1}}{2} + \frac{v_{x_1} - 4\frac{v_{x_1}}{2}}{8} = 6$$

$$\Rightarrow \quad \frac{v_{x_1}}{2} + \frac{v_{x_1} - 2v_{x_1}}{8} = 6$$

$$\Rightarrow \quad 4v_{x_1} + v_{x_1} - 2v_{x_1} = 48$$

$$\Rightarrow \quad v_{x_1} = \frac{48}{3} = 16\text{V}$$

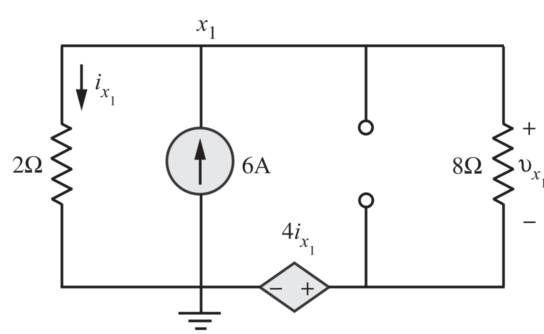


Figure 3.18

To find v_{x2}, deactivate the 6A current source, resulting in a circuit diagram as shown in Fig. 3.19.

KCL at node x_2 :

$$\frac{v_{x2}}{8}+\frac{v_{x2}-(-4i_{x2})}{2}=4$$

$$\Rightarrow \qquad \frac{v_{x2}}{8}+\frac{v_{x2}+4i_{x2}}{2}=4 \tag{3.3}$$

Applying KVL along dotted path, we get

$$v_{x2}+4i_{x2}-2i_{x2}=0$$

$$\Rightarrow \qquad v_{x2}=-2i_{x2} \quad \text{or} \quad i_{x2}=\frac{-v_{x2}}{2} \tag{3.4}$$

Substituting equation (3.4) in equation (3.3), we get

$$\frac{v_{x2}}{8}+\frac{v_{x2}+4\left(\frac{-v_{x2}}{2}\right)}{2}=4$$

$$\Rightarrow \qquad \frac{v_{x2}}{8}+\frac{v_{x2}-2v_{x2}}{2}=4$$

$$\Rightarrow \qquad \frac{v_{x2}}{8}-\frac{v_{x2}}{2}=4$$

$$\Rightarrow \qquad v_{x2}-4v_{x2}=32$$

$$\Rightarrow \qquad v_{x2}=-\frac{32}{3}\text{V}$$

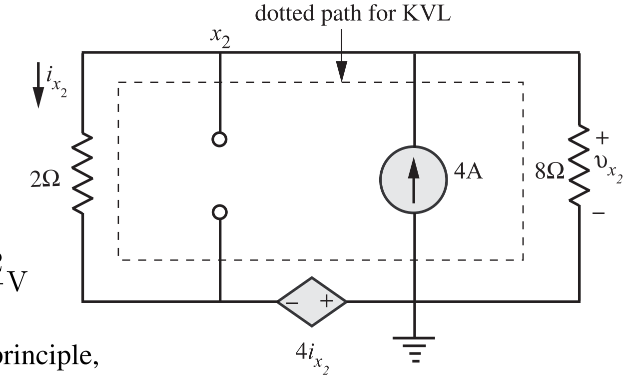


Figure 3.19

Hence, according to the superposition principle,

$$v_x=v_{x_1}+v_{x2}$$
$$=16-\frac{32}{2}=\mathbf{5.33V}$$

EXAMPLE 3.7

Which of the source in Fig. 3.20 contributes most of the power dissipated in the 2 Ω resistor ? The least ? What is the power dissipated in 2 Ω resistor ?

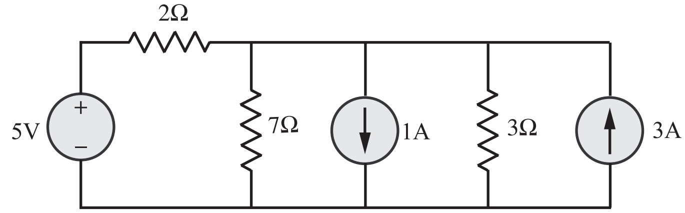


Figure 3.20

SOLUTION

The Superposition theorem cannot be used to identify the individual contribution of each source to the power dissipated in the resistor. However, the superposition theorem can be used to find the total power dissipated in the 2 Ω resistor.

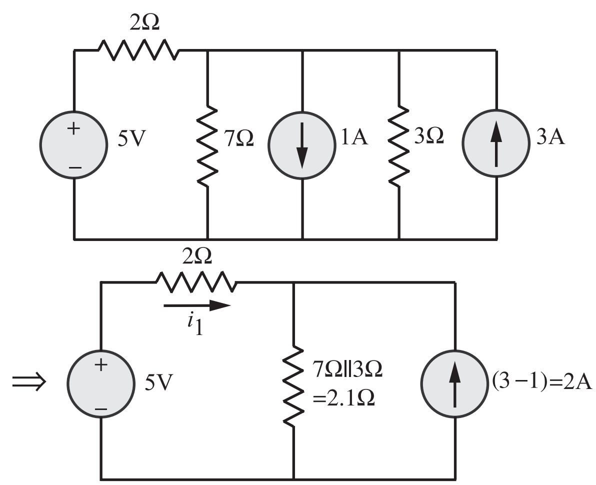


Figure 3.21

According to the superposition principle,

$$i_1 = i_1' + i_2'$$

where i_1' = Contribution to i_1 from 5V source alone.
and i_2' = Contribution to i_1 from 2A source alone.

Let us first find i_1'. This needs the deactivation of 2A source. Refer to Fig. 3.22.

$$i_1' = \frac{5}{2 + 2.1} = 1.22\text{A}$$

Similarly to find i_2' we have to disable the 5V source by shorting it.

Referring to Fig. 3.23, we find that

$$i_2' = \frac{-2 \times 2.1}{2 + 2.1} = -1.024 \text{ A}$$

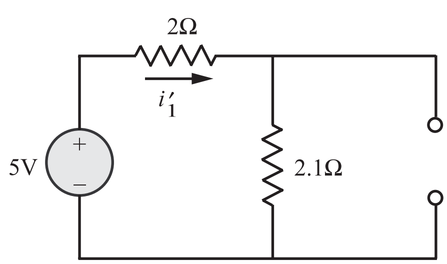


Figure 3.22

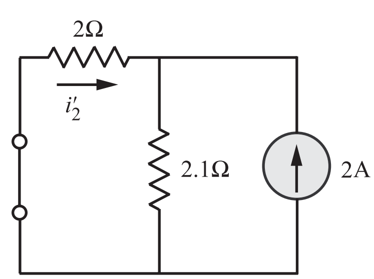


Figure 3.23

Total current,

$$\begin{aligned} i_1 &= i_1' + i_2' \\ &= 1.22 - 1.024 \\ &= 0.196 \text{ A} \end{aligned}$$

Thus,
$$\begin{aligned} P_{2\Omega} &= (0.196)^2 \times 2 \\ &= 0.0768 \text{ Watts} \\ &= \mathbf{76.8} \text{ mW} \end{aligned}$$

EXAMPLE 3.8

Find the voltage V_1 using the superposition principle. Refer the circuit shown in Fig.3.24.

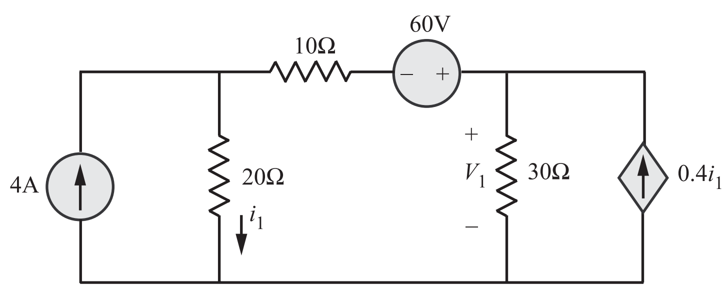


Figure 3.24

SOLUTION

According to the superposition principle,

$$V_1 = V_1' + V_1''$$

where V_1' is the contribution from 60V source alone and V_1'' is the contribution from 4A current source alone.

To find V_1', the 4A current source is opened, resulting in a circuit as shown in Fig. 3.25.

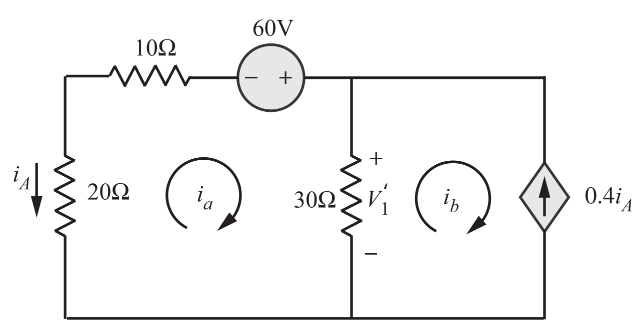


Figure 3.25

Applying KVL to the left mesh:

$$30i_a - 60 + 30\,(i_a - i_b) = 0 \tag{3.5}$$

Also

$$\begin{aligned} i_b &= -0.4 i_A \\ &= -0.4\,(-i_a) = 0.4 i_a \end{aligned} \tag{3.6}$$

Substituting equation (3.6) in equation (3.5), we get

$$30i_a - 60 + 30i_a - 30 \times 0.4i_a = 0$$

$$\Rightarrow \qquad i_a = \frac{60}{48} = 1.25\text{A}$$

$$\begin{aligned} i_b &= 0.4i_a = 0.4 \times 1.25 \\ &= 0.5\text{A} \end{aligned}$$

Hence,

$$\begin{aligned} V_1' &= (i_a - i_b) \times 30 \\ &= 22.5 \text{ V} \end{aligned}$$

To find, V_1'', the 60V source is shorted as shown in Fig. 3.26.

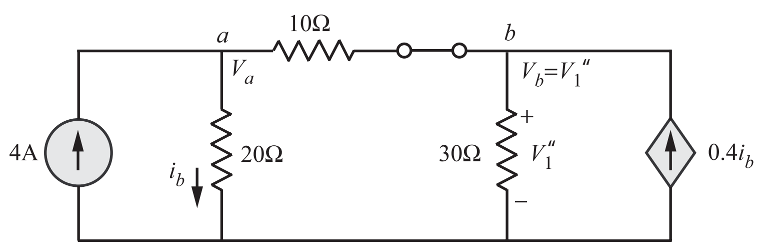


Figure 3.26

Applying KCL at node a:

$$\frac{V_a}{20} + \frac{V_a - V_1''}{10} = 4$$

$$\Rightarrow \qquad 30V_a - 20V_1'' = 800 \tag{3.7}$$

Applying KCL at node b:

$$\frac{V_1''}{30} + \frac{V_1'' - V_a}{10} = 0.4i_b$$

Also,

$$V_a = 20i_a \quad \Rightarrow \quad i_b = \frac{V_a}{20}$$

Hence,

$$\frac{V_1''}{30} + \frac{V_1'' - V_a}{10} = \frac{0.4V_a}{20}$$

$$\Rightarrow \qquad -7.2V_a + 8V_1'' = 0 \tag{3.8}$$

Solving the equations (3.7) and (3.8), we find that

$$V_1'' = 60\text{V}$$

Hence
$$\begin{aligned} V_1 &= V_1' + V_1'' \\ &= 22.5 + 60 = \mathbf{82.5V} \end{aligned}$$

EXAMPLE 3.9

(a) Refer to the circuit shown in Fig. 3.27. Before the 10 mA current source is attached to terminals $x - y$, the current i_a is found to be 1.5 mA. Use the superposition theorem to find the value of i_a after the current source is connected.

(b) Verify your solution by finding i_a, when all the three sources are acting simultaneously.

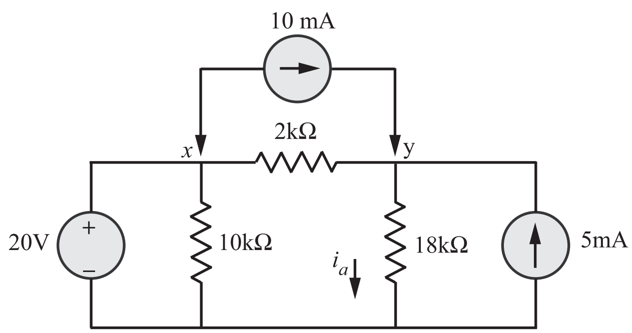


Figure 3.27

SOLUTION

According to the principle of superposition,

$$i_a = i_{a_1} + i_{a_2} + i_{a_3}$$

where i_{a_1}, i_{a_2} and i_{a_3} are the contributions to i_a from 20V source, 5 mA source and 10 mA source respectively.

As per the statement of the problem,

$$i_{a_1} + i_{a_2} = 1.5 \text{ mA}$$

To find i_{a_3}, deactivate 20V source and the 5 mA source. The resulting circuit diagram is shown in Fig 3.28.

$$i_{a_3} = \frac{10\text{mA} \times 2\text{k}}{18\text{k} + 2\text{k}} = 1 \text{ mA}$$

Hence, total current

$$\begin{aligned} i_a &= i_{a_1} + i_{a_2} + i_{a_3} \\ &= 1.5 + 1 = 2.5 \textbf{ mA} \end{aligned}$$

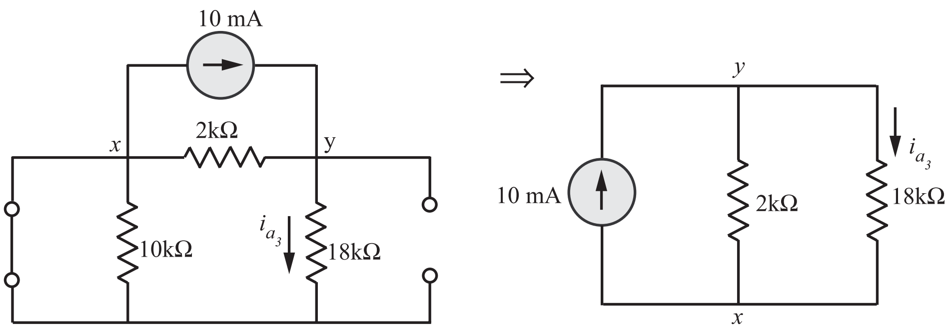


Figure 3.28

(b) Refer to Fig. 3.29

KCL at node y:

$$\frac{V_y}{18\times 10^3}+\frac{V_y-20}{2\times 10^3}=(10+5)\times 10^{-3}$$

Solving, we get $\quad V_y = 45\text{V}.$

$$\text{Hence,}\quad i_a=\frac{V_y}{18\times 10^3}=\frac{45}{18\times 10^3}$$
$$=2.5\text{ mA}$$

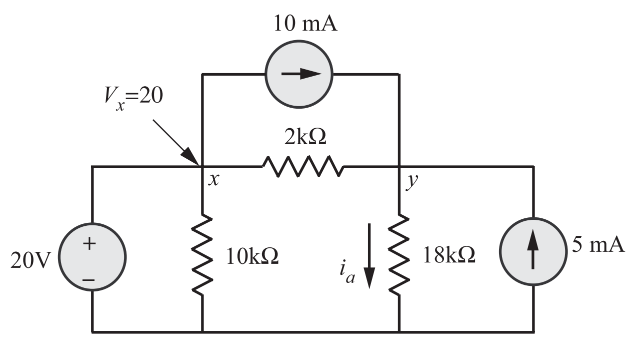


Figure 3.29

3.2 Thevenin's theorem

In section 3.1, we saw that the analysis of a circuit may be greatly reduced by the use of superposition principle. The main objective of Thevenin's theorem is to reduce some portion of a circuit to an equivalent source and a single element. This reduced equivalent circuit connected to the remaining part of the circuit will allow us to find the desired current or voltage. Thevenin's theorem is based on circuit equivalence. A circuit equivalent to another circuit exhibits identical characteristics at identical terminals.

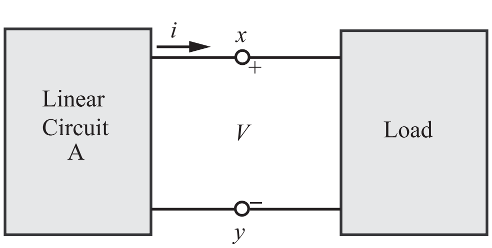


Figure 3.30 A Linear two terminal network

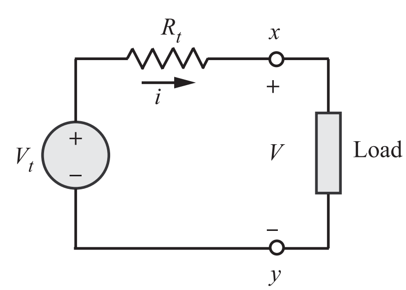


Figure 3.31 The Thevenin's equivalent circuit

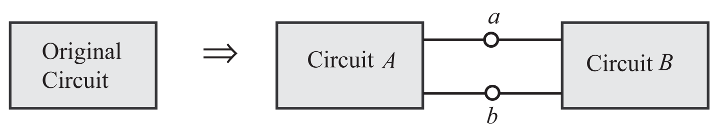
According to Thevenin's theorem, the linear circuit of Fig. 3.30 can be replaced by the one shown in Fig. 3.31 (The load resistor may be a single resistor or another circuit). The circuit to the left of the terminals $x-y$ in Fig. 3.31 is known as the Thevenin's equivalent circuit.

The Thevenin's theorem may be stated as follows:

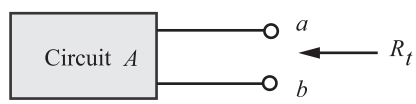
A linear two–terminal circuit can be replaced by an equivalent circuit consisting of a voltage source V_t in series with a resistor R_t, Where V_t is the open–circuit voltage at the terminals and R_t is the input or equivalent resistance at the terminals when the independent sources are turned off or R_t is the ratio of open–circuit voltage to the short–circuit current at the terminal pair.

Action plan for using Thevenin's theorem :

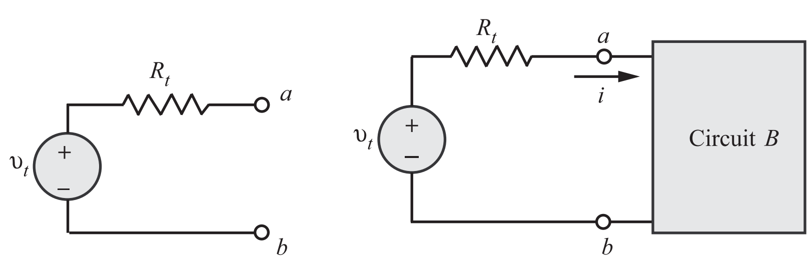
1. Divide the original circuit into circuit A and circuit B.



In general, circuit B is the load which may be linear or non-linear. Circuit A is the balance of the original network exclusive of load and must be linear. In general, circuit A may contain independent sources, dependent sources and resistors or other linear elements.



2. Separate the circuit A from circuit B.
3. Replace circuit A with its Thevenin's equivalent.
4. Reconnect circuit B and determine the variable of interest (e.g. current 'i' or voltage 'v').



Procedure for finding R_t:

Three different types of circuits may be encountered in determining the resistance, R_t :

(i) If the circuit contains only independent sources and resistors, deactivate the sources and find R_t by circuit reduction technique. Independent current sources, are deactivated by opening them while independent voltage sources are deactivated by shorting them.

(ii) If the circuit contains resistors, dependent and independent sources, follow the instructions described below:

(a) Determine the open circuit voltage v_{oc} with the sources activated.

(b) Find the short circuit current i_{sc} when a short circuit is applied to the terminals $a - b$

(c) $R_t = \dfrac{v_{oc}}{i_{sc}}$

(iii) If the circuit contains resistors and only dependent sources, then

(a) $v_{oc} = 0$ (since there is no energy source)

(b) Connect 1A current source to terminals $a - b$ and determine v_{ab}.

(c) $R_t = \dfrac{v_{ab}}{1}$

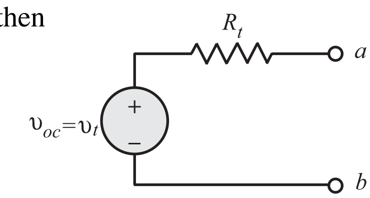


Figure 3.32

For all the cases discussed above, the Thevenin's equivalent circuit is as shown in Fig. 3.32.

EXAMPLE 3.10

Using the Thevenin's theorem, find the current i through $R = 2\ \Omega$. Refer Fig. 3.33.

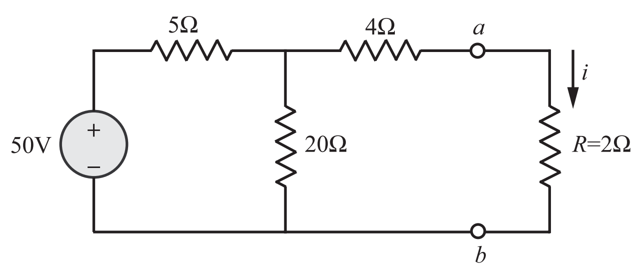


Figure 3.33

SOLUTION

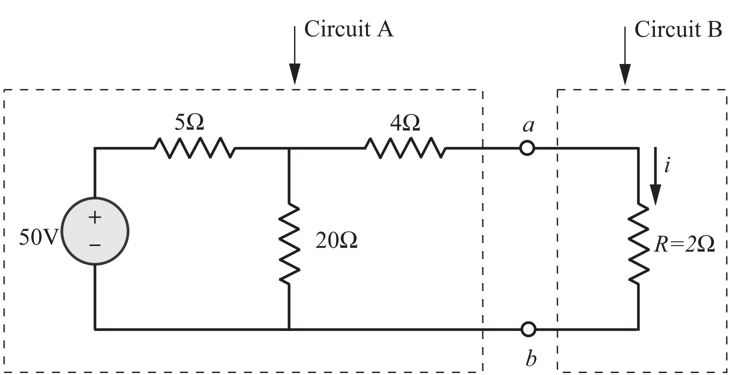


Figure 3.34

Since we are interested in the current i through R, the resistor R is identified as circuit B and the remainder as circuit A. After removing the circuit B, circuit A is as shown in Fig. 3.35.

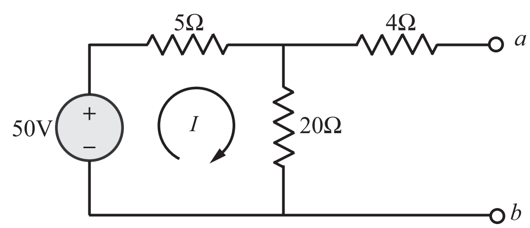


Figure 3.35

To find R_t, we have to deactivate the independent voltage source. Accordingly, we get the circuit in Fig. 3.36.

$$R_t = (5\ \Omega || 20\ \Omega) + 4\ \Omega$$
$$= \frac{5 \times 20}{5 + 20} + 4 = 8\ \Omega$$

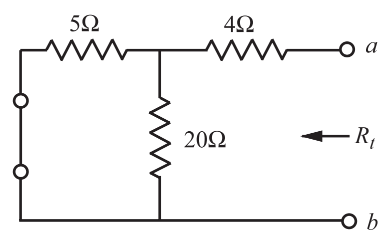


Figure 3.36

Referring to Fig. 3.35,

$$-50 + 25I = 0 \quad \Rightarrow \quad I = 2\text{A}$$

Hence $$V_{ab} = V_{oc} = 20(I) = 40\text{V}$$

Thus, we get the Thevenin's equivalent circuit which is as shown in Fig.3.37.

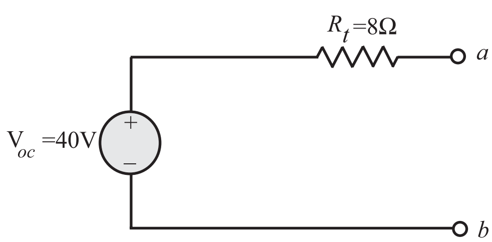


Figure 3.37

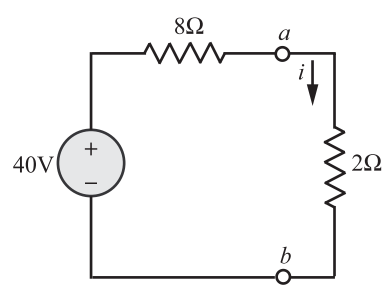


Figure 3.38

Reconnecting the circuit B to the Thevenin's equivalent circuit as shown in Fig. 3.38, we get

$$i = \frac{40}{2 + 8} = \mathbf{4A}$$

EXAMPLE 3.11

(a) Find the Thevenin's equivalent circuit with respect to terminals $a - b$ for the circuit shown in Fig. 3.39 by finding the open-circuit voltage and the short–circuit current.

(b) Solve the Thevenin resistance by removing the independent sources. Compare your result with the Thevenin resistance found in part (a).

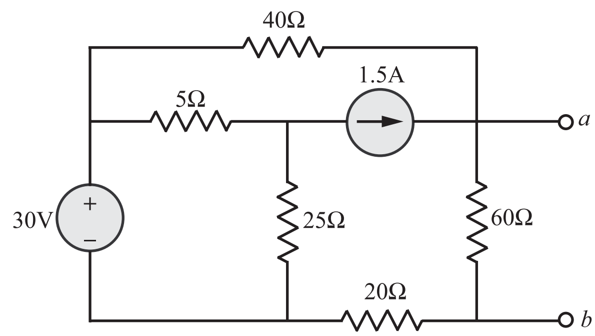
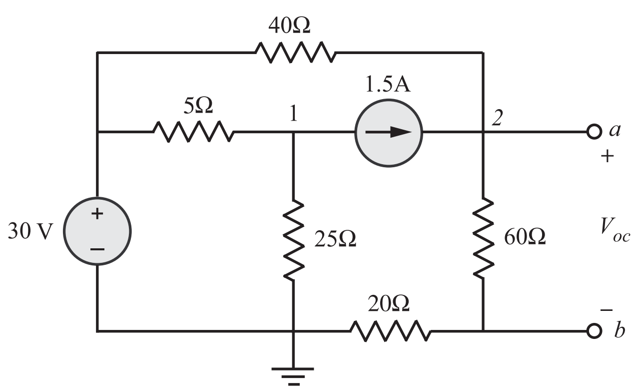


Figure 3.39

SOLUTION



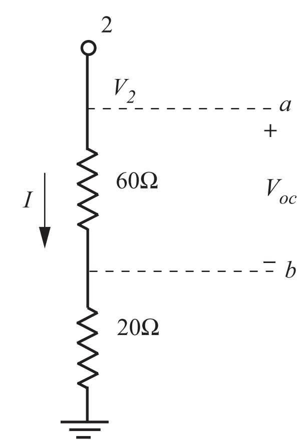


Figure 3.40

(a) *To find* V_{oc} *:*

Apply KCL at node 2 :

$$\frac{V_2}{60+20} + \frac{V_2 - 30}{40} - 1.5 = 0$$

$$\Rightarrow \quad V_2 = 60 \text{ Volts}$$

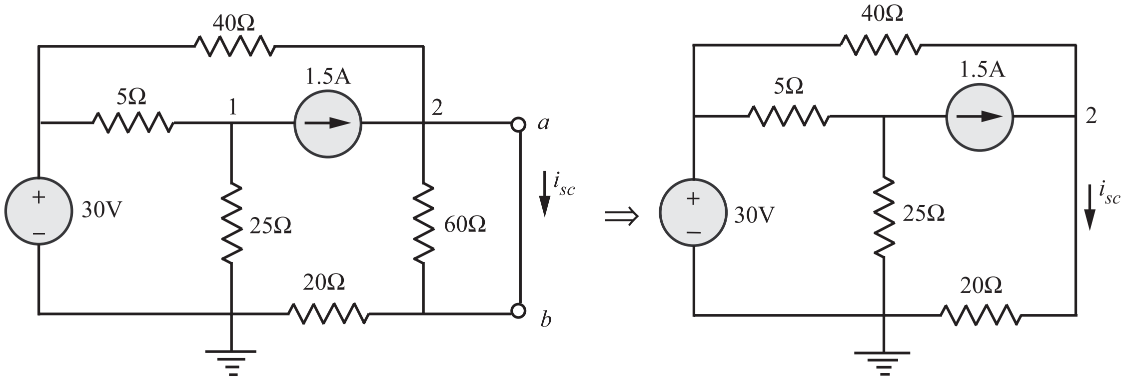
Hence,

$$V_{oc} = I \times 60$$

$$= \left[\frac{V_2 - 0}{60+20}\right] \times 60$$

$$= 60 \times \frac{60}{80} = 45 \text{ V}$$

To find i_{sc} :



Applying KCL at node 2:

$$\frac{V_2}{20} + \frac{V_2 - 30}{40} - 1.5 = 0$$

$$\Rightarrow \qquad V_2 = 30\text{V}$$

$$i_{sc} = \frac{V_2}{20} = 1.5\text{A}$$

Therefore,

$$R_t = \frac{V_{oc}}{i_{sc}} = \frac{45}{1.5}$$
$$= 30\ \Omega$$

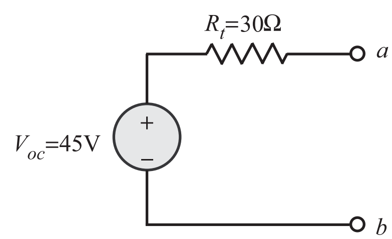
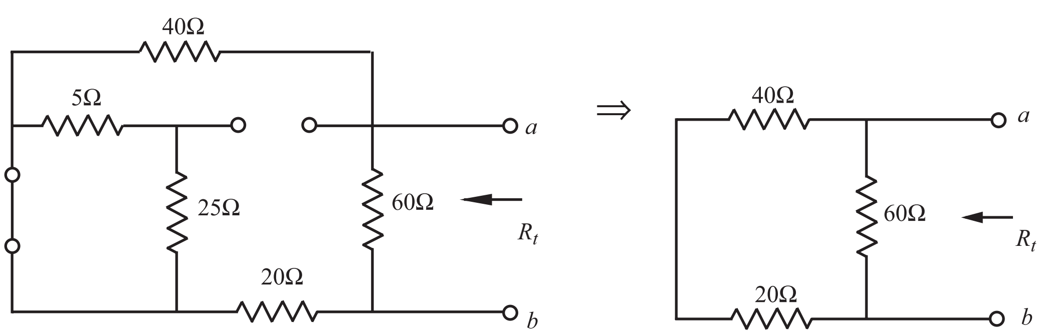


Figure 3.40 (a)

The Thevenin equivalent circuit with respect to the terminals $a - b$ is as shown in Fig. 3.40(a).
(b) Let us now find Thevenin resistance R_t by deactivating all the independent sources,



$$R_t = 60\ \Omega || (40 + 20)\ \Omega$$
$$= \frac{60}{2} = 30\ \Omega\ (\text{verified})$$

It is seen that, if only independent sources are present, it is easy to find R_t by deactivating all the independent sources.

EXAMPLE 3.12

Find the Thevenin equivalent for the circuit shown in Fig. 3.41 with respect to terminals $a - b$.

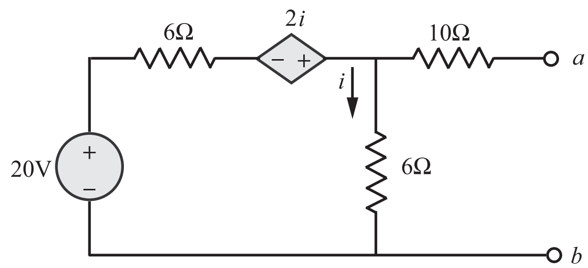


Figure 3.41

SOLUTION

To find $V_{oc} = V_{ab}$:

Applying KVL around the mesh of Fig. 3.42, we get

$$-20 + 6i - 2i + 6i = 0$$

$$\Rightarrow \qquad i = 2\text{A}$$

Since there is no current flowing in 10 Ω resistor, $V_{oc} = 6i = 12$ V

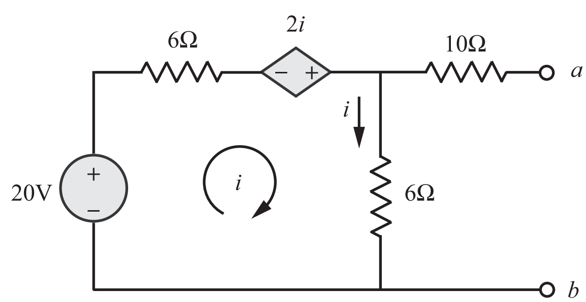


Figure 3.42

To find R_t*:* (Refer Fig. 3.43)

Since both dependent and independent sources are present, Thevenin resistance is found using the relation,

$$R_t = \frac{v_{oc}}{i_{sc}}$$

Applying KVL clockwise for mesh 1 :

$$-20 + 6i_1 - 2i + 6(i_1 - i_2) = 0$$

$$\Rightarrow \qquad 12i_1 - 6i_2 = 20 + 2i$$

Since $i = i_1 - i_2$, we get

$$12i_1 - 6i_2 = 20 + 2(i_1 - i_2)$$

$$\Rightarrow \qquad 10i_1 - 4i_2 = 20$$

Applying KVL clockwise for mesh 2 :

$$10i_2 + 6(i_2 - i_1) = 0$$

$$\Rightarrow \qquad -6i_1 + 16i_2 = 0$$

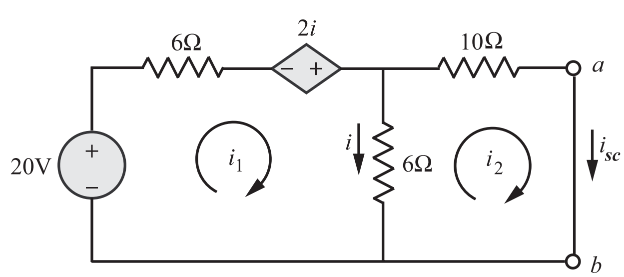


Figure 3.43

Solving the above two mesh equations, we get

$$i_2 = \frac{120}{136}\text{A} \quad \Rightarrow \quad i_{sc} = i_2 = \frac{120}{136}\text{A}$$
$$R_t = \frac{v_{oc}}{i_{sc}} = \frac{12}{\frac{120}{136}} = 13.6\ \Omega$$

EXAMPLE 3.13

Find V_o in the circuit of Fig. 3.44 using Thevenin's theorem.

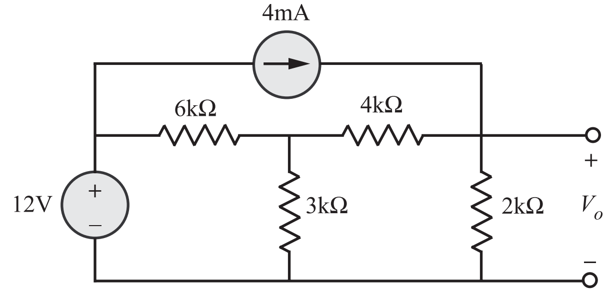


Figure 3.44

SOLUTION

To find V_{oc} :

Since we are interested in the voltage across 2 kΩ resistor, it is removed from the circuit of Fig. 3.44 and so the circuit becomes as shown in Fig. 3.45.

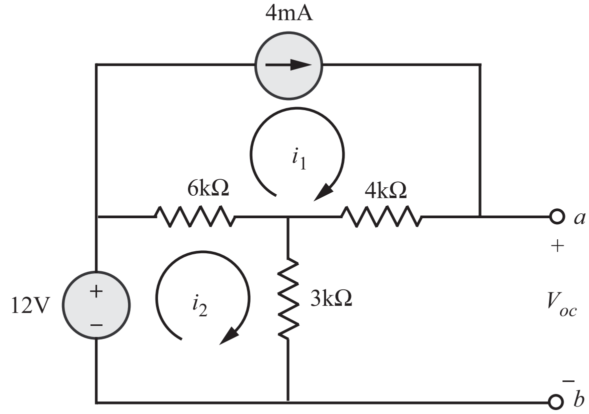


Figure 3.45

By inspection, $\quad i_1 = 4$ mA

Applying KVL to mesh 2 :

$$-12 + 6 \times 10^3 (i_2 - i_1) + 3 \times 10^3 i_2 = 0$$
$$\Rightarrow \quad -12 + 6 \times 10^3 \left(i_2 - 4 \times 10^{-3}\right) + 3 \times 10^3 i_2 = 0$$

Solving, we get $\qquad i_2 = 4 \text{ mA}$

Applying KVL to the path $4 \text{ k}\Omega \rightarrow$ a–b $\rightarrow 3 \text{ k}\Omega$, we get

$$-4 \times 10^3 i_1 + V_{oc} - 3 \times 10^3 i_2 = 0$$

$$\Rightarrow \qquad V_{oc} = 4 \times 10^3 i_1 + 3 \times 10^3 i_2$$

$$= 4 \times 10^3 \times 4 \times 10^{-3} + 3 \times 10^3 \times 4 \times 10^{-3} = 28\text{V}$$

To find R_t :

Deactivating all the independent sources, we get the circuit diagram shown in Fig. 3.46.

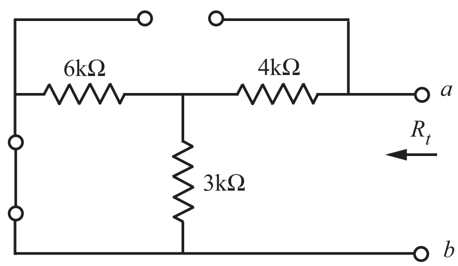


Figure 3.46

$$R_t = R_{ab} = 4 \text{ k}\Omega + (6 \text{ k}\Omega || 3 \text{ k}\Omega) = 6 \text{ k}\Omega$$

Hence, the Thevenin equivalent circuit is as shown in Fig. 3.47.

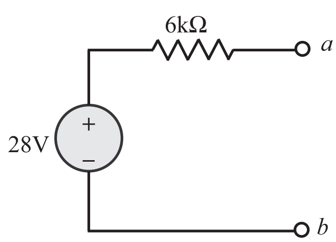


Figure 3.47

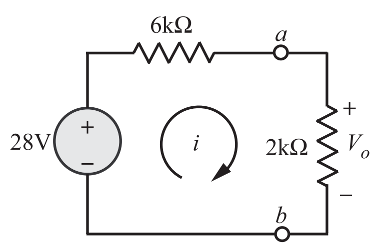


Figure 3.48

If we connect the $2 \text{ k}\Omega$ resistor to this equivalent network, we obtain the circuit of Fig. 3.48.

$$V_o = i\left(2 \times 10^3\right)$$

$$= \frac{28}{(6+2) \times 10^3} \times 2 \times 10^3 = 7\text{V}$$

EXAMPLE 3.14

The wheatstone bridge in the circuit shown in Fig. 3.49 (a) is balanced when $R_2 = 1200 \ \Omega$. If the galvanometer has a resistance of $30 \ \Omega$, how much current will be detected by it when the bridge is unbalanced by setting R_2 to $1204 \ \Omega$?

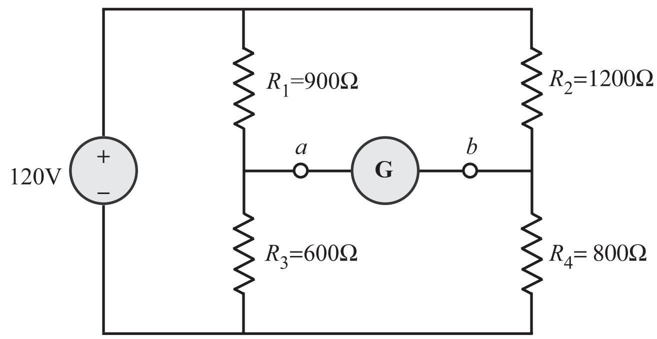


Figure 3.49(a)

SOLUTION

To find V_{oc} :

We are interested in the galavanometer current. Hence, it is removed from the circuit of Fig. 3.49 (a) to find V_{oc} and we get the circuit shown in Fig. 3.49 (b).

$$i_1 = \frac{120}{900+600} = \frac{120}{1500}\text{A}$$

$$i_2 = \frac{120}{1204+800} = \frac{120}{2004}\text{A}$$

Applying KVL clockwise along the path $1204\Omega \rightarrow\ b - a \rightarrow 900\ \ \Omega$, we get

$$1204 i_2 - V_t - 900 i_1 = 0$$

$$\Rightarrow \quad V_t = 1204 i_2 - 900 i_1$$

$$= 1204 \times \frac{120}{2004} - 900 \times \frac{120}{1500}$$

$$= 95.8\ \text{mV}$$

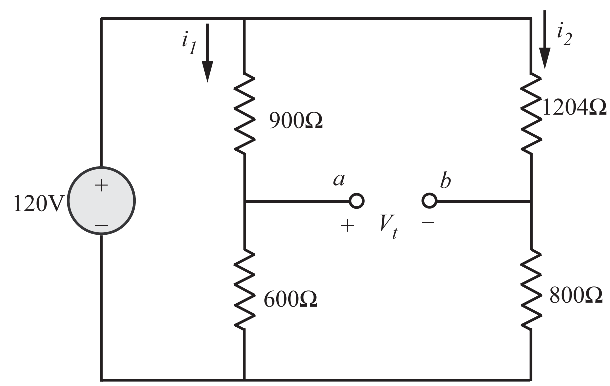


Figure 3.49(b)

To find R_t :

Deactivate all the independent sources and look into the terminals $a - b$ to determine the Thevenin's resistance.

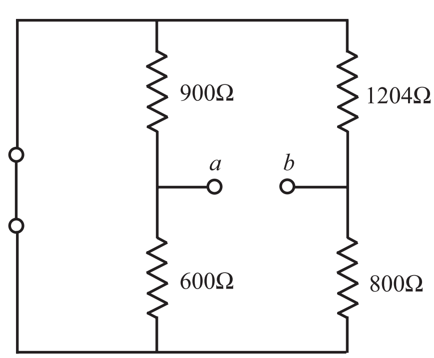


Figure 3.49(c)

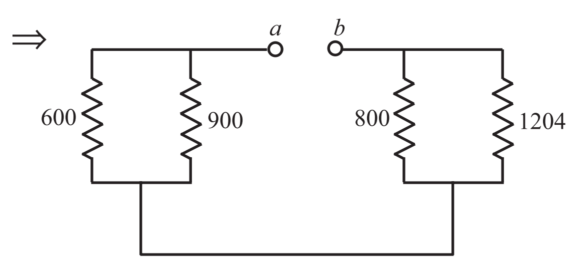


Figure 3.49(d)

$$R_t = R_{ab} = 600||900 + 800||1204$$
$$= \frac{900 \times 600}{1500} + \frac{1204 \times 800}{2004}$$
$$= 840.64\ \Omega$$

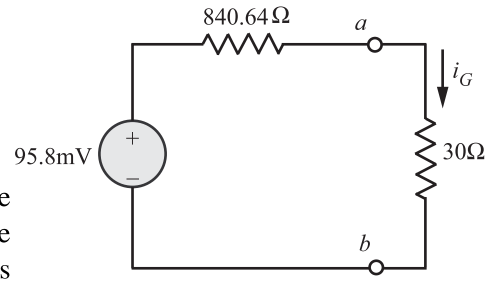


Figure 3.50

Hence, the Thevenin equivalent circuit consists of the 95.8 mV source in series with 840.64Ω resistor. If we connect 30Ω resistor (galvanometer resistance) to this equivalent network, we obtain the circuit in Fig. 3.50.

$$i_G = \frac{95.8 \times 10^{-3}}{840.64 + 30\ \Omega} \quad = 110.03\ \mu\text{A}$$

EXAMPLE 3.15

For the circuit shown in Fig. 3.51, find the Thevenin's equivalent circuit between terminals a and b.

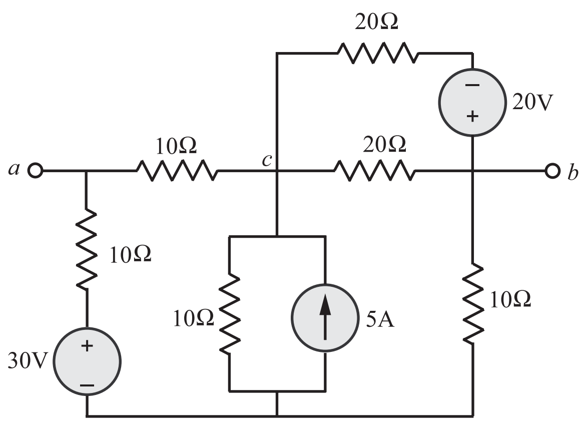


Figure 3.51

SOLUTION

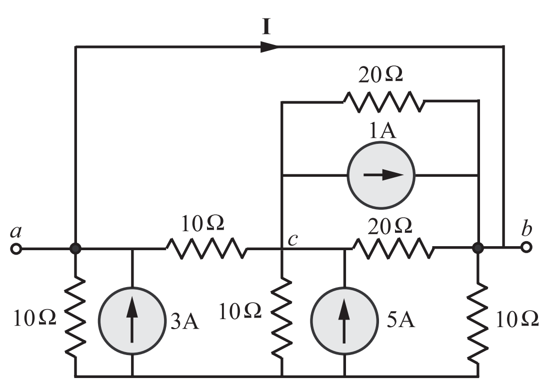


Figure 3.52

With ab shorted, let $I_{sc} = I$. The circuit after transforming voltage sources into their equivalent current sources is as shown in Fig 3.52. Writing node equations for this circuit,

At a : $\quad 0.2V_a - 0.1\ V_c + I = 3$

At c : $\quad -0.1V_a + 0.3\ V_c - 0.1\ V_b = 4$

At b : $\quad -0.1V_c + 0.2\ V_b - I = 1$

As the terminals a and b are shorted $V_a = V_b$ and the above equations become

$$0.2V_a - 0.1\,V_c + I = 3$$
$$-0.2V_a + 0.3\,V_c = 4$$
$$0.2V_a - 0.1\,V_c - 1 = 1$$

Solving the above equations, we get the short circuit current, $I = I_{sc} = 1$ A.

Next let us open circuit the terminals a and b and this makes $I = 0$. And the node equations written earlier are modified to

$$0.2V_a - 0.1\,V_c = 3$$
$$-\,0.1V_a + 0.3\,V_c - 0.1\,V_b = 4$$
$$-0.1V_c + 0.2\,V_b = 1$$

Solving the above equations, we get

$$V_a = 30\text{V and } V_b = 20\text{V}$$

Hence, $V_{ab} = 30 - 20 = 10$ V $= V_{oc} = V_t$

Therefore $R_t = \dfrac{V_{oc}}{I_{sc}} = \dfrac{10}{1} = 10\Omega$

The Thevenin's equivalent is as shown in Fig 3.53

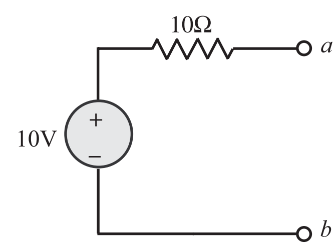


Figure 3.53

EXAMPLE 3.16

Refer to the circuit shown in Fig. 3.54. Find the Thevenin equivalent circuit at the terminals $a - b$.

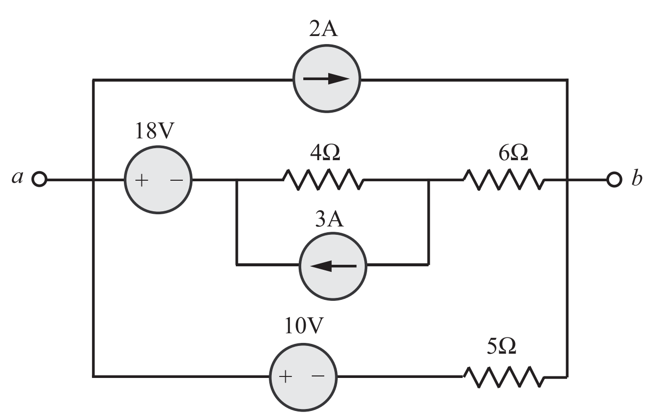


Figure 3.54

SOLUTION

To begin with let us transform 3 A current source and 10 V voltage source. This results in a network as shown in Fig. 3.55 (a) and further reduced to Fig. 3.55 (b).

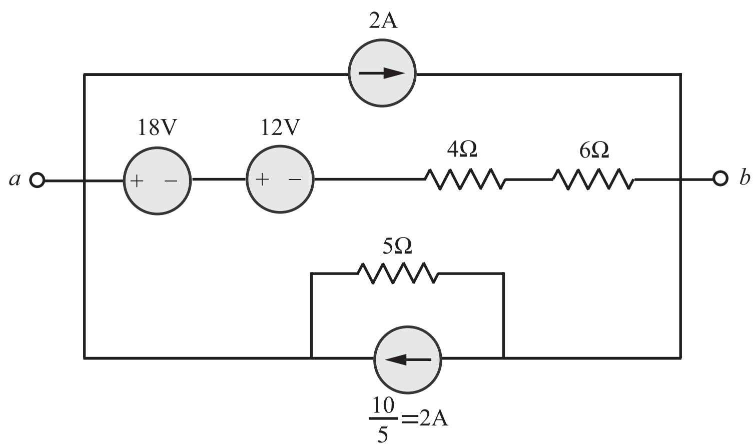


Figure 3.55(a)

Again transform the 30 V source and following the reduction procedure step by step from Fig. 3.55 (b) to 3.55 (d), we get the Thevenin's equivalent circuit as shown in Fig. 3.56.

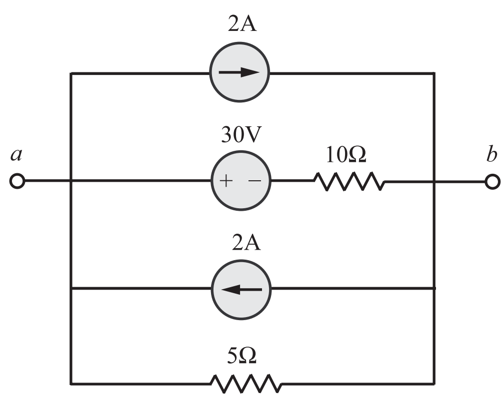


Figure 3.55(b)

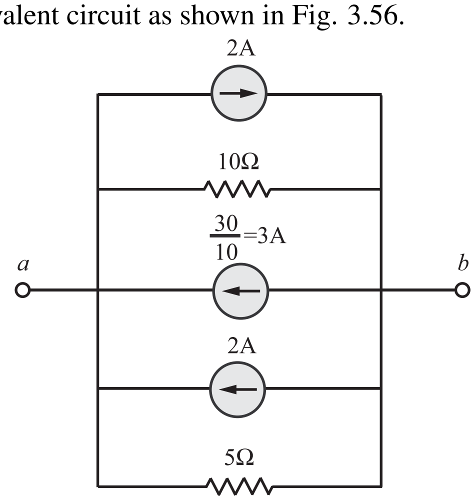


Figure 3.55(c)

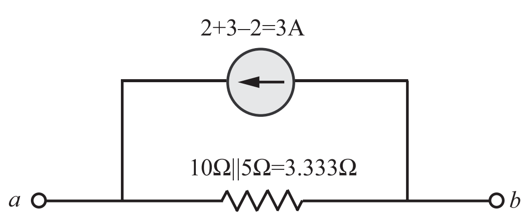


Figure 3.55(d)

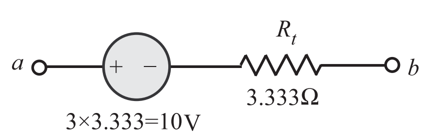


Figure 3.56 Thevenin equivalent circuit

EXAMPLE 3.17

Find the Thevenin equivalent circuit as seen from the terminals $a - b$. Refer the circuit diagram shown in Fig. 3.57.

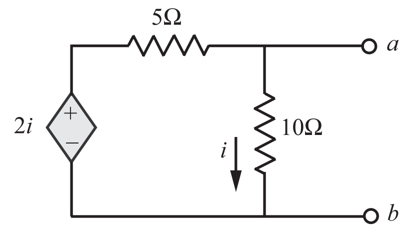


Figure 3.57

SOLUTION

Since the circuit has no independent sources, $i = 0$ when the terminals $a - b$ are open. Therefore, $V_{oc} = 0$.

The onus is now to find R_t. Since $V_{oc} = 0$ and $i_{sc} = 0$, R_t cannot be determined from $R_t = \dfrac{V_{oc}}{i_{sc}}$. Hence, we choose to connect a source of 1 A at the terminals $a - b$ as shown in Fig. 3.58. Then, after finding V_{ab}, the Thevenin resistance is,

$$R_t = \frac{V_{ab}}{1}$$

KCL at node a :

$$\frac{V_a - 2i}{5} + \frac{V_a}{10} - 1 = 0$$

Also,

$$i = \frac{V_a}{10}$$

Hence,

$$\frac{V_a - 2\left(\frac{V_a}{10}\right)}{5} + \frac{V_a}{10} - 1 = 0$$

$$\Rightarrow \qquad V_a = \frac{50}{13}\text{V}$$

Hence,

$$R_t = \frac{V_a}{1} = \frac{50}{13}\ \Omega$$

Alternatively one could find R_t by connecting a 1V source at the terminals $a - b$ and then find the current from b to a. Then $R_t = \dfrac{1}{i_{ba}}$. The concept of finding R_t by connecting a 1A source between the terminals $a - b$ may also be used for circuits containing independent sources. Then set all the independent sources to zero and use 1A source at the terminals $a - b$ to find V_{ab} and hence, $R_t = \dfrac{V_{ab}}{1}$.

For the present problem, the Thevenin equivalent circuit as seen between the terminals $a - b$ is shown in Fig. 3.58 (a).

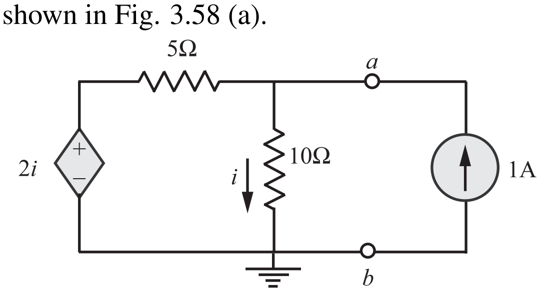


Figure 3.58

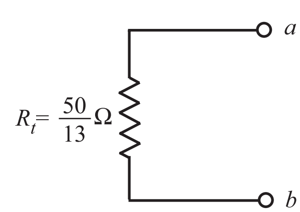


Figure 3.58 (a)

EXAMPLE 3.18

Determine the Thevenin equivalent circuit between the terminals $a - b$ for the circuit of Fig. 3.59.

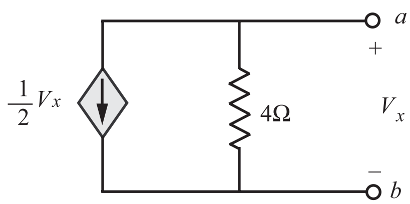


Figure 3.59

SOLUTION

As there are no independent sources in the circuit, we get $V_{oc} = V_t = 0$.

To find R_t, connect a 1V source to the terminals $a - b$ and measure the current I that flows from b to a. (Refer Fig. 3.60 a).

$$R_t = \frac{1}{I}\ \Omega$$

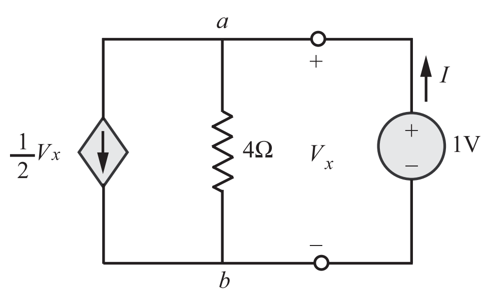


Figure 3.60(a)

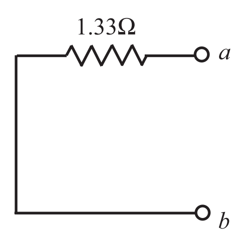


Figure 3.60(b)

Applying KCL at node a:

$$I = 0.5V_x + \frac{V_x}{4}$$

Since, $\quad V_x = 1\text{V}$

we get, $\quad I = 0.5 + \frac{1}{4} = 0.75\ \text{A}$

Hence, $\quad R_t = \frac{1}{0.75} = 1.33\ \Omega$

The Thevenin equivalent circuit is shown in 3.60(b).

Alternatively, sticking to our strategy, let us connect 1A current source between the terminals $a - b$ and then measure V_{ab} (Fig. 3.60 (c)). Consequently, $R_t = \frac{V_{ab}}{1} = V_{ab}\ \Omega$.

Applying KCL at node a:

$$0.5V_x + \frac{V_x}{4} = 1 \Rightarrow V_x = 1.33\text{V}$$

$$\text{Hence} \quad R_t = \frac{V_{ab}}{1} = \frac{V_x}{1} = 1.33\ \Omega$$

The corresponding Thevenin equivalent circuit is same as shown in Fig. 3.60(b)

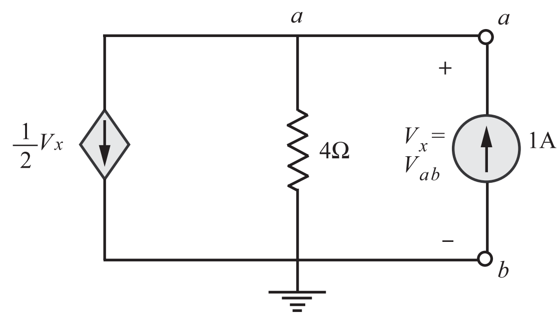


Figure 3.60(c)

3.3 Norton's theorem

An American engineer, E.L. Norton at Bell Telephone Laboratories, proposed a theorem similar to Thevenin's theorem.

Norton's theorem states that a linear two-terminal network can be replaced by an equivalent circuit consisting of a current source i_N in parallel with resistor R_N, where i_N is the short-circuit current through the terminals and R_N is the input or equivalent resistance at the terminals when the independent sources are turned off. If one does not wish to turn off the independent sources, then R_N is the ratio of open circuit voltage to short–circuit current at the terminal pair.

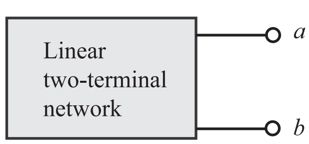


Figure 3.61(a) Original circuit

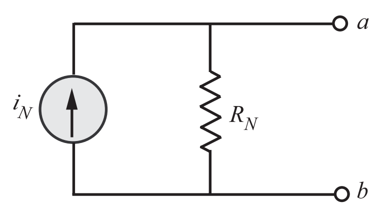


Figure 3.61(b) Norton's equivalent circuit

Figure 3.61(b) shows Norton's equivalent circuit as seen from the terminals $a - b$ of the original circuit shown in Fig. 3.61(a). Since this is the dual of the Thevenin circuit, it is clear that $R_N = R_t$ and $i_N = \frac{v_{oc}}{R_t}$. In fact, source transformation of Thevenin equivalent circuit leads to Norton's equivalent circuit.

Procedure for finding Norton's equivalent circuit:

(1) If the network contains resistors and independent sources, follow the instructions below:

(a) Deactivate the sources and find R_N by circuit reduction techniques.

(b) Find i_N with sources activated.

(2) If the network contains resistors, independent and dependent sources, follow the steps given below:

(a) Determine the short-circuit current i_N with all sources activated.

(b) Find the open-circuit voltage v_{oc}.

(c) $R_t = R_N = \dfrac{v_{oc}}{i_N}$

(3) If the network contains only resistors and dependent sources, follow the procedure described below:

(a) Note that $i_N = 0$.

(b) Connect 1A current source to the terminals $a - b$ and find v_{ab}.

(c) $R_t = \dfrac{v_{ab}}{1}$

Note: Also, since $v_t = v_{oc}$ and $i_N = i_{sc}$

$$R_t = \frac{v_{oc}}{i_{sc}} = R_N$$

The open–circuit and short–circuit test are sufficient to find any Thevenin or Norton equivalent.

3.3.1 PROOF OF THEVENIN'S AND NORTON'S THEOREMS

The principle of superposition is employed to provide the proof of Thevenin's and Norton's theorems.

Derivation of Thevenin's theorem:

Let us consider a linear circuit having two accessible terminals $x - y$ and excited by an external current source i. The linear circuit is made up of resistors, dependent and independent sources. For the sake of simplified analysis, let us assume that the linear circuit contains only two independent voltage sources v_1 and v_2 and two independent current sources i_1 and i_2. The terminal voltage v may be obtained, by applying the principle of superposition. That is, v is made up of contributions due to the external source and independent sources within the linear network.

Hence,
$$v = a_0 i + a_1 v_1 + a_2 v_2 + a_3 i_1 + a_4 i_2 \tag{3.9}$$
$$= a_0 i + b_0 \tag{3.10}$$

where
$$b_0 = a_1 v_1 + a_2 v_2 + a_3 i_1 + a_4 i_2$$
$= $ contribution to the terminal voltage v by independent sources within the linear network.

Let us now evaluate the values of constants a_0 and b_0.

(i) When the terminals x and y are open–circuited, $i = 0$ and $v = v_{oc} = v_t$. Making use of this fact in equation 3.10, we find that $b_0 = v_t$.

(ii) When all the internal sources are deactivated, $b_0 = 0$. This enforces equation 3.10 to become

$$v = a_0 i = R_t i \Rightarrow a_0 = R_t$$

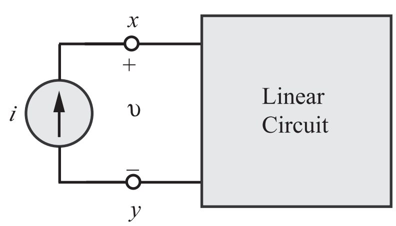


Figure 3.62 Current-driven circuit

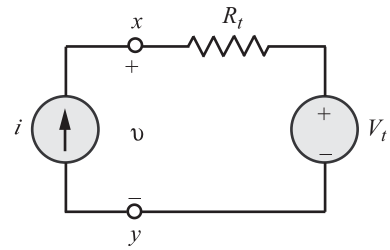


Figure 3.63 Thevenin's equivalent circuit of Fig. 3.62

where R_t is the equivalent resistance of the linear network as viewed from the terminals $x - y$. Also, a_0 must be R_t in order to obey the ohm's law. Substuting the values of a_0 and b_0 in equation 3.10, we find that

$$v = R_t i + v_1$$

which expresses the voltage-current relationship at terminals $x - y$ of the circuit in Fig. 3.63. Thus, the two circuits of Fig. 3.62 and 3.63 are equivalent.

Derivation of Norton's theorem:

Let us now assume that the linear circuit described earlier is driven by a voltage source v as shown in Fig. 3.64.

The current flowing into the circuit can be obtained by superposition as

$$i = c_0 v + d_0 \tag{3.11}$$

where $c_0 v$ is the contribution to i due to the external voltage source v and d_0 contains the contributions to i due to all independent sources within the linear circuit. The constants c_0 and d_0 are determined as follows :

(i) When terminals $x - y$ are short-circuited, $v = 0$ and $i = -i_{sc}$. Hence from equation (3.11), we find that $i = d_0 = -i_{sc}$, where i_{sc} is the short-circuit current flowing out of terminal x, which is same as Norton current i_N

Thus, $$d_0 = -i_N$$

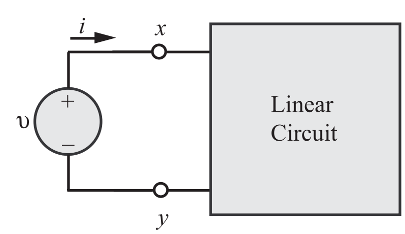


Figure 3.64
Voltage-driven circuit

(ii) Let all the independent sources within the linear network be turned off, that is $d_0 = 0$. Then, equation (3.11) becomes

$$i = c_0 v$$

For dimensional validity, c_0 must have the dimension of conductance. This enforces $c_0 = \frac{1}{R_t}$ where R_t is the equivalent resistance of the linear network as seen from the terminals $x - y$. Thus, equation (3.11) becomes

$$\begin{aligned} i &= \frac{1}{R_t}v - i_{sc} \\ &= \frac{1}{R_t}v - i_N \end{aligned}$$

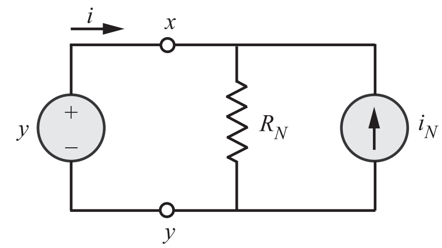


Figure 3.65 Norton's equivalent of voltage driven circuit

This expresses the voltage-current relationship at the terminals $x - y$ of the circuit in Fig. (3.65), validating that the two circuits of Figs. 3.64 and 3.65 are equivalents.

EXAMPLE 3.19

Find the Norton equivalent for the circuit of Fig. 3.66.

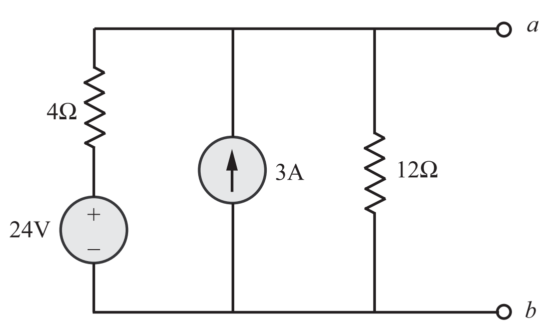


Figure 3.66

SOLUTION

As a first step, short the terminals $a - b$. This results in a circuit diagram as shown in Fig. 3.67. *Applying KCL at node a*, we get

$$\begin{aligned} \frac{0-24}{4} - 3 + i_{sc} = 0 \\ \Rightarrow i_{sc} = 9\text{A} \end{aligned}$$

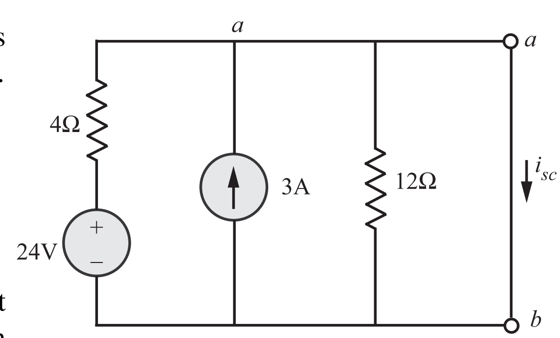


Figure 3.67

To find R_N, deactivate all the independent sources, resulting in a circuit diagram as shown in Fig. 3.68 (a). We find R_N in the same way as R_t in the Thevenin equivalent circuit.

$$R_N = \frac{4 \times 12}{4 + 12} = 3\ \Omega$$

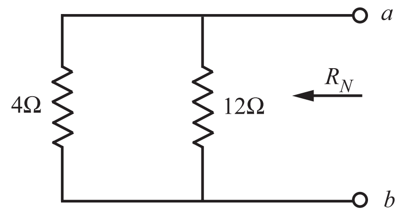


Figure 3.68(a)

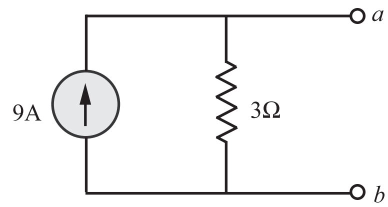


Figure 3.68(b)

Thus, we obtain Nortion equivalent circuit as shown in Fig. 3.68(b).

EXAMPLE 3.20

Refer the circuit shown in Fig. 3.69. Find the value of i_b using Norton equivalent circuit. Take $R = 667\ \Omega$.

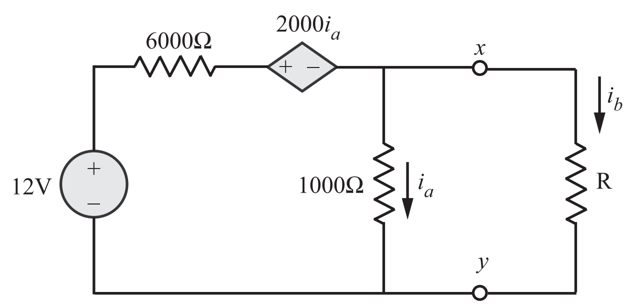


Figure 3.69

SOLUTION

Since we want the current flowing through R, remove R from the circuit of Fig. 3.69. The resulting circuit diagram is shown in Fig. 3.70.

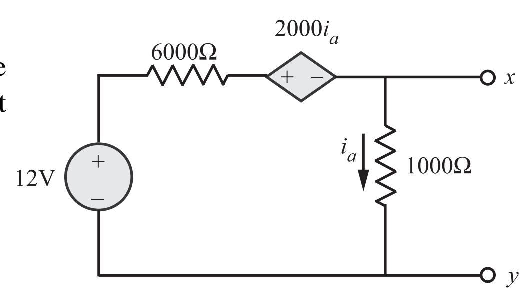


Figure 3.70

To find i_{ac} or i_N referring Fig 3.70(a) :

$$i_a = \frac{0}{1000} = 0\text{A}$$

$$i_{sc} = \frac{12}{6000}\text{A} = 2\text{ mA}$$

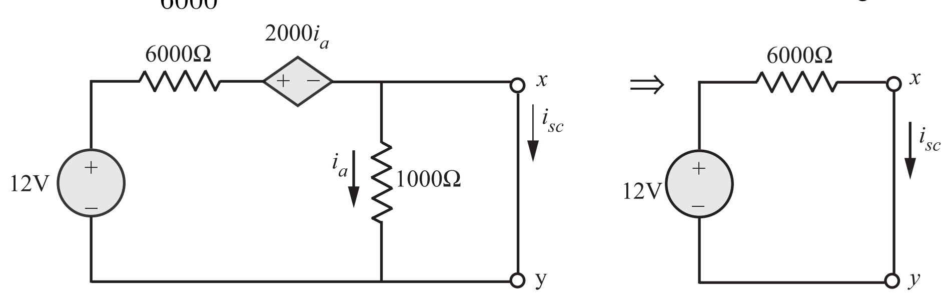


Figure 3.70(a)

To find R_N :
The procedure for finding R_N is same that of R_t in the Thevenin equivalent circuit.

$$R_t = R_N = \frac{v_{oc}}{i_{sc}}$$

To find v_{oc}, make use of the circuit diagram shown in Fig. 3.71. Do not deactivate any source.
Applying KVL clockwise, we get

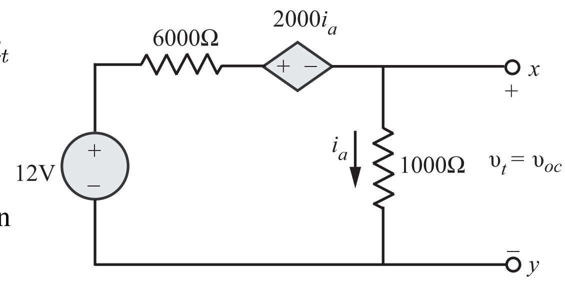


Figure 3.71

$$-12 + 6000i_a + 2000i_a + 1000i_a = 0$$

$$\Rightarrow \qquad i_a = \frac{4}{3000}\text{A}$$

$$\Rightarrow \qquad v_{\text{oc}} = i_a \times 1000 = \frac{4}{3}\text{V}$$

$$\text{Therefore,} \qquad R_N = \frac{v_{oc}}{i_{sc}} = \frac{\frac{4}{3}}{2 \times 10^{-3}} = 667\ \Omega$$

The Norton equivalent circuit along with resistor R is as shown below:

$$i_b = \frac{i_{sc}}{2} = \frac{2\text{mA}}{2} = 1\text{mA}$$

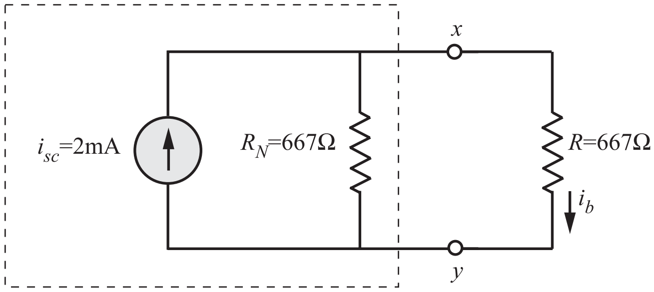


Figure : Norton equivalent circuit with load R

EXAMPLE 3.21

Find I_o in the network of Fig. 3.72 using Norton's theorem.

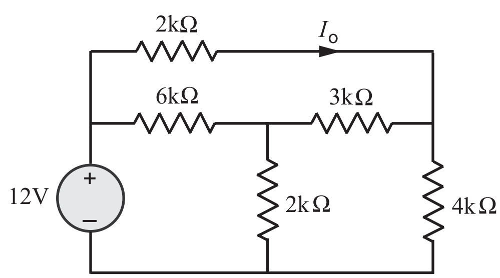


Figure 3.72

SOLUTION

We are interested in I_o, hence the 2 kΩ resistor is removed from the circuit diagram of Fig. 3.72. The resulting circuit diagram is shown in Fig. 3.73(a).

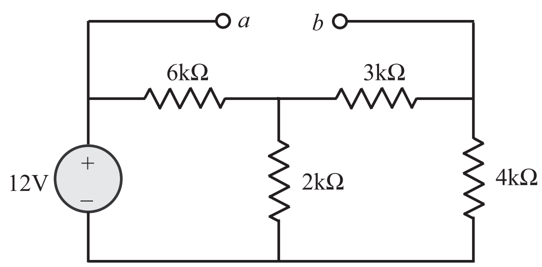


Figure 3.73(a)

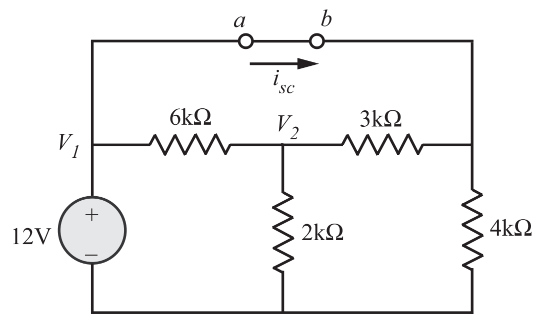


Figure 3.73(b)

To find i_N or i_{sc}:

Refer Fig. 3.73(b). By inspection, $V_1 = 12$ V

Applying KCL at node V_2 :

$$\frac{V_2 - V_1}{6\ \text{k}\Omega} + \frac{V_2}{2\ \text{k}\Omega} + \frac{V_2 - V_1}{3\ \text{k}\Omega} = 0$$

Substituting $V_1 = 12$ V and solving, we get

$$V_2 = 6\text{V}$$

$$i_{sc} = \frac{V_1 - V_2}{3\ \text{k}\Omega} + \frac{V_1}{4\ \text{k}\Omega} = 5\ \text{mA}$$

To find R_N:

Deactivate all the independent sources (refer Fig. 3.73(c)).

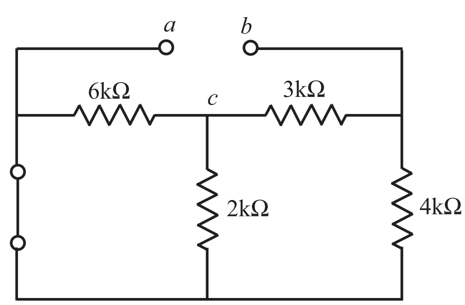


Figure 3.73(c)

⇒

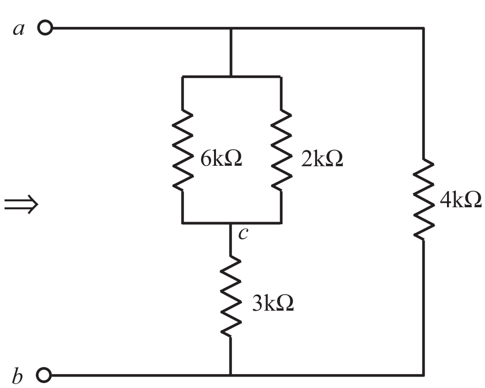


Figure 3.73(d)

Referring to Fig. 3.73 (d), we get

$$R_N = R_{ab} = 4\,\text{k}\Omega||\,[3\,\text{k}\Omega + (6\,\text{k}\Omega||2\,\text{k}\Omega)] = 2.12\,\text{k}\Omega$$

Hence, the Norton equivalent circuit along with 2 kΩ resistor is as shown in Fig. 3.73(e).

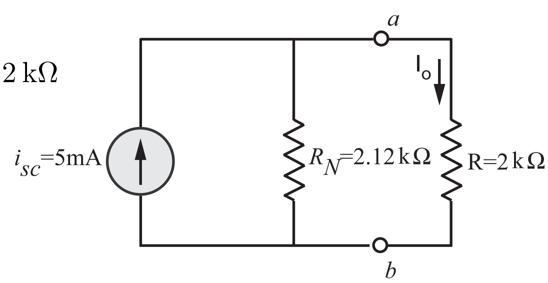


Figure 3.73(e)

$$I_o = \frac{i_{sc} \times R_N}{R + R_N} = 2.57\text{mA}$$

EXAMPLE 3.22

Find V_o in the circuit of Fig. 3. 74.

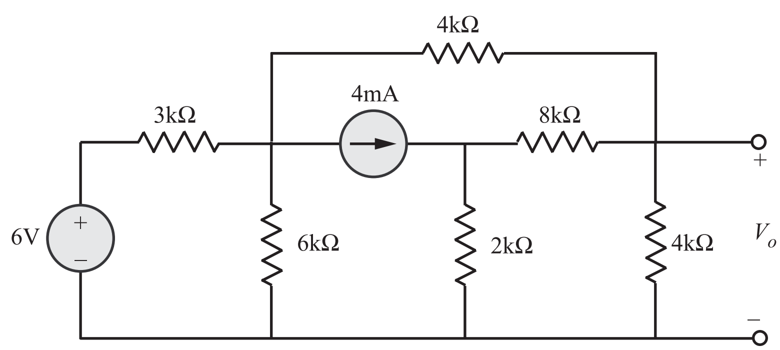


Figure 3.74

SOLUTION

Since we are interested in V_o, the voltage across 4 kΩ resistor, remove this resistance from the circuit. This results in a circuit diagram as shown in Fig. 3.75.

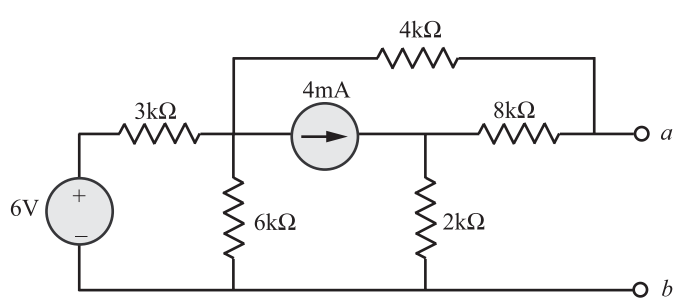
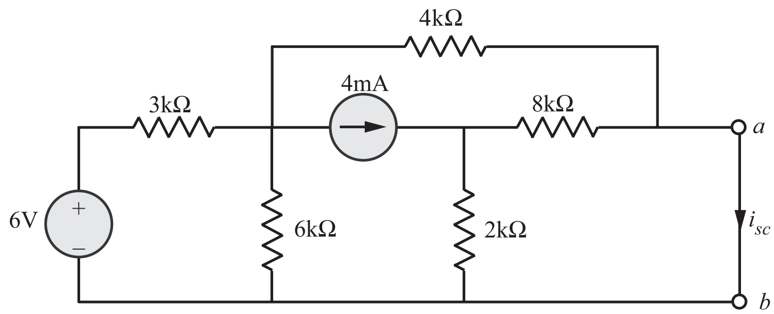


Figure 3.75

To find i_{sc}, short the terminals $a - b$:



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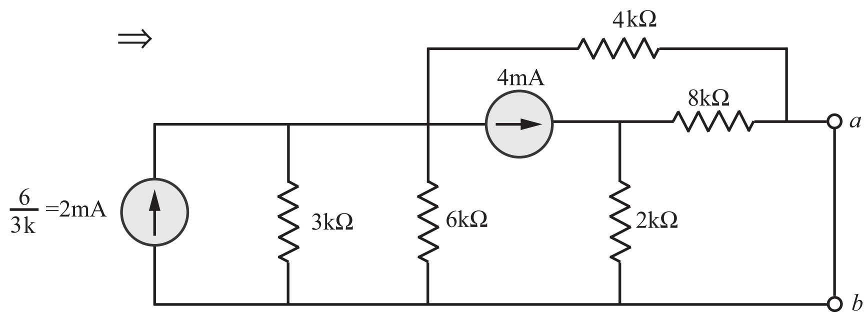
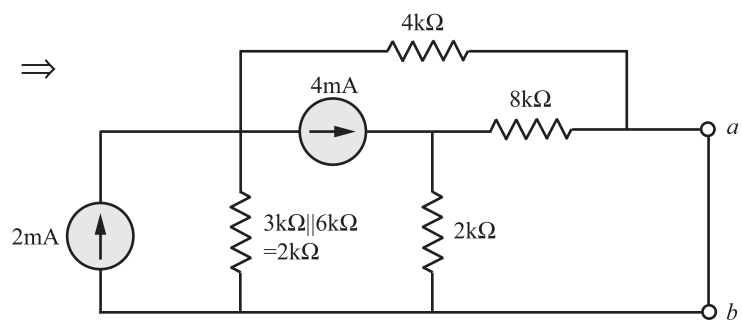
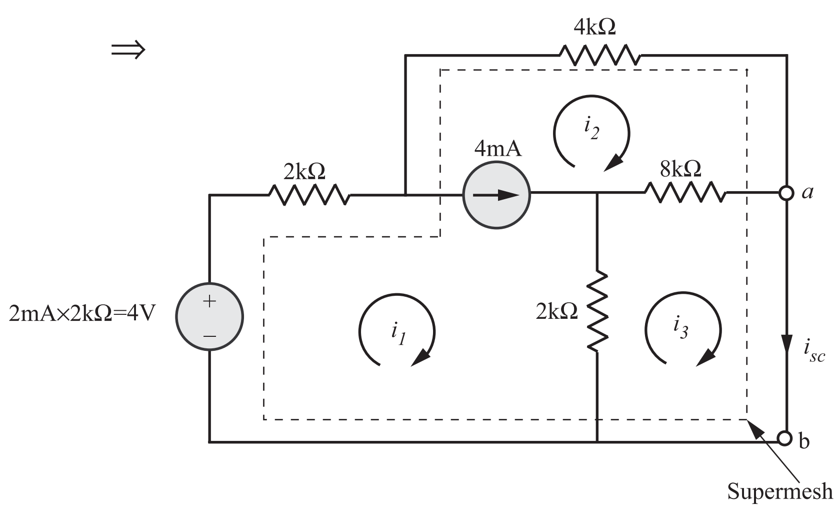


⇒



⇒



Constraint equation :

$$i_1 - i_2 = 4\text{mA} \tag{3.12}$$

KVL around supermesh :

$$-4 + 2 \times 10^3 i_1 + 4 \times 10^3 i_2 = 0 \tag{3.13}$$

KVL around mesh 3 :

$$8 \times 10^3 (i_3 - i_2) + 2 \times 10^3 (i_3 - i_1) = 0$$

Since $i_3 = i_{sc}$, the above equation becomes,

$$8 \times 10^3 (i_{sc} - i_2) + 2 \times 10^3 (i_{sc} - i_1) = 0 \tag{3.14}$$

Solving equations (3.12), (3.13) and (3.14) simultaneously, we get $i_{sc} = 0.1333$ mA.
To find R_N:
Deactivate all the sources in Fig. 3.75. This yields a circuit diagram as shown in Fig. 3.76.

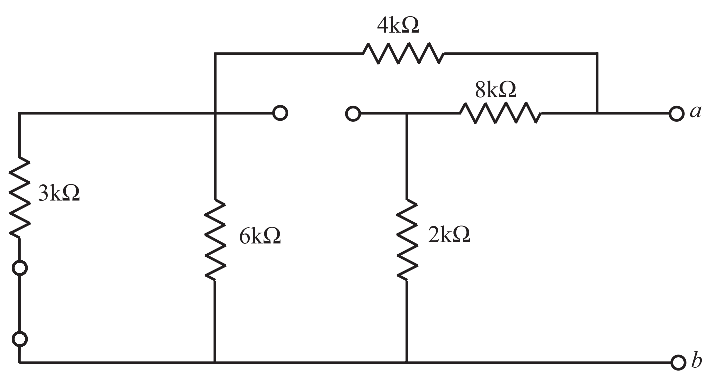
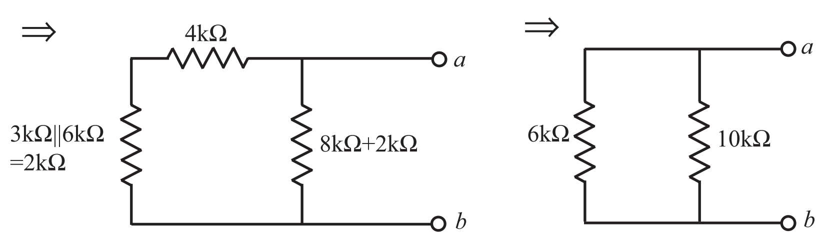


Figure 3.76



$$R_N = 6 \text{ k}\Omega || 10 \text{ k}\Omega$$
$$= \frac{6 \times 10}{6 + 10} = 3.75 \text{ k}\Omega$$

Hence, the Norton equivalent circuit is as shown in Fig 3.76 (a).
To the Norton equivalent circuit, now connect the 4 kΩ resistor that was removed earlier to get the network shown in Fig. 3.76(b).

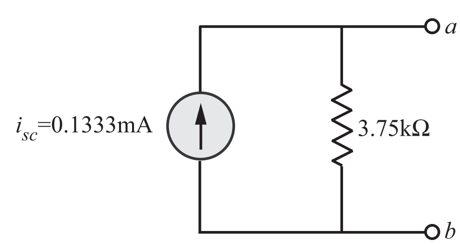


Figure 3.76(a)

$$V_o = i_{sc}\,(R_N||R)$$
$$= i_{sc}\frac{R_N R}{R_N + R}$$
$$= 258 \text{ mV}$$

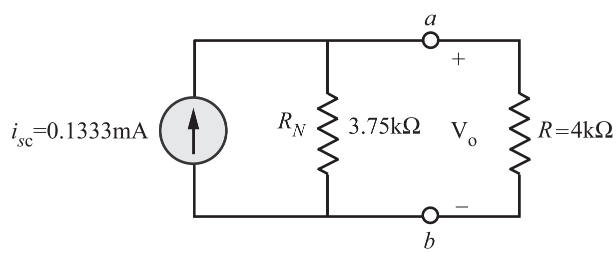


Figure 3.76(b) Norton equivalent circuit with $R = 4$ kΩ

EXAMPLE 3.23

Find the Norton equivalent to the left of the terminals $a - b$ for the circuit of Fig. 3.77.

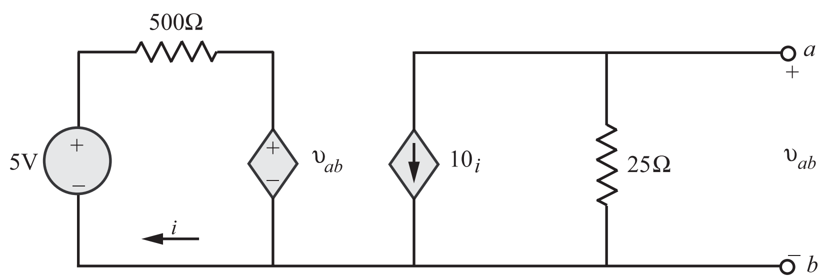
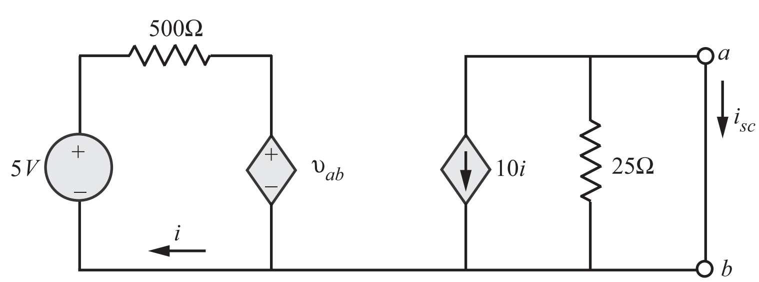


Figure 3.77

SOLUTION

To find i_{sc}:

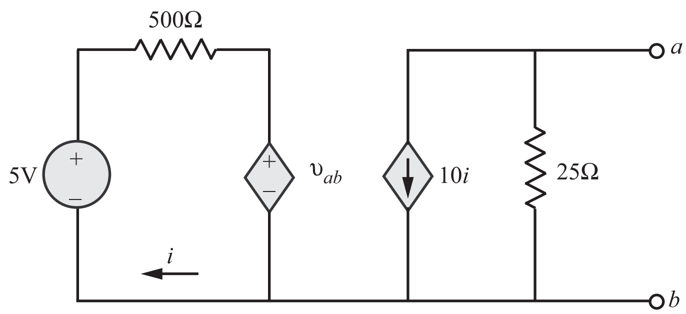


Note that $v_{ab} = 0$ when the terminals $a - b$ are short-circuited.

Then
$$i = \frac{5}{500} = 10 \text{ mA}$$

Therefore, for the right–hand portion of the circuit, $i_{sc} = -10i = -100$ mA.

To find R_N or R_t :



Writing the KVL equations for the left-hand mesh, we get

$$-5 + 500i + v_{ab} = 0 \tag{3.15}$$

Also for the right-hand mesh, we get

$$v_{ab} = -25(10i) = -250i$$

Therefore $$i = \frac{-v_{ab}}{250}$$

Substituting i into the mesh equation (3.15), we get

$$-5 + 500\left(\frac{-v_{ab}}{250}\right) + v_{ab} = 0$$

$$\Rightarrow \qquad v_{ab} = -5 \text{ V}$$

$$R_N = R_t \triangleq \frac{v_{oc}}{i_{sc}} = \frac{v_{ab}}{i_{sc}} = \frac{-5}{-0.1} = 50\ \Omega$$

The Norton equivalent circuit is shown in Fig 3.77 (a).

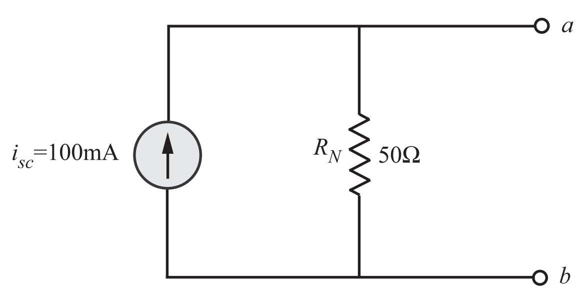


Figure 3.77 (a)

EXAMPLE 3.24

Find the Norton equivalent of the network shown in Fig. 3.78.

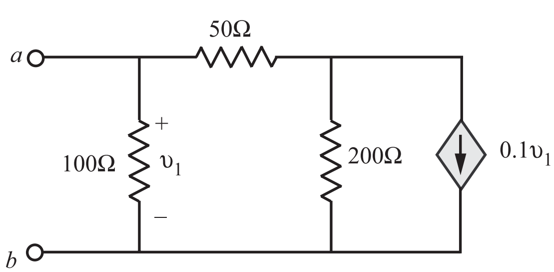


Figure 3.78

SOLUTION

Since there are no independent sources present in the network of Fig. 3.78, $i_N = i_{sc} = 0$.

To find R_N, we inject a current of 1A between the terminals $a - b$. This is illustrated in Fig. 3.79.

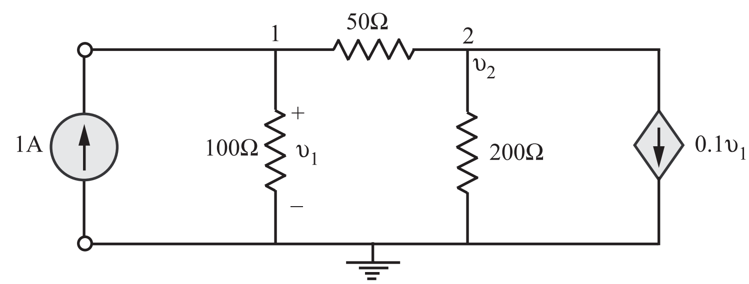


Figure 3.79

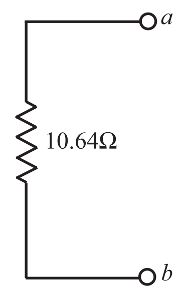


Figure 3.79(a) Norton equivalent circuit

KCL at node 1:

$$1 = \frac{v_1}{100} + \frac{v_1 - v_2}{50}$$

$$\Rightarrow \qquad 0.03v_1 - 0.02v_2 = 1$$

KCL at node 2:

$$\frac{v_2}{200} + \frac{v_2 - v_1}{50} + 0.1v_1 = 0$$

$$\Rightarrow \qquad 0.08v_1 + 0.025v_2 = 0$$

Solving the above two nodal equations, we get

$$v_1 = 10.64 \text{ volts} \quad \Rightarrow \quad v_{oc} = 10.64 \text{ volts}$$

Hence,
$$R_N = R_t = \frac{v_{oc}}{1} = \frac{10.64}{1} = 10.64\ \Omega$$

Norton equivalent circuit for the network shown in Fig. 3.78 is as shown in Fig. 3.79(a).

EXAMPLE 3.25

Find the Thevenin and Norton equivalent circuits for the network shown in Fig. 3.80 (a).

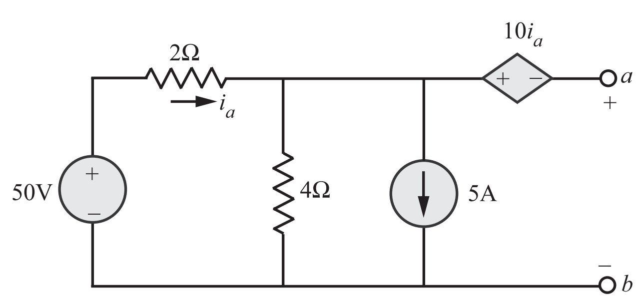


Figure 3.80(a)

SOLUTION

To find V_{oc} :

Performing source transformation on 5A current source, we get the circuit shown in Fig. 3.80 (b).

Applying KVL around Left mesh :

$$-50 + 2i_a - 20 + 4i_a = 0$$

$$\Rightarrow \qquad i_a = \frac{70}{6}\text{A}$$

Applying KVL around right mesh:

$$20 + 10i_a + V_{oc} - 4i_a = 0$$

$$\Rightarrow \qquad V_{oc} = -90 \text{ V}$$

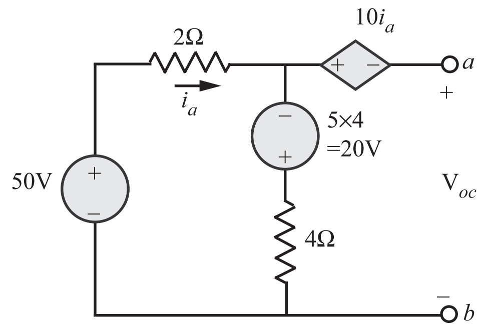


Figure 3.80(b)

To find i_{sc}(referring Fig 3.80 (c)) :

KVL around Left mesh :

$$-50 + 2i_a - 20 + 4(i_a - i_{sc}) = 0$$

$$\Rightarrow \qquad 6i_a - 4i_{sc} = 70$$

KVL around right mesh :

$$4(i_{sc} - i_a) + 20 + 10i_a = 0$$

$$\Rightarrow \qquad 6i_a + 4i_{sc} = -20$$

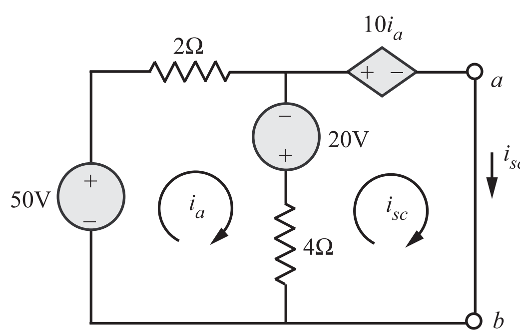
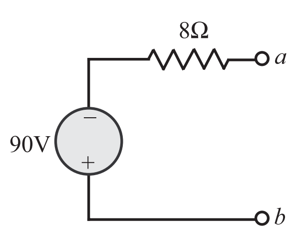


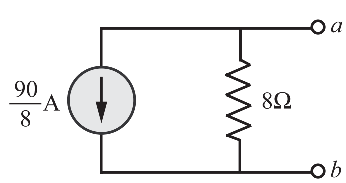
Figure 3.80(c)

Solving the two mesh equations simultaneously, we get $i_{sc} = -11.25$ A

Hence, $R_t = R_N = \dfrac{v_{oc}}{i_{sc}} = \dfrac{-90}{-11.25} = 8\ \Omega$

Performing source transformation on Thevenin equivalent circuit, we get the norton equivalent circuit (both are shown below).





Thevenin equivalent circuit

Norton equivalent circuit

EXAMPLE 3.26

If an 8 kΩ load is connected to the terminals of the network in Fig. 3.81, $V_{AB} = 16$ V. If a 2 kΩ load is connected to the terminals, $V_{AB} = 8$V. Find V_{AB} if a 20 kΩ load is connected across the terminals.

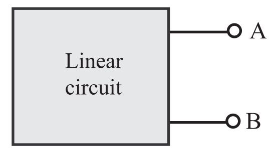
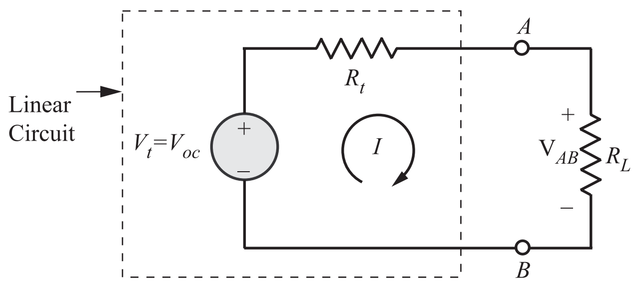


Figure 3.81

SOLUTION



Applying KVL around the mesh, we get $(R_t + R_L)\, I = V_{oc}$

If $\quad R_L = 2\text{ k}\Omega,\ I = 10\text{ mA} \Rightarrow V_{oc} = 20 + 0.01R_t$

If $\quad R_L = 10\text{ k}\Omega,\ I = 6\text{ mA} \Rightarrow V_{oc} = 60 + 0.006R_t$

Solving, we get $V_{oc} = 120$ V, $R_t = 10$ kΩ.

If $$R_L = 20\text{ k}\Omega,\ I = \frac{V_{oc}}{(R_L + R_t)} = \frac{120}{(20 \times 10^3 + 10 \times 10^3)} = 4\text{ mA}$$

3.4 Maximum Power Transfer Theorem

In circuit analysis, we are some times interested in determining the maximum power that a circuit can supply to the load. Consider the linear circuit A as shown in Fig. 3.82.

Circuit A is replaced by its Thevenin equivalent circuit as seen from a and b (Fig 3.83).

We wish to find the value of the load R_L such that the maximum power is delivered to it.

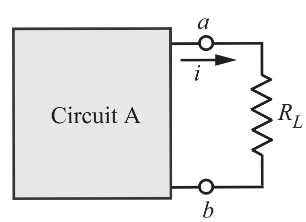


Figure 3.82 Circuit A with load R_L

The power that is delivered to the load is given by

$$p = i^2 R_L = \left[\frac{V_t}{R_t + R_L}\right]^2 R_L \tag{3.16}$$

Assuming that V_t and R_t are fixed for a given source, the maximum power is a function of R_L. In order to determine the value of R_L that maximizes p, we differentiate p with respect to R_L and equate the derivative to zero.

$$\frac{dp}{dR_L} = V_t^2\left[\frac{(R_t + R_L)^2 - 2\,(R_t + R_L)}{(R_L + R_t)^2}\right] = 0$$

which yields $$R_L = R_t \tag{3.17}$$

To confirm that equation (3.17) is a maximum, it should be shown that $\frac{d^2p}{dR_L^2} < 0$. Hence, maximum power is transferred to the load when R_L is equal to the Thevenin equivalent resistance R_t. The maximum power transferred to the load is obtained by substituting $R_L = R_t$ in equation 3.16.

Accordingly,

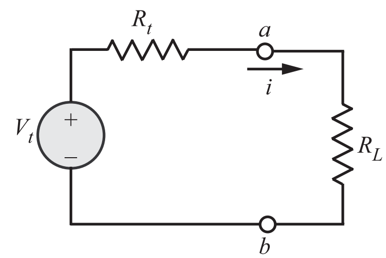


Figure 3.83 Thevenin equivalent circuit is substituted for circuit A

$$P_{\max} = \frac{V_t^2 R_L}{(2R_L)^2} = \frac{V_t^2}{4R_L}$$

The maximum power transfer theorem states that the maximum power delivered by a source represented by its Thevenin equivalent circuit is attained when the load R_L is equal to the Thevenin resistance R_t.

EXAMPLE 3.27

Find the load R_L that will result in maximum power delivered to the load for the circuit of Fig. 3.84. Also determine the maximum power $P_{\max}$.

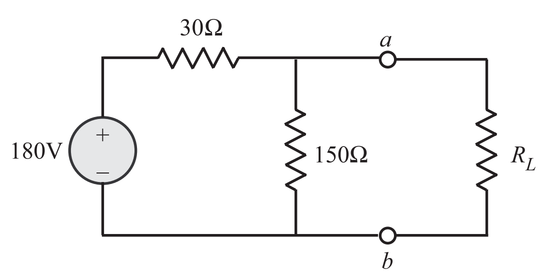


Figure 3.84

SOLUTION

Disconnect the load resistor R_L. This results in a circuit diagram as shown in Fig. 3.85(a).

Next let us determine the Thevenin equivalent circuit as seen from $a - b$.

$$i = \frac{180}{150 + 30} = 1\text{A}$$
$$V_{oc} = V_t = 150 \times i = 150 \text{ V}$$

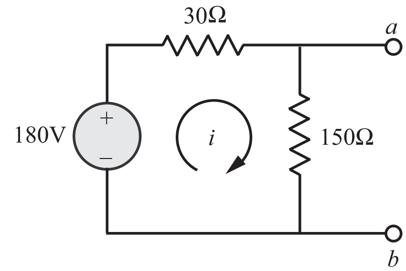


Figure 3.85(a)

To find R_t, deactivate the 180 V source. This results in the circuit diagram of Fig. 3.85(b).

$$R_t = R_{ab} = 30\ \Omega || 150\ \Omega$$
$$= \frac{30 \times 150}{30 + 150} = 25\ \Omega$$

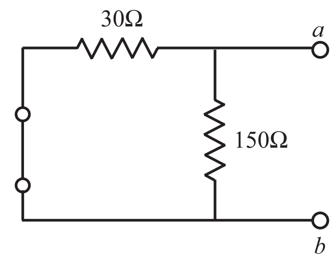


Figure 3.85(b)

The Thevenin equivalent circuit connected to the load resistor is shown in Fig. 3.86.
Maximum power transfer is obtained when $R_L = R_t = 25\ \Omega$.
Then the maximum power is

$$P_{\max} = \frac{V_t^2}{4R_L} = \frac{(150)^2}{4 \times 25}$$
$$= 2.25 \text{ Watts}$$

The Thevenin source V_t actually provides a total power of

$$P_t = 150 \times i$$
$$= 150 \times \left(\frac{150}{25 + 25}\right)$$
$$= 450 \text{ Watts}$$

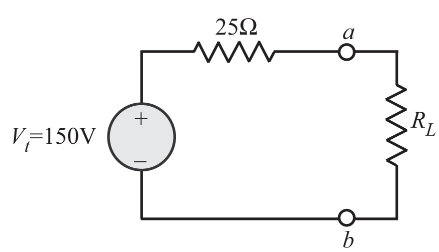


Figure 3.86

Thus, we note that one-half the power is dissipated in R_L.

EXAMPLE 3.28

Refer to the circuit shown in Fig. 3.87. Find the value of R_L for maximum power transfer. Also find the maximum power transferred to R_L.

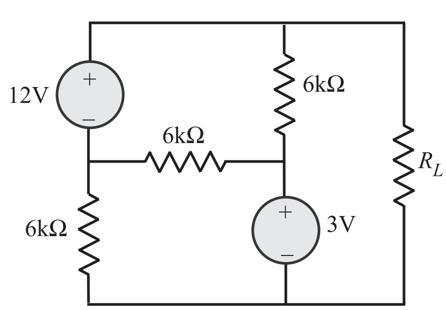


Figure 3.87

SOLUTION

Disconnecting R_L, results in a circuit diagram as shown in Fig. 3.88(a).

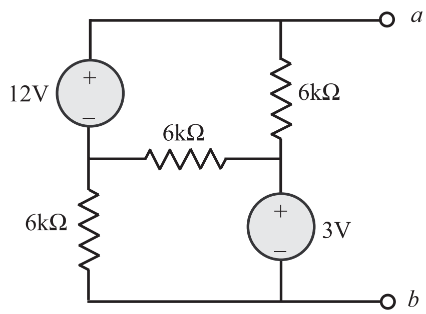


Figure 3.88(a)

To find R_t, deactivate all the independent voltage sources as in Fig. 3.88(b).

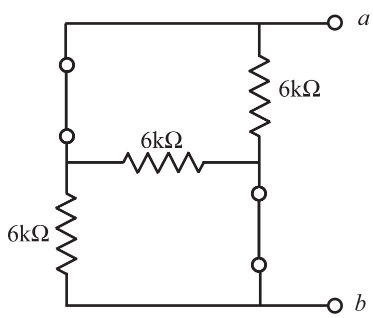


Figure 3.88(b)

$\Rightarrow$

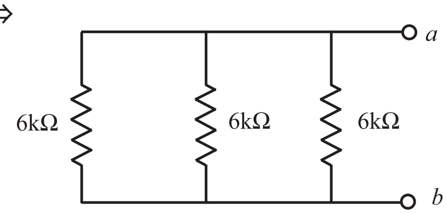


Figure 3.88(c)

$$R_t = R_{ab} = 6\ \text{k}\Omega || 6\ \text{k}\Omega || 6\ \text{k}\Omega$$
$$= 2\ \text{k}\Omega$$

To find V_t :

Refer the Fig. 3.88(d).

Constraint equation :

$$V_3 - V_1 = 12\ \text{V}$$

By inspection, $V_2 = 3\ \text{V}$

KCL at supernode :

$$\frac{V_3 - V_2}{6\text{k}} + \frac{V_1}{6\text{k}} + \frac{V_1 - V_2}{6\text{k}} = 0$$

$$\Rightarrow \quad \frac{V_3 - 3}{6\text{k}} + \frac{V_3 - 12}{6\text{k}} + \frac{V_3 - 12 - 3}{6\text{k}} = 0$$

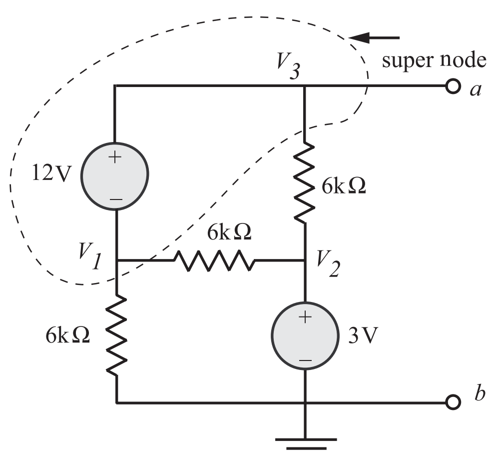


Figure 3.88(d)

$$\Rightarrow \quad V_3 - 3 + V_3 - 12 + V_3 - 15 = 0$$
$$\Rightarrow \quad 3V_3 = 30$$
$$\Rightarrow \quad V_3 = 10$$
$$\Rightarrow \quad V_t = V_{ab} = V_3 = 10 \text{ V}$$

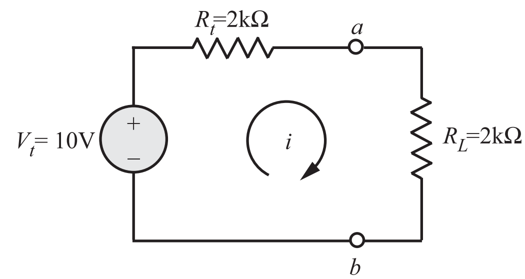


Figure 3.88(e)

The Thevenin equivalent circuit connected to the load resistor R_L is shown in Fig. 3.88(e).

$$\begin{aligned} P_{\max} &= i^2 R_L \\ &= \left[\frac{V_t}{2R_L}\right]^2 R_L \\ &= 12.5 \text{ mW} \end{aligned}$$

Alternate method :
It is possible to find $P_{\max}$, without finding the Thevenin equivalent circuit. However, we have to find R_t. For maximum power transfer, $R_L = R_t = 2$ kΩ. Insert the value of R_L in the original circuit given in Fig. 3.87. Then use any circuit reduction technique of your choice to find power dissipated in R_L.

Refer Fig. 3.88(f). By inspection we find that, $V_2 = 3$ V.

Constraint equation :

$$V_3 - V_1 = 12$$
$$\Rightarrow \quad V_1 = V_3 - 12$$

KCL at supernode :

$$\frac{V_3 - V_2}{6\text{k}} + \frac{V_1 - V_2}{6\text{k}} + \frac{V_3}{2\text{k}} + \frac{V_1}{6\text{k}} = 0$$
$$\Rightarrow \frac{V_3 - 3}{6\text{k}} + \frac{V_3 - 12 - 3}{6\text{k}} + \frac{V_3}{2\text{k}} + \frac{V_3 - 12}{6\text{k}} = 0$$
$$\Rightarrow \quad V_3 - 3 + V_3 - 15 + 3V_3 + V_3 - 12 = 0$$
$$\Rightarrow \quad 6V_3 = 30$$
$$\Rightarrow \quad V_3 = 5 \text{ V}$$

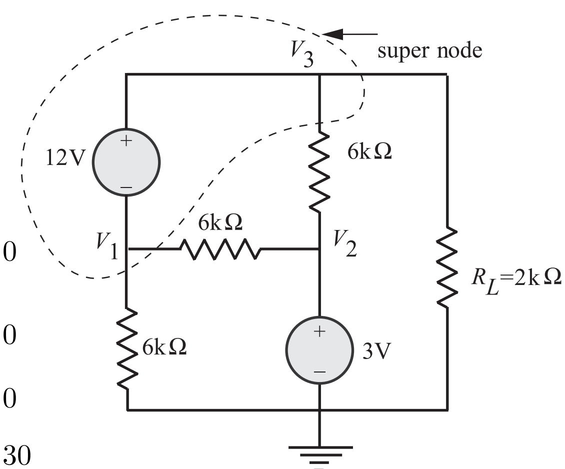


Figure 3.88(f)

Hence, $$P_{\max} = \frac{V_3^2}{R_L} = \frac{25}{2\text{k}} = \mathbf{12.5\ mW}$$

EXAMPLE 3.29

Find R_L for maximum power transfer and the maximum power that can be transferred in the network shown in Fig. 3.89.

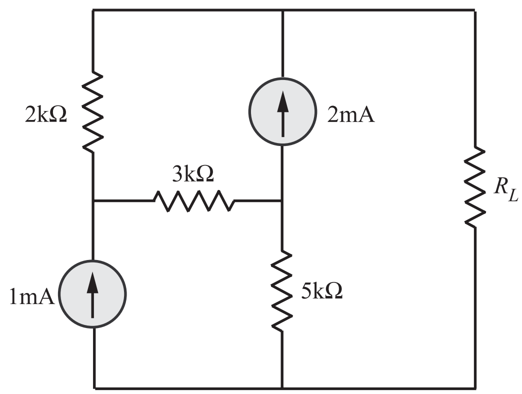


Figure 3.89

SOLUTION

Disconnect the load resistor R_L. This results in a circuit as shown in Fig. 3.89(a).

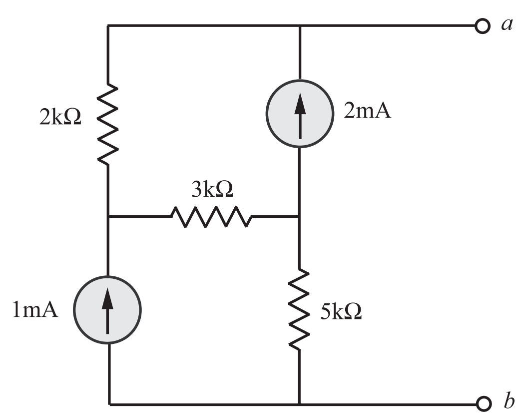


Figure 3.89(a)

To find R_t, let us deactivate all the independent sources, which results the circuit as shown in Fig. 3.89(b).

$$R_t = R_{ab} = 2\ \mathrm{k\Omega} + 3\ \mathrm{k\Omega} + 5\ \mathrm{k\Omega} = 10\ \mathrm{k\Omega}$$

For maximum power transfer $R_L = R_t = 10\ \mathrm{k\Omega}$.

Let us next find V_{oc} or V_t.

Refer Fig. 3.89 (c). By inspection, $i_1 = -2$ mA & $i_2 = 1$ mA.

Figure 3.89(b)

Applying KVL clockwise to the loop 5 kΩ → 3 kΩ → 2 kΩ → $a - b$, we get

$$-5\text{k} \times i_2 + 3\text{k}\,(i_1 - i_2) + 2\text{k} \times i_1 + V_t = 0$$

$$\Rightarrow -5\times 10^3\left(1 \times 10^{-3}\right) + 3\times 10^3\left(-2 \times 10^{-3} - 1 \times 10^{-3}\right) + 2\times 10^3\left(-2 \times 10^{-3}\right) + V_t = 0$$

$$\Rightarrow \quad -5 - 9 - 4 + V_t = 0$$

$$\Rightarrow \quad V_t = 18 \text{ V}.$$

The Thevenin equivalent circuit with load resistor R_L is as shown in Fig. 3.89 (d).

$$i = \frac{18}{(10+10) \times 10^3} = 0.9 \text{ mA}$$

Then,

$$P_{\max} = P_L = (0.9 \text{ mA})^2 \times 10 \text{ k}\Omega$$
$$= 8.1 \text{ mW}$$

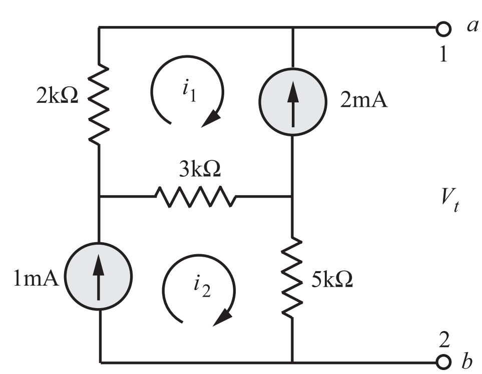


Figure 3.89(c)

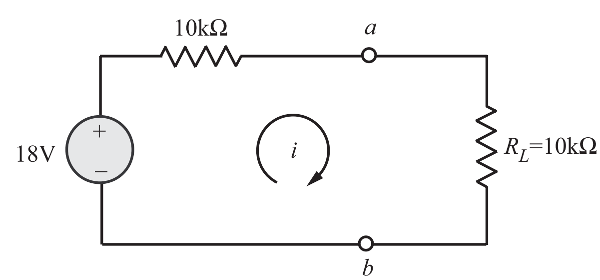


Figure 3.89(d)

EXAMPLE 3.30

Find the maximum power dissipated in R_L. Refer the circuit shown in Fig. 3.90.

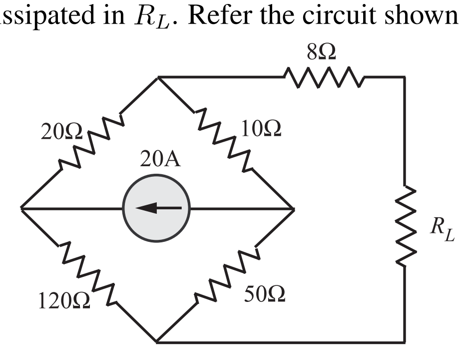


Figure 3.90

SOLUTION

Disconnecting the load resistor R_L from the original circuit results in a circuit diagram as shown in Fig. 3.91.

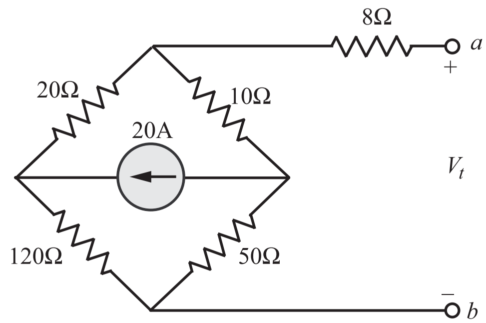


Figure 3.91

As a first step in the analysis, let us find R_t. While finding R_t, we have to deactivate all the independent sources. This results in a network as shown in Fig 3.91 (a) :

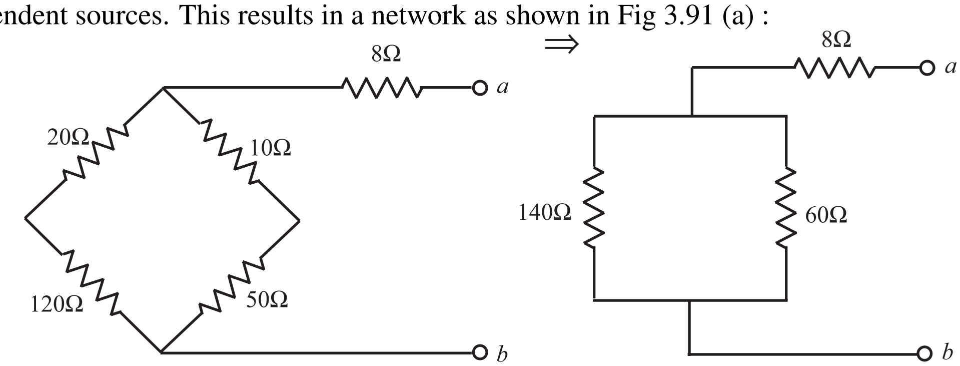


Figure 3.91(a)

$$R_t = R_{ab} = [140\ \Omega||60\ \Omega] + 8\ \Omega$$
$$= \frac{140 \times 60}{140 + 60} + 8 = 50\ \Omega.$$

For maximum power transfer, $R_L = R_t = 50\ \Omega$. Next step in the analysis is to find V_t. Refer Fig 3.91(b), using the principle of current division,

$$i_1 = \frac{i \times R_2}{R_1 + R_2}$$
$$= \frac{20 \times 170}{170 + 30} = 17\text{ A}$$
$$i_2 = \frac{i \times R_1}{R_1 + R_2} = \frac{20 \times 30}{170 + 30}$$
$$= \frac{600}{200} = 3\text{A}$$

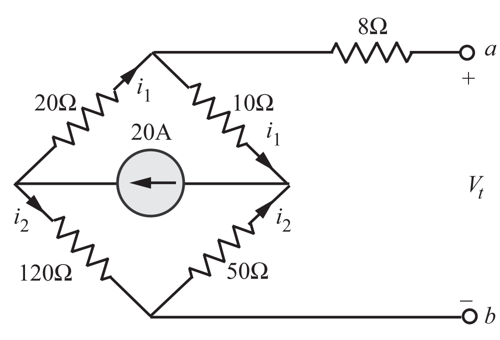


Figure 3.91(a)

Applying KVL clockwise to the loop comprising of $50\ \Omega \to 10\ \Omega \to 8\ \Omega \to a - b$, we get

$$\begin{aligned}
50i_2 - 10i_1 + 8 \times 0 + V_t &= 0 \\
\Rightarrow 50(3) - 10\,(17) + V_t &= 0 \\
\Rightarrow \qquad\qquad V_t &= 20\ \text{V}
\end{aligned}$$

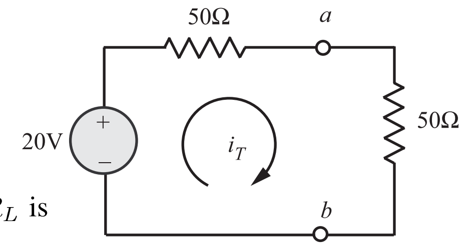


Figure 3.91(c)

The Thevenin equivalent circuit with load resistor R_L is as shown in Fig. 3.91(c).

$$i_T = \frac{20}{50+50} = 0.2\text{A}$$

$$P_{\max} = i_T^2 \times 50 = 0.04 \times 50 = \mathbf{2\ W}$$

EXAMPLE 3.31

Find the value of R_L for maximum power transfer in the circuit shown in Fig. 3.92. Also find $P_{\max}$.

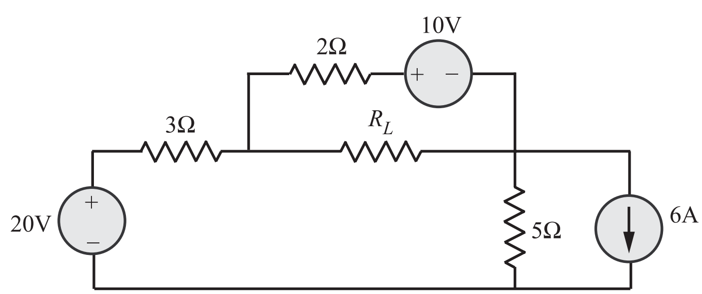


Figure 3.92

SOLUTION

Disconnecting R_L from the original circuit, we get the network shown in Fig. 3.93.

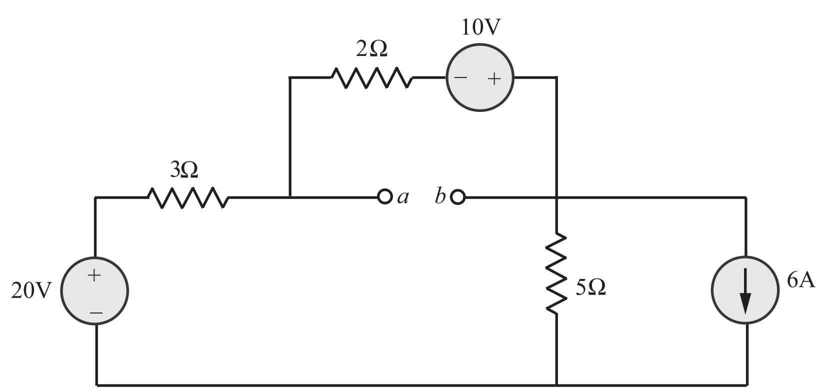


Figure 3.93

Let us draw the Thevenin equivalent circuit as seen from the terminals $a - b$ and then insert the value of $R_L = R_t$ between the terminals $a - b$. To find R_t, let us deactivate all independent sources which results in the circuit as shown in Fig. 3.94.

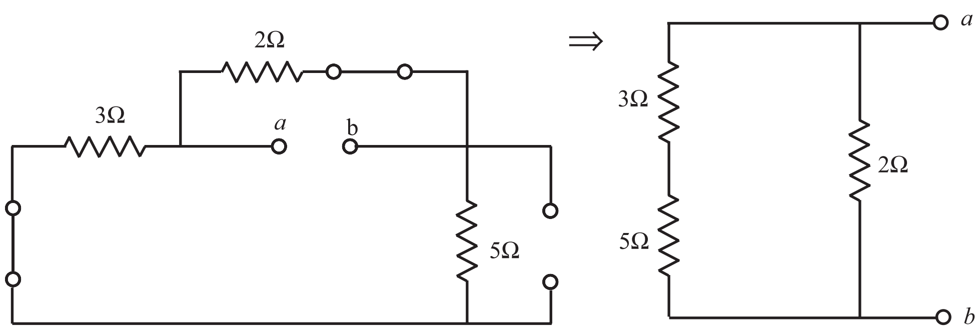


Figure 3.94

$$\begin{aligned} R_t &= R_{ab} \\ &= 8\ \Omega || 2\ \Omega \\ &= \frac{8 \times 2}{8 + 2} = 1.6\ \Omega \end{aligned}$$

Next step is to find V_{oc} or V_t.

By performing source transformation on the circuit shown in Fig. 3.93, we obtain the circuit shown in Fig. 3.95.

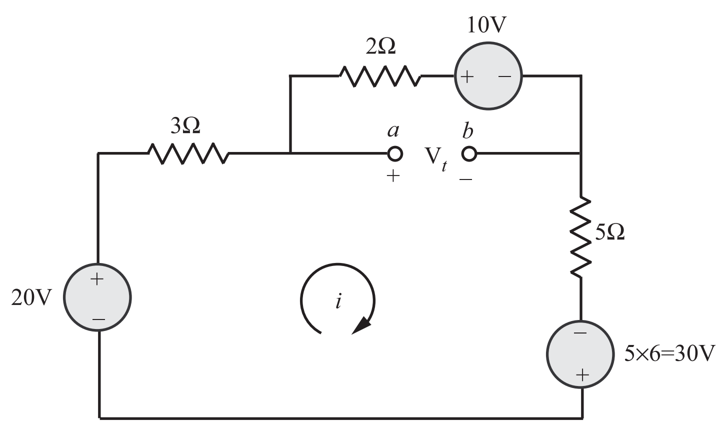


Figure 3.95

Applying KVL to the loop made up of 20 V → 3 Ω → 2 Ω → 10 V → 5 Ω → 30 V, we get

$$-20 + 10i - 10 - 30 = 0$$

$$\Rightarrow \qquad i = \frac{60}{10} = 6\text{A}$$

Again applying KVL clockwise to the path $2\ \Omega \to 10\ \text{V} \to a - b$, we get

$$2i - 10 - V_t = 0$$

$$\Rightarrow \qquad V_t = 2i - 10$$

$$= 2(6) - 10 = 2\ \text{V}$$

The Thevenin equivalent circuit with load resistor R_L is as shown in Fig. 3.95 (a).

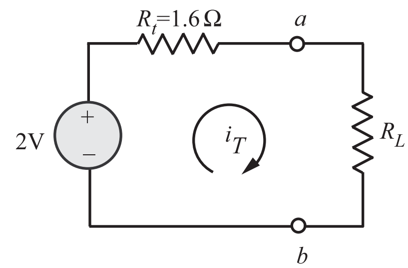


Figure 3.95(a) Thevenin equivalent circuit

$$P_{\max} = i_T^2 R_L$$

$$= \frac{V_t^2}{4R_t} = \mathbf{625}\ \text{mW}$$

EXAMPLE 3.32

Find the value of R_L for maximum power transfer. Hence find $P_{\max}$.

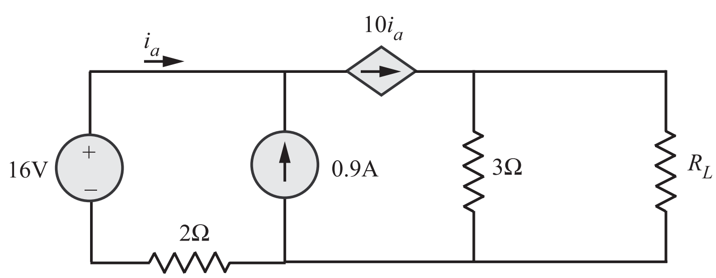


Figure 3.96

SOLUTION

Removing R_L from the original circuit gives us the circuit diagram shown in Fig. 3.97.

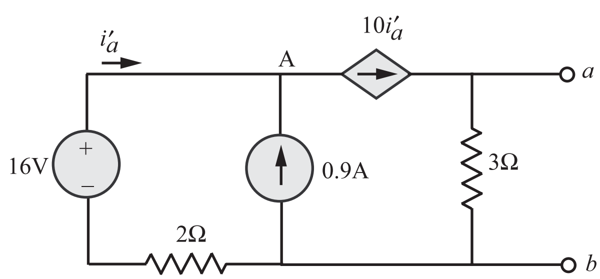


Figure 3.97

To find V_{oc} :

KCL at node A :

$$-i'_a - 0.9 + 10i'_a = 0$$

$$\Rightarrow \qquad i'_a = 0.1\ \text{A}$$

Hence, $$V_{oc} = 3\left(10i'_a\right)$$

$$= 3 \times 10 \times 0.1 = 3\ \text{V}$$

To find R_t, we need to compute i_{sc} with all independent sources activated.

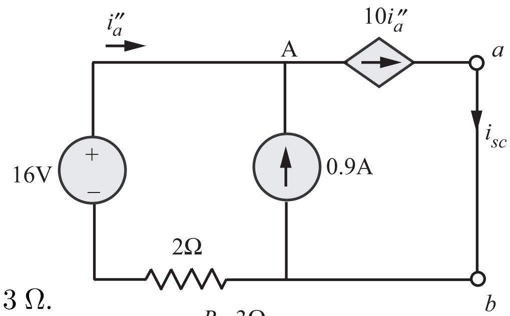
KCL at node A:

$$-i_a{''} - 0.9 + 10i_a{''} = 0$$

$$\Rightarrow \quad i_a{''} = 0.1 \text{ A}$$

Hence $i_{sc} = 10i_a{''} = 10 \times 0.1 = 1$ A

$$R_t = \frac{V_{oc}}{i_{sc}} = \frac{3}{1} = 3\ \Omega$$



Hence, for maximum power transfer $R_L = R_t = 3\ \Omega$. The Thevenin equivalent circuit with $R_L = 3\ \Omega$ inserted between the terminals $a - b$ gives the network shown in Fig. 3.97(a).

$$i_T = \frac{3}{3+3} = 0.5 \text{ A}$$

$$P_{\max} = i_T^2 R_L$$

$$= (0.5)^2 \times 3$$

$$= \mathbf{0.75} \text{ W}$$

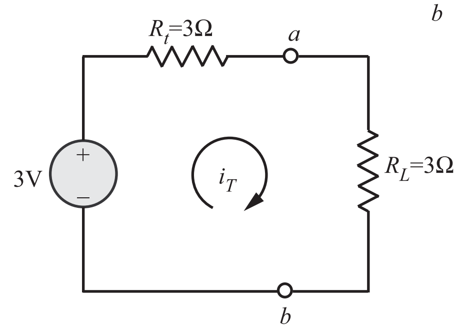


Figure 3.97(a)

EXAMPLE 3.33

Find the value of R_L in the network shown that will achieve maximum power transfer, and determine the value of the maximum power.

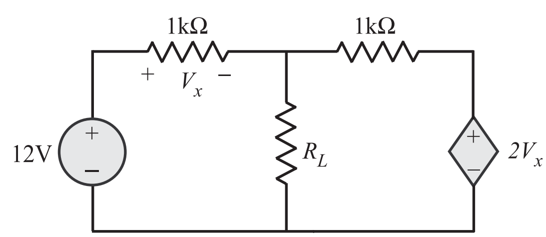


Figure 3.98(a)

SOLUTION

Removing R_L from the circuit of Fig. 3.98(a), we get the circuit of Fig 3.98(b).

Applying KVL clockwise we get

$$-12 + 2 \times 10^3 i + 2V_x' = 0$$

Also $\quad V_x' = 1 \times 10^3 i$

Hence, $\quad -12 + 2 \times 10^3 i + 2\left(1 \times 10^3 i\right) = 0$

$$i = \frac{12}{4 \times 10^3} = 3 \text{ mA}$$

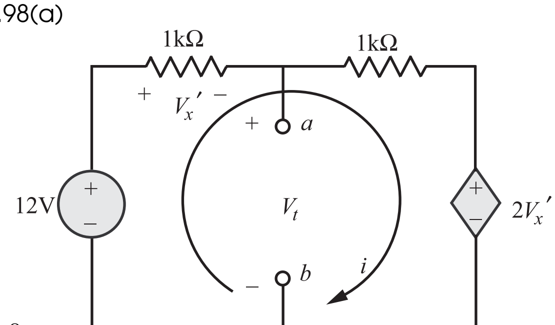


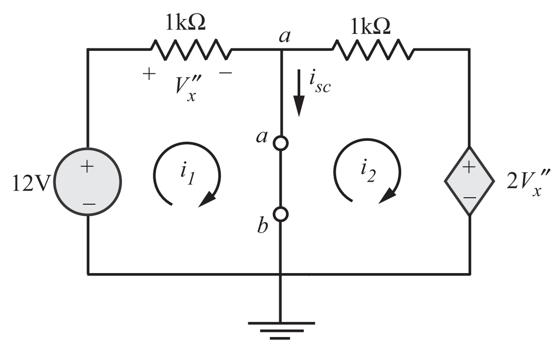
Figure 3.98(b)

Applying KVL to loop $1\ \text{k}\Omega \to 2V_x' \to b - a$, we get

$$1 \times 10^3 i + 2V_x' - V_t = 0$$

$$\Rightarrow \quad V_t = 1 \times 10^3 i + 2\left(1 \times 10^3 i\right)$$
$$= \left(1 \times 10^3 + 2 \times 10^3\right) i$$
$$= 3 \times 10^3 \left(3 \times 10^{-3}\right)$$
$$= 9\ \text{V}$$

To find R_t, we need to find i_{sc}. While finding i_{sc}, none of the independent sources must be deactivated.



Applying KVL to mesh 1:

$$-12 + V_x'' + 0 = 0$$
$$\Rightarrow \quad V_x'' = 12$$
$$\Rightarrow \quad 1 \times 10^3 i_1 = 12 \ \Rightarrow\ i_1 = 12\ \text{mA}$$

Applying KVL to mesh 2:

$$1 \times 10^3 i_2 + 2V_x'' = 0$$
$$\Rightarrow \quad 1 \times 10^3 i_2 = -24$$
$$i_2 = -24\ \text{mA}$$

Applying KCL at node a:

$$i_{sc} = i_1 - i_2$$
$$= 12 + 24 = 36\ \text{mA}$$

Hence,
$$R_t = \frac{V_t}{i_{sc}} = \frac{V_{oc}}{i_{sc}}$$
$$= \frac{9}{36 \times 10^{-3}}$$
$$= 250\ \Omega$$

For maximum power transfer, $R_L = R_t = 250\ \Omega$. Thus, the Thevenin equivalent circuit with R_L is as shown in Fig 3.98 (c) :

$$i_T = \frac{9}{250 + 250} = \frac{9}{500}\text{A}$$
$$P_{\max} = i_T^2 \times 250$$
$$= \left(\frac{9}{500}\right)^2 \times 250$$
$$= \mathbf{81\ mW}$$

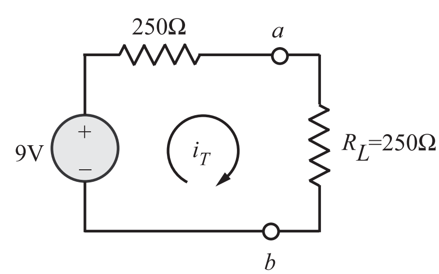


Figure 3.98 (c) Thevenin equivalent circuit

EXAMPLE 3.34

The variable resistor R_L in the circuit of Fig. 3.99 is adjusted untill it absorbs maximum power from the circuit.

(a) Find the value of R_L.

(b) Find the maximum power.

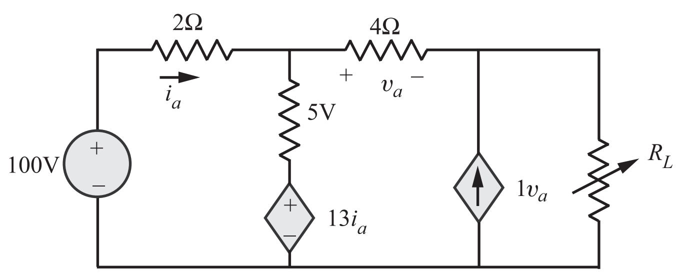


Figure 3.99

SOLUTION

Disconnecting the load resistor R_L from the original circuit, we get the circuit shown in Fig. 3.99(a).

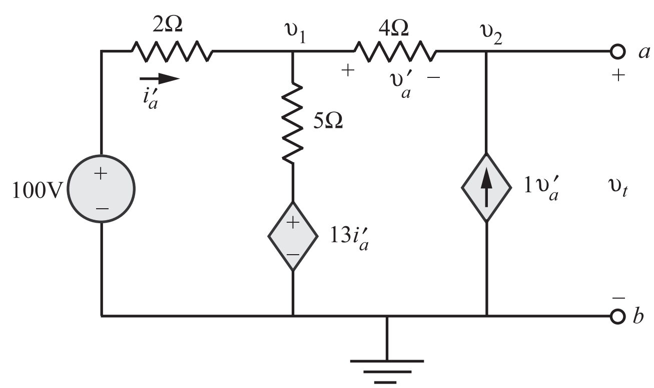


Figure 3.99(a)

KCL at node v_1 :

$$\frac{v_1 - 100}{2} + \frac{v_1 - 13i'_a}{5} + \frac{v_1 - v_2}{4} = 0 \tag{3.18}$$

Constraint equations :

$$i'_a = \frac{100 - v_1}{2} \tag{3.19}$$

$$\frac{v_2 - v_1}{4} = v'_a \qquad (applying\ KCL\ at\ v_2) \tag{3.20}$$

$$v'_a = v_1 - v_2 \qquad (potential\ across\ 4\,\Omega) \tag{3.21}$$

From equations (3.20) and (3.21), we have

$$\frac{v_2 - v_1}{4} = v_1 - v_2$$

$$\Rightarrow \quad v_2 - v_1 = 4v_1 - 4v_2$$

$$\Rightarrow \quad 5v_1 - 5v_2 = 0$$

$$\Rightarrow \quad v_1 = v_2 \tag{3.22}$$

Making use of equations (3.19) and (3.22) in (3.18), we get

$$\frac{v_1 - 100}{2} + \frac{v_2 - 13\dfrac{(100 - v_1)}{2}}{5} + \frac{v_1 - v_1}{4} = 0$$

$$\Rightarrow \quad 5\left(v_1 - 100\right) + 2\left[v_1 - 13\frac{(100 - v_1)}{2}\right] = 0$$

$$\Rightarrow \quad 5v_1 - 500 + 2v_1 - 13 \times 100 + 13v_1 = 0$$

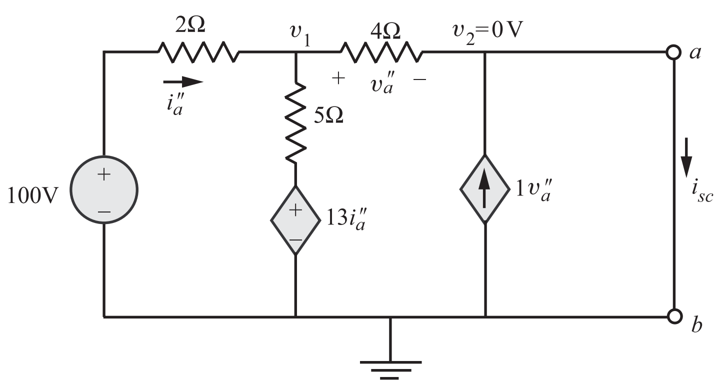
$$\Rightarrow \quad 20v_1 = 1800$$

$$\Rightarrow \quad v_1 = 90 \text{ Volts}$$

Hence, $\quad v_t = v_2 = v_1 = 90$ Volts

We know that, $\quad R_t = \dfrac{v_{oc}}{i_{sc}} = \dfrac{v_t}{i_{sc}}$

The short circuit current is calculated using the circuit shown below:



Here $\quad i_a'' = \dfrac{100 - v_1}{2}$

Applying KCL at node v_1 :

$$\frac{v_1 - 100}{2} + \frac{v_1 - 13i_a''}{5} + \frac{v_1 - 0}{4} = 0$$

$$\Rightarrow \quad \frac{v_1 - 100}{2} + \frac{v_1 - 13\dfrac{(100 - v_1)}{2}}{5} + \frac{v_1}{4} = 0$$

Solving we get $v_1 = 80$ volts $= v_a''$
Applying KCL at node a :

$$\frac{0 - v_1}{4} + i_{sc} = v_a''$$

$$\Rightarrow \qquad i_{sc} = \frac{v_1}{4} + v_a''$$

$$= \frac{80}{4} + 80 = 100 \text{ A}$$

Hence,

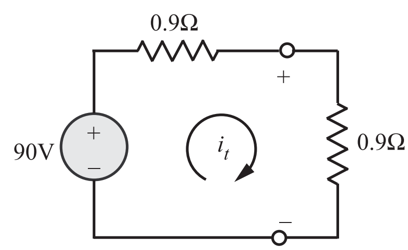
$$R_t = \frac{v_{oc}}{i_{sc}} = \frac{v_t}{i_{sc}}$$

$$= \frac{90}{100} = 0.9\ \Omega$$

Hence for maximum power transfer,

$$R_L = R_t = \mathbf{0.9\ \Omega}$$

The Thevenin equivalent circuit with $R_L = 0.9\ \Omega$ is as shown.



$$i_t = \frac{90}{0.9 + 0.9} = \frac{90}{1.8}$$

$$P_{\max} = i_t^2 \times 0.9$$

$$= \left(\frac{90}{1.8}\right)^2 \times 0.9 = \mathbf{2250\ W}$$

EXAMPLE 3.35

Refer to the circuit shown in Fig. 3.100 :

(a) Find the value of R_L for maximum power transfer.

(b) Find the maximum power that can be delivered to R_L.

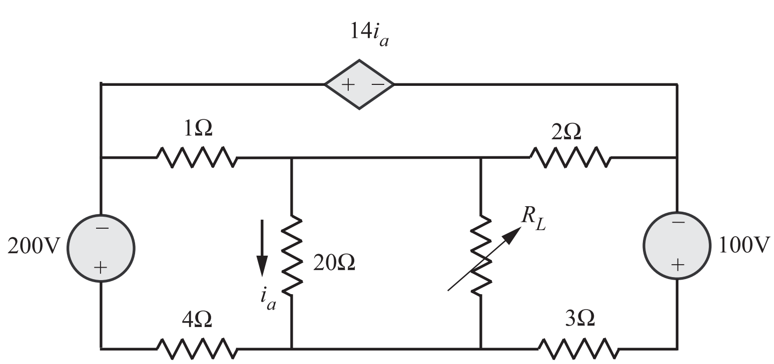


Figure 3.100

SOLUTION

Removing the load resistor R_L, we get the circuit diagram shown in Fig. 3.100(a). Let us proceed to find V_t.

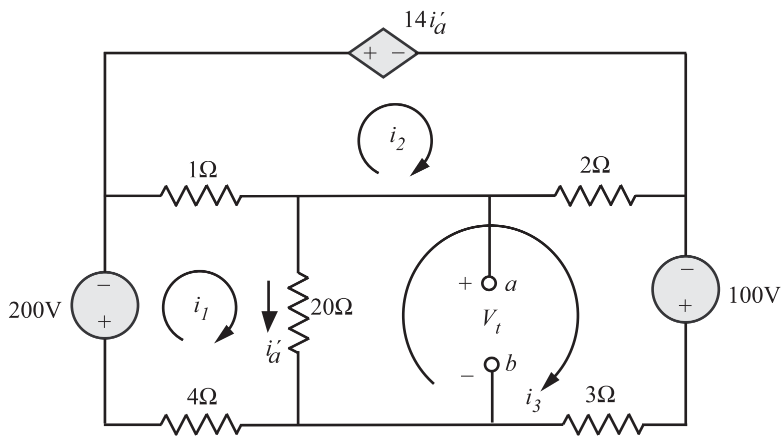


Figure 3.100(a)

Constraint equation :

$$i_a' = i_1 - i_3$$

KVL clockwise to mesh 1 :

$$200 + 1(i_1 - i_2) + 20(i_1 - i_3) + 4i_1 = 0$$

$$\Rightarrow \quad 25i_1 - i_2 - 20i_3 = -200$$

KVL clockwise to mesh 2 :

$$14i_a' + 2(i_2 - i_3) + 1(i_2 - i_1) = 0$$

$$\Rightarrow \quad 14(i_1 - i_3) + 2(i_2 - i_3) + 1(i_2 - i_1) = 0$$

$$\Rightarrow \quad 13i_1 + 3i_2 - 16i_3 = 0$$

KVL clockwise to mesh 3 :

$$2(i_3 - i_2) - 100 + 3i_3 + 20(i_3 - i_1) = 0$$

$$\Rightarrow \quad -20i_1 - 2i_2 + 25i_3 = 100$$

Solving the mesh equations, we get

$$i_1 = -2.5\text{A}, i_3 = 5\text{A}$$

Applying KVL clockwise to the path comprising of $a - b \rightarrow 20\ \Omega$, we get

$$V_t - 20i_a' = 0$$

$$\Rightarrow \quad V_t = 20i_a'$$

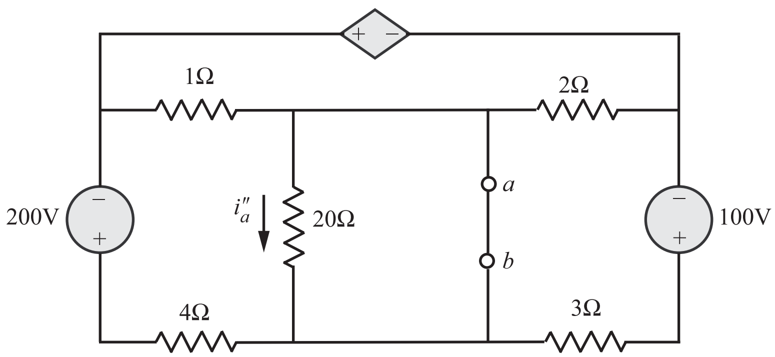
$$= 20(i_1 - i_3)$$

$$= 20(-2.5 - 5)$$

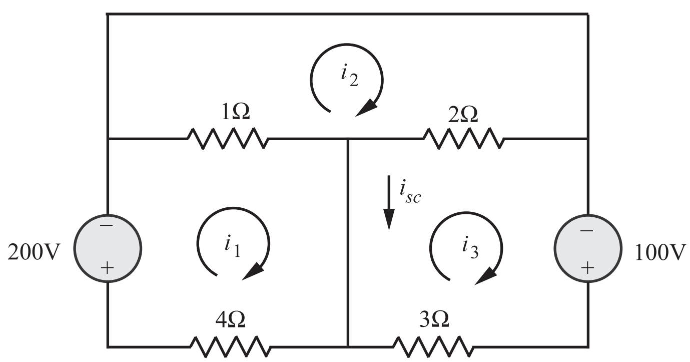
$$= -150\text{ V}$$

Next step is to find R_t.

$$R_t = \frac{V_{oc}}{i_{sc}} = \frac{V_t}{i_{sc}}$$



When terminals $a - b$ are shorted, $i''_a = 0$. Hence, $14\ i''_a$ is also zero.



KVL clockwise to mesh 1 :

$$200 + 1\,(i_1 - i_2) + 4i_1 = 0$$
$$\Rightarrow \qquad 5i_1 - i_2 = -200$$

KVL clockwise to mesh 2 :

$$2\,(i_2 - i_3) + 1\,(i_2 - i_1) = 0$$
$$\Rightarrow \qquad -\,i_1 + 3i_2 - 2i_3 = 0$$

KVL clockwise to mesh 3 :

$$-100 + 3i_3 + 2\,(i_3 - i_2) = 0$$
$$\Rightarrow \qquad -\,2i_2 + 5i_3 = 100$$

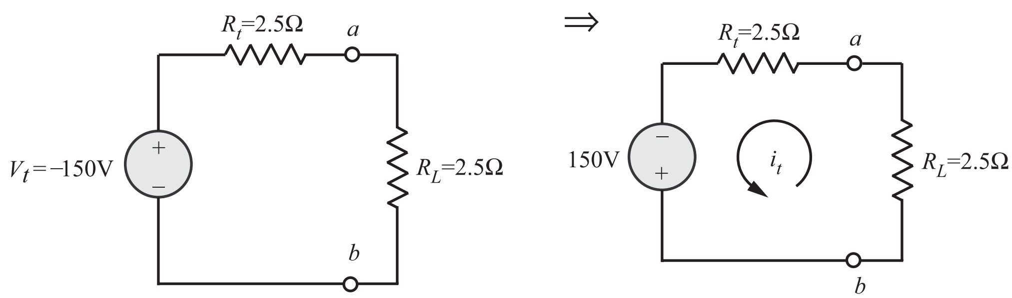
Solving the mesh equations, we find that

$$i_1 = -40\text{A},\ i_3 = 20\text{A},$$

$$\Rightarrow \qquad i_{sc} = i_1 - i_3 = -60\text{A}$$

$$R_t = \frac{V_t}{i_{sc}} = \frac{-150}{-60} = 2.5\ \Omega$$

For maximum power transfer, $R_L = R_t = 2.5\ \Omega$. The Thevenin equivalent circuit with R_L is as shown below :



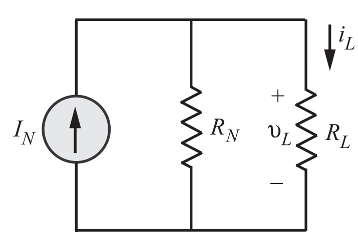
$$\begin{aligned} P_{\max} &= i_1^2 R_L \\ &= \left[\frac{150}{2.5+2.5}\right]^2 \times 2.5 \\ &= \mathbf{2250\ W} \end{aligned}$$

EXAMPLE 3.36

A practical current source provides 10 W to a 250 Ω load and 20 W to an 80 Ω load. A resistance R_L, with voltage v_L and current i_L, is connected to it. Find the values of R_L, v_L and i_L if (a) $v_L i_L$ is a maximum, (b) v_L is a maximum and (c) i_L is a maximum.

SOLUTION

Load current calculation:



$$10\text{W to } 250\ \Omega \text{ corresponds to } i_L = \sqrt{\frac{10}{250}} = 200\text{ mA}$$

$$20\text{W to } 80\ \Omega \text{ corresponds to } i_L = \sqrt{\frac{20}{80}} = 500\text{ mA}$$

Using the formula for division of current between two parallel branches :

$$i_2 = \frac{i \times R_1}{R_1 + R_2}$$

In the present context, $$0.2 = \frac{I_N R_N}{R_N + 250} \tag{3.23}$$

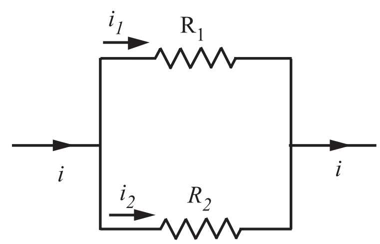
and $$0.5 = \frac{I_N R_N}{R_N + 80} \tag{3.24}$$

Solving equations (3.23) and (3.24), we get

$$I_N = 1.7 \text{ A}$$
$$R_N = 33.33 \, \Omega$$

(a) If $v_L i_L$ is maximum,

$$R_L = R_N = 33.33 \, \Omega$$
$$i_L = 1.7 \times \frac{33.33}{33.33 + 33.33}$$
$$= 850 \text{ mA}$$
$$v_L = i_L R_L = 850 \times 10^{-3} \times 33.33$$
$$= 28.33 \text{ V}$$



(b) $v_L = I_N (R_N || R_L)$ is a maximum when $R_N || R_L$ is a maximum, which occurs when $R_L = \infty$.

Then, $i_L = 0$ and

$$v_L = 1.7 \times R_N$$
$$= 1.7 \times 33.33$$
$$= 56.66 \text{ V}$$

(c) $i_L = \dfrac{I_N R_N}{R_N + R_L}$ is maxmimum when $R_L = 0 \, \Omega$

$\Rightarrow$ $\qquad i_L = 1.7\text{A}$ and $v_L = 0$ V

3.5 Sinusoidal steady state analysis using superposition, Thevenin and Norton equivalents

Circuits in the frequency domain with phasor currents and voltages and impedances are analogous to resistive circuits.

To begin with, let us consider the principle of superposition, which may be restated as follows :

For a linear circuit containing two or more independent sources, any circuit voltage or current may be calculated as the algebraic sum of all the individual currents or voltages caused by each independent source acting alone.

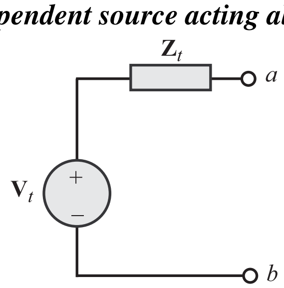


Figure 3.101 Thevenin equivalent circuit

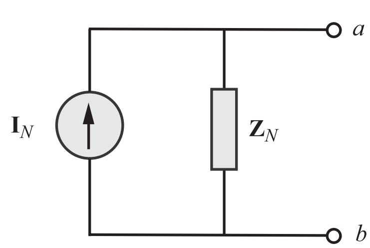


Figure 3.102 Norton equivalent circuit

The superposition principle is particularly useful if a circuit has two or more sources acting at different frequencies. The circuit will have one set of impedance values at one frequency and a different set of impedance values at another frequency. Phasor responses corresponding to different frequencies cannot be superposed; only their corresponding sinusoids can be superposed. That is, when frequencies differ, the principle of superposition applies to the summing of time domain components, not phasors. Within a component, problem corresponding to a single frequency, however phasors may be superposed.

Thevenin and Norton equivalents in phasor circuits are found exactly in the same manner as described earlier for resistive circuits, except for the subtitution of impedance $\mathbf{Z}$ in place of resistance R and subsequent use of complex arithmetic. The Thevenin and Norton equivalent circuits are shown in Fig. 3.101 and 3.102.

The Thevenin and Norton forms are equivalent if the relations

$$\text{(a) } \mathbf{Z}_t = \mathbf{Z}_N \qquad\qquad \text{(b) } \mathbf{V}_t = \mathbf{Z}_N\mathbf{I}_N$$

hold between the circuits.

A step by step procedure for finding the Thevenin equivalent circuit is as follows:

1. Identify a seperate circuit portion of a total circuit.
2. Find $\mathbf{V}_t = \mathbf{V}_{oc}$ at the terminals.
3. (a) If the circuit contains only impedances and independent sources, then deactivate all the independent sources and then find $\mathbf{Z}_t$ by using circuit reduction techniques.

   (b) If the circuit contains impedances, independent sources and dependent sources, then either short–circuit the terminals and determine $\mathbf{I}_{sc}$ from which

$$\mathbf{Z}_t = \frac{\mathbf{V}_{oc}}{\mathbf{I}_{sc}}$$

or deactivate the independent sources, connect a voltage or current source at the terminals, and determine both $\mathbf{V}$ and $\mathbf{I}$ at the terminals from which

$$\mathbf{Z}_t = \frac{\mathbf{V}}{\mathbf{I}}$$

A step by step procedure for finding Norton equivalent circuit is as follows:

(i) Identify a seperate circuit portion of the original circuit.

(ii) Short the terminals after seperating a portion of the original circuit and find the current through the short circuit at the terminals, so that $\mathbf{I}_N = \mathbf{I}_{sc}$.

(iii) (a) If the circuit contains only impedances and independent sources, then deactivate all the independent sources and then find $\mathbf{Z}_N = \mathbf{Z}_t$ by using circuit reduction techniques.

(b) If the circuit contains impedances, independent sources and one or more dependent sources, find the open–circuit voltage at the terminals, $\mathbf{V}_{oc}$, so that $\mathbf{Z}_N = \mathbf{Z}_t = \dfrac{\mathbf{V}_{oc}}{\mathbf{I}_{sc}}$.

EXAMPLE 3.37

Find the Thevenin and Norton equivalent circuits at the terminals $a - b$ for the circuit in Fig. 3.103.

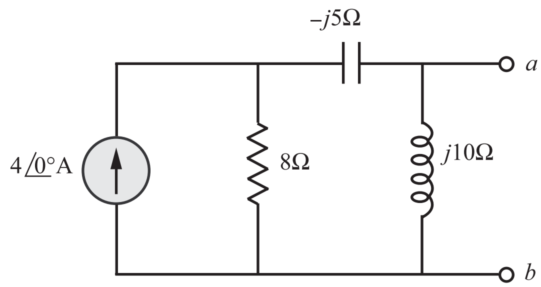
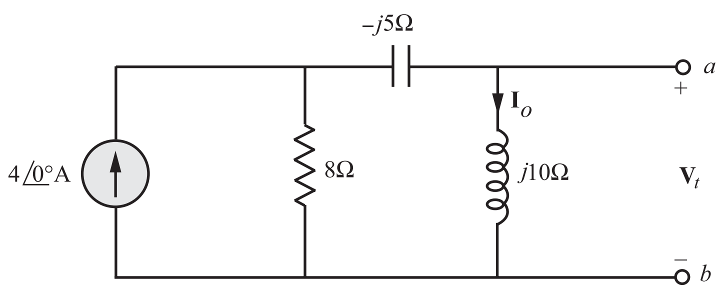


Figure 3.103

SOLUTION

As a first step in the analysis, let us find $\mathbf{V}_t$.



Using the principle of current division,

$$\mathbf{I}_o = \frac{8\,(4\,\underline{/0^\circ}\,)}{8 + j10 - j5} = \frac{32}{8 + j5}$$

$$\mathbf{V}_t = \mathbf{I}_o(j10) = \frac{j320}{8 + j5} = 33.92\,\underline{/58^\circ}\ \ \mathrm{V}$$

To find $\mathbf{Z}_t$, deactivate all the independent sources. This results in a circuit diagram as shown in Fig. 3.103 (a).

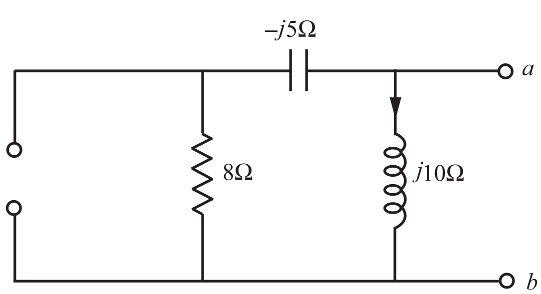


Figure 3.103(a)

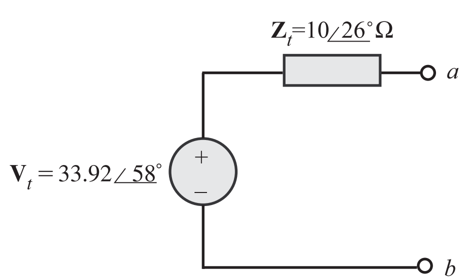


Figure 3.103(b) Thevenin equivalent circuit

$$\begin{aligned}\mathbf{Z}_t &= j10 || (8 - j5)\ \Omega \\ &= \frac{(j10)(8-j5)}{j10+8-j5} \\ &= 10\,\underline{/26^\circ}\ \ \Omega\end{aligned}$$

The Thevenin equivalent circuit as viewed from the terminals $a - b$ is as shown in Fig 3.103(b). Performing source transformation on the Thevenin equivalent circuit, we get the Norton equivalent circuit.

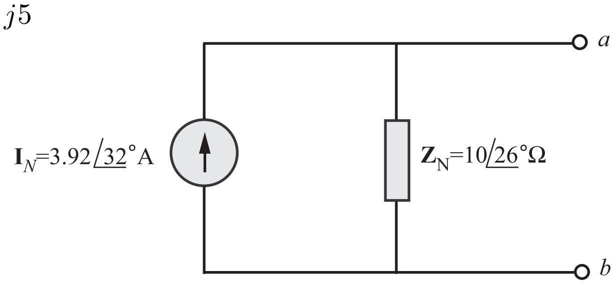


Figure : Norton equivalent circuit

$$\begin{aligned}\mathbf{I}_N &= \frac{\mathbf{V}_t}{\mathbf{Z}_t} = \frac{33.92\,\underline{/58^\circ}}{10\,\underline{/26^\circ}} \\ &= 3.392\,\underline{/32^\circ}\ \text{A} \\ \mathbf{Z}_N &= \mathbf{Z}_t = 10\,\underline{/26^\circ}\ \ \Omega\end{aligned}$$

EXAMPLE 3.38

Find v_o using Thevenin's theorem. Refer to the circuit shown in Fig. 3.104.

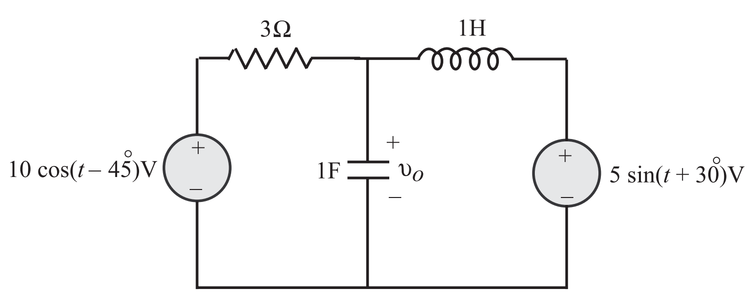


Figure 3.104

SOLUTION

Let us convert the circuit given in Fig. 3.104 into a frequency domain equiavalent or phasor circuit (shown in Fig. 3.105(a)). $\omega = 1$

$$\begin{aligned}10\cos(t - 45^\circ) &\rightarrow 10\,\underline{/-45^\circ}\ \ \text{V} \\ 5\sin(t+30^\circ) = 5\cos(t-60^\circ) &\rightarrow 5\,\underline{/-60^\circ}\ \ \text{V}\end{aligned}$$

$$\begin{aligned}L = 1\text{H} &\rightarrow j\,\omega L = j \times 1 \times 1 = j1\ \Omega \\ C = 1\text{F} &\rightarrow \frac{1}{j\,\omega C} = \frac{1}{j \times 1 \times 1} = -j1\ \Omega\end{aligned}$$

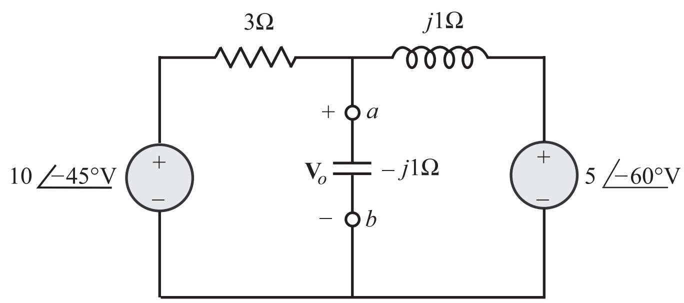


Figure 3.105(a)

Disconnecting the capicator from the original circuit, we get the circuit shown in Fig. 3.105(b). This circuit is used for finding $\mathbf{V}_t$.

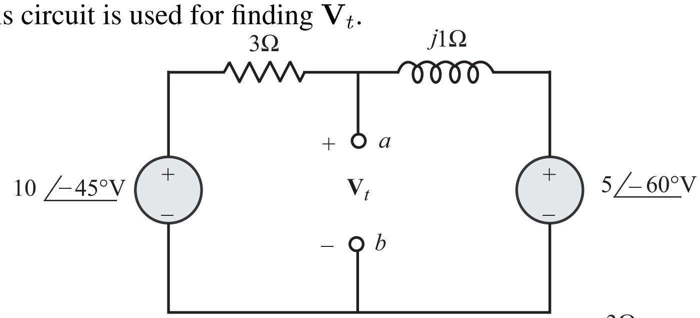


Figure 3.105(b)

KCL at node a :

$$\frac{\mathbf{V}_t - 10\,\underline{/-45^\circ}}{3} + \frac{\mathbf{V}_t - 5\,\underline{/-60^\circ}}{j1} = 0$$

Solving, $\quad \mathbf{V}_t = 4.97\,\underline{/-40.54^\circ}$

To find $\mathbf{Z}_t$ deactivate all the independent sources in Fig. 3.105(b). This results in a network as shown in Fig. 3.105(c) :

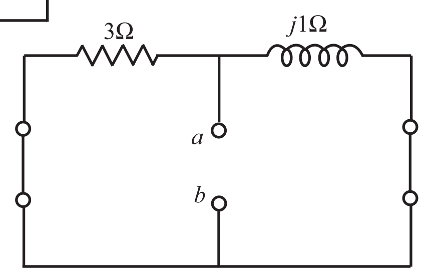


Figure 3.105(c)

$$\begin{aligned}\mathbf{Z}_t = \mathbf{Z}_{ab} &= 3\Omega || j1\ \Omega \\ &= \frac{j3}{3+j} = \frac{3}{10}(1+j3)\ \Omega\end{aligned}$$

The Thevenin equivalent circuit along with the capicator is as shown in Fig 3.105(d).

$$\begin{aligned}\mathbf{V}_o &= \frac{\mathbf{V}_t}{\mathbf{Z}_t - j1}(-j1) \\ &= \frac{4.97\,\underline{/-40.54^\circ}}{0.3(1+j3) - j1}(-j1) \\ &= 15.73\,\underline{/247.9^\circ}\ \text{V}\end{aligned}$$

Hence, $\quad v_o = 15.73\cos(t + 247.9^\circ)\ \text{V}$

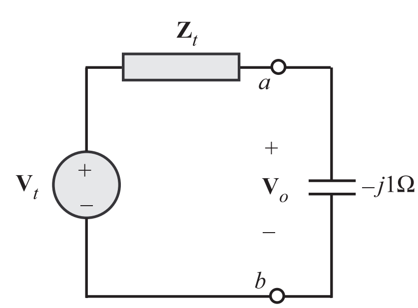


Figure 3.105(d) Thevenin equivalent circuit

EXAMPLE 3.39

Find the Thevenin equivalent circuit of the circuit shown in Fig. 3.106.

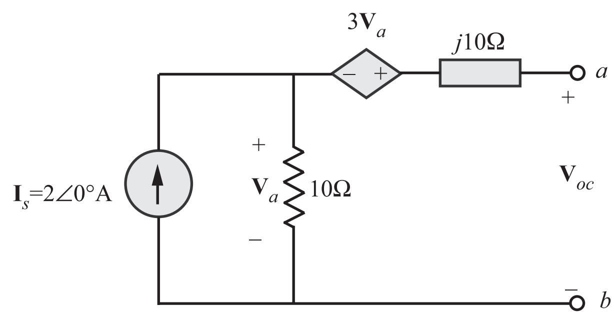


Figure 3.106

SOLUTION

Since terminals $a - b$ are open,

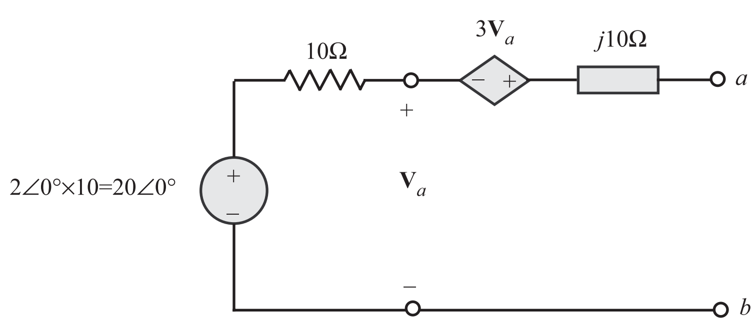
$$\mathbf{V}_a = \mathbf{I}_s \times 10$$
$$= 20 \underline{/0^\circ} \text{ V}$$

Applying KVL clockwise for the mesh on the right hand side of the circuit, we get

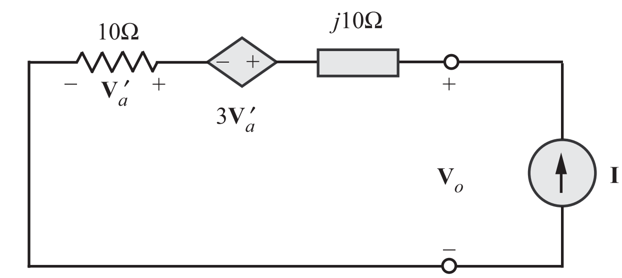
$$-3\mathbf{V}_a + 0\,(j10) + \mathbf{V}_{oc} - \mathbf{V}_a = 0$$

$$\mathbf{V}_{oc} = 4\mathbf{V}_a$$
$$= 80 \underline{/0^\circ} \text{ V}$$

Let us transform the current source with 10 Ω parallel resistance to a voltage source with 10 Ω series resistance as shown in figure below :



To find $\mathbf{Z}_t$, the independent voltage source is deactivated and a current source of $\mathbf{I}$ A is connected at the terminals as shown below :



Applying KVL clockwise we get,

$$-\mathbf{V}'_a - 3\mathbf{V}'_a - j10\mathbf{I} + \mathbf{V}_o = 0$$

$$\Rightarrow \quad -4\mathbf{V}'_a - j10\mathbf{I} + \mathbf{V}_o = 0$$

Since $\mathbf{V}'_a = 10\mathbf{I}$

we get $-40\mathbf{I} - j10\mathbf{I} = -\mathbf{V}_o$

Hence, $$\mathbf{Z}_t = \frac{\mathbf{V}_o}{\mathbf{I}} = 40 + j10\Omega$$

Hence the Thevenin equivalent circuit is as shown in Fig 3.106(a) :

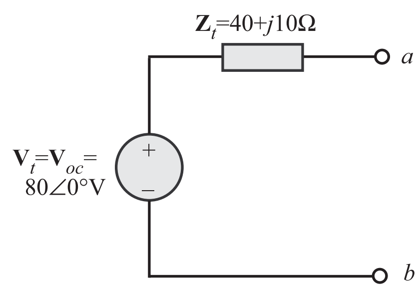


Figure 3.106(a)

EXAMPLE 3.40

Find the Thevenin and Norton equivalent circuits for the circuit shown in Fig. 3.107.

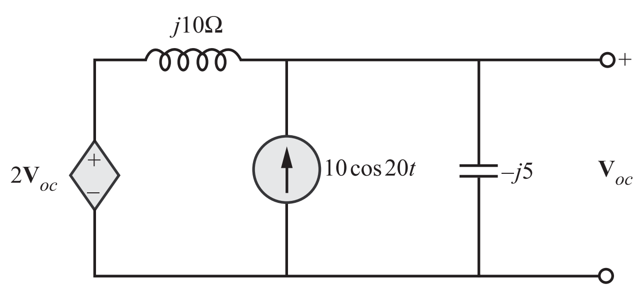


Figure 3.107

SOLUTION

The phasor equivalent circuit of Fig. 3.107 is shown in Fig. 3.108.

KCL at node a :

$$\frac{\mathbf{V}_{oc} - 2\mathbf{V}_{oc}}{j10} - 10 + \frac{\mathbf{V}_{oc}}{-j5} = 0$$

$$\Rightarrow \quad \mathbf{V}_{oc} = -j\frac{100}{3} = \frac{100}{3}\underline{/-90^\circ}\ \text{V}$$

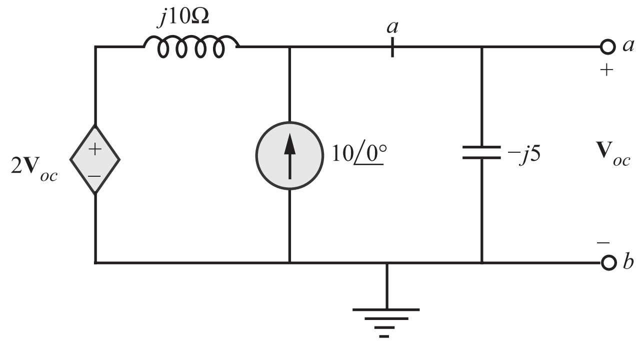


Figure 3.108

To find $\mathbf{I}_{sc}$, short the terminals $a - b$ of Fig. 3.108 as in Fig. 3.108(a).

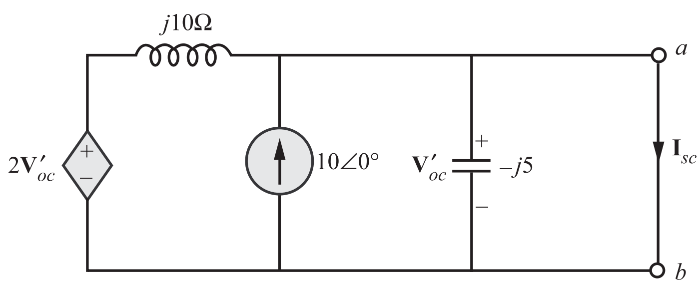


Figure 3.108 (a)

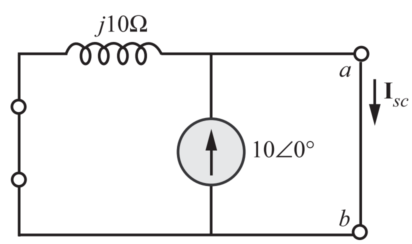


Figure 3.108 (b)

Since $\mathbf{V}_{oc} = 0$, the above circuit takes the form shown in Fig 3.108 (b).

$$\mathbf{I}_{sc} = 10 \underline{/0^\circ} \text{ A}$$

Hence,

$$\mathbf{Z}_t = \frac{\mathbf{V}_{oc}}{\mathbf{I}_{sc}} = \frac{\frac{100}{3} \underline{/-90^\circ}}{10 \underline{/0^\circ}} = \frac{10}{3} \underline{/-90^\circ} \ \Omega$$

The Thevenin equivalent and the Norton equivalent circuits are as shown below.

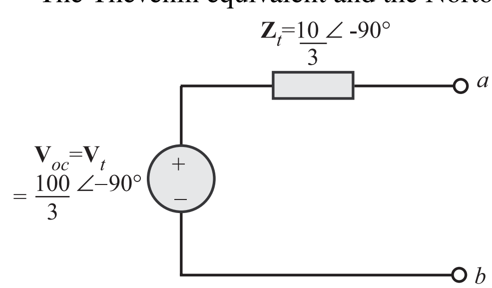


Figure Thevenin equivalent

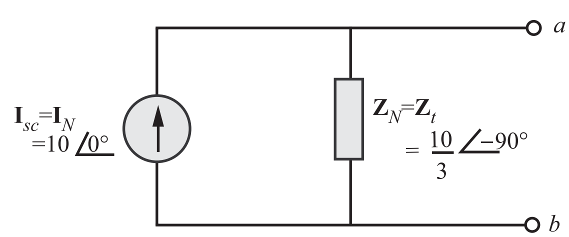


Figure Norton equivalent

EXAMPLE 3.41

Find the Thevenin and Norton equivalent circuits in frequency domain for the network shown in Fig. 3.109.

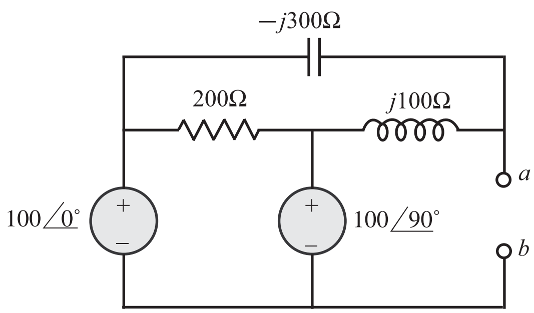
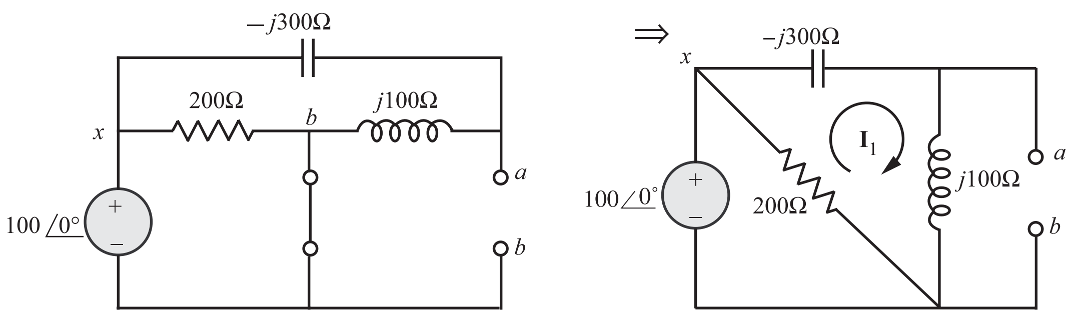


Figure 3.109

SOLUTION

Let us find $\mathbf{V}_t = \mathbf{V}_{ab}$ using superpostion theorem.

(i) $\mathbf{V}_{ab}$ due to $100\,\underline{/0°}$

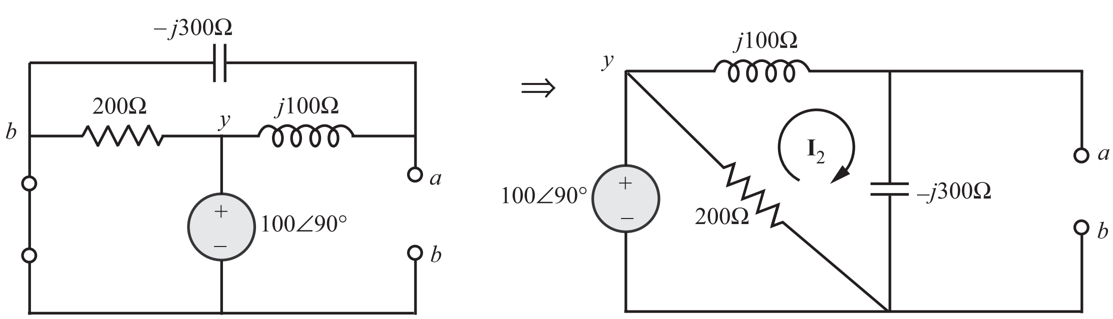


$$\mathbf{I}_1 = \frac{100\,\underline{/0°}}{-j300 + j100} = \frac{100}{-j200}\text{A}$$

$$\mathbf{V}_{ab_1} = \mathbf{I}_1\,(j100)$$

$$= \frac{100}{-j200}\,(j100) = -50\,\underline{/0°}\ \text{Volts}$$

(ii) $\mathbf{V}_{ab}$ due to $100\,\underline{/90°}$



$$\mathbf{I}_2 = \frac{100\,\underline{/90^\circ}}{j100 - j300}$$

$$\mathbf{V}_{ab_2} = \mathbf{I}_2\,(-j300)$$

$$= \frac{100\,\underline{/90^\circ}}{j100 - j300}(-j300) = j150\text{ V}$$

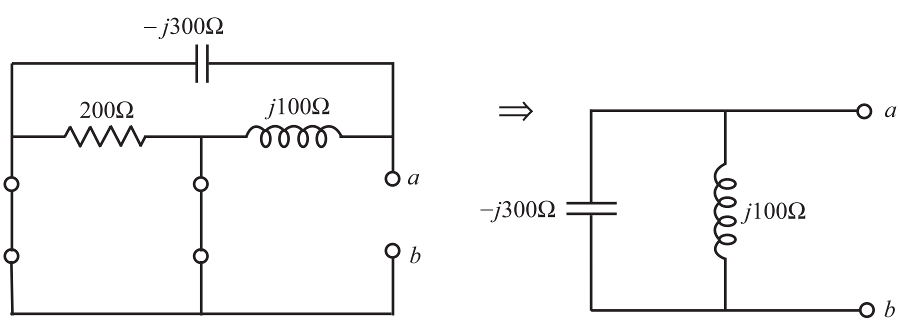
Hence,

$$\mathbf{V}_t = \mathbf{V}_{ab_1} + \mathbf{V}_{ab_2}$$

$$= -50 + j150$$

$$= 158.11\,\underline{/108.43^\circ}\text{ V}$$

To find $\mathbf{Z}_t$, deactivate all the independent sources.



$$\mathbf{Z}_t = j100\ \Omega || - j300\ \Omega$$

$$= \frac{j100(-j300)}{j100 - j300} = j150\ \Omega$$

Hence the Thevenin equivalent circuit is as shown in Fig. 3.109(a). Performing source transformation on the Thevenin equivalent circuit, we get the Norton equivalent circuit.

$$\mathbf{I}_N = \frac{\mathbf{V}_t}{\mathbf{Z}_t} = \frac{158.11\,\underline{/108.43^\circ}}{150\,\underline{/90^\circ}} = 1.054\,\underline{/18.43^\circ}\text{ A}$$

$$\mathbf{Z}_N = \mathbf{Z}_t = j150\ \Omega$$

The Norton equivalent circuit is as shown in Fig. 3.109(b).

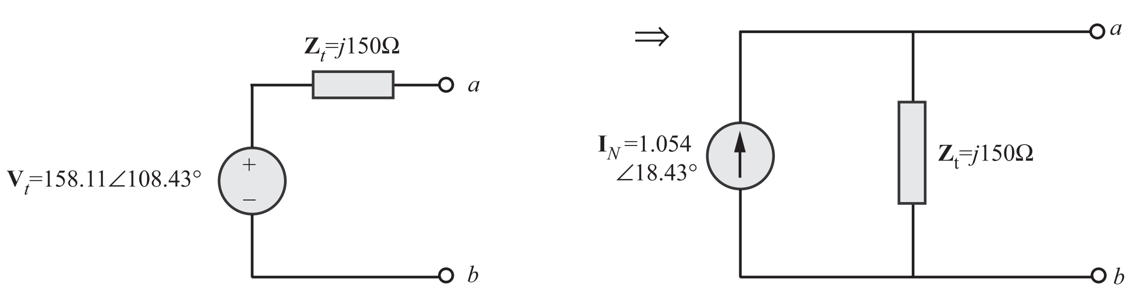


Figure 3.109(a) Figure 3.109(b)

3.6 Maximum power transfer theorem

We have earlier shown that for a resistive network, maximum power is transferred from a source to the load, when the load resistance is set equal to the Thevenin resistance with Thevenin equivalent source. Now we extend this result to the ac circuits.

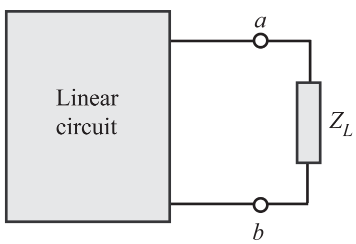


Figure 3.110 Linear circuit

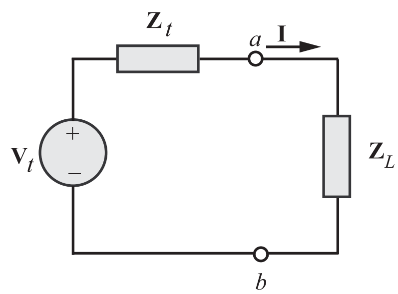


Figure 3.111 Thevenin equivalent circuit

In Fig. 3.110, the linear circuit is made up of impedances, independent and dependent sources. This linear circuit is replaced by its Thevenin equivalent circuit as shown in Fig. 3.111. The load impedance could be a model of an antenna, a TV, and so forth. In rectangular form, the Thevenin impedance $\mathbf{Z}_t$ and the load impedance $\mathbf{Z}_L$ are

$$\mathbf{Z}_t = R_t + jX_t$$

and

$$\mathbf{Z}_L = R_L + jX_L$$

The current through the load is

$$\mathbf{I} = \frac{\mathbf{V}_t}{\mathbf{Z}_t + \mathbf{Z}_L} = \frac{\mathbf{V}_t}{(R_t + jX_t) + (R_L + jX_L)}$$

The phasors $\mathbf{I}$ and $\mathbf{V}_t$ are the maximum values. The corresponding RMS values are obtained by dividing the maximum values by $\sqrt{2}$. Also, the RMS value of phasor current flowing in the load must be taken for computing the average power delivered to the load. The average power delivered to the load is given by

$$\begin{aligned} P &= \frac{1}{2}|\mathbf{I}|^2 R_L \\ &= \frac{|\mathbf{V}_t|^2 \frac{R_L}{2}}{(R_t + R_L)^2 \, (X_t + X_L)^2} \end{aligned} \tag{3.25}$$

Our idea is to adjust the load parameters R_L and X_L so that P is maximum. To do this, we get $\frac{\partial P}{\partial R_L}$ and $\frac{\partial P}{\partial X_L}$ equal to zero.

$$\frac{\partial P}{\partial X_L} = \frac{-|V_t|^2 R_L (X_t + X_L)}{\left[(R_t + R_L)^2 + (X_t + X_L)^2\right]^2}$$

$$\frac{\partial P}{\partial R_L} = \frac{|V_t|^2 \left[(R_t + R_L)^2 + (X_t + X_L)^2 - 2R_L (R_t + R_L)\right]}{2\left[(R_t + R_L)^2 + (X_t + X_L)^2\right]^2}$$

Setting $\dfrac{\partial P}{\partial X_L} = 0$ gives

$$X_L = -X_t \tag{3.26}$$

and Setting $\dfrac{\partial P}{\partial R_L} = 0$ gives

$$R_L = \sqrt{R_t^2 + (X_t + X_L)^2} \tag{3.27}$$

Combining equations (3.26) and (3.27), we can conclude that for maximum average power transfer, $\mathbf{Z}_L$ must be selected such that $X_L = -X_t$ and $R_L = R_t$. That is the maximum average power of a circuit with an impedance $\mathbf{Z}_t$ that is obtained when $\mathbf{Z}_L$ is set equal to complex conjugate of $\mathbf{Z}_t$.

Setting $R_L = R_t$ and $X_L = -X_t$ in equation (3.25), we get the maximum average power as

$$P = \frac{|V_t|^2}{8R_t}$$

In a situation where the load is purely real, the condition for maximum power transfer is obtained by putting $X_L = 0$ in equation (3.27). That is,

$$R_L = \sqrt{R_t^2 + X_t^2} = |\mathbf{Z}_t|$$

Hence for maximum average power transfer to a purely resistive load, the load resistance is equal to the magnitude of Thevenin impedance.

3.6.1 Maximum Power Transfer When Z is Restricted

Maximum average power can be delivered to $\mathbf{Z}_L$ only if $\mathbf{Z}_L = \mathbf{Z}_t^*$. There are few situations in which this is not possible. These situations are described below :

(i) R_L and X_L may be restricted to a limited range of values. With this restriction, choose X_L as close as possible to $-X_t$ and then adjust R_L as close as possible to $\sqrt{R_t^2 + (X_L + X_t)^2}$.

(ii) Magnitude of $\mathbf{Z}_L$ can be varied but its phase angle cannot be. Under this restriction, greatest amount of power is transferred to the load when $[\mathbf{Z}_L] = |\mathbf{Z}_t|$.

Z_t^* is the complex conjugate of Z_t.

EXAMPLE 3.42

Find the load impedance that transfers the maximum power to the load and determine the maximum power quantity obtained for the circuit shown in Fig. 3.112.

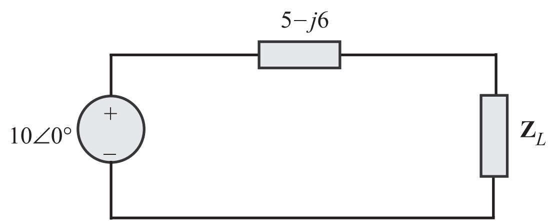
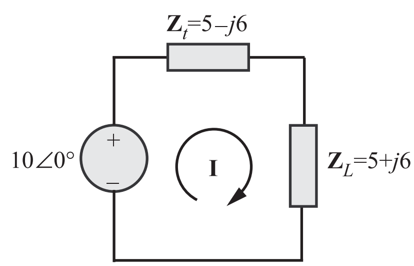


Figure 3.112

SOLUTION

We select, $\mathbf{Z}_L = \mathbf{Z}_t^*$ for maximum power transfer.



$$\text{Hence} \quad \mathbf{Z}_L = 5 + j6$$

$$\mathbf{I} = \frac{10 \,\underline{/0^\circ}}{5+5} = 1 \,\underline{/0^\circ}$$

Hence, the maximum average power transfered to the load is

$$P = \frac{1}{2}|\mathbf{I}|^2 R_L$$

$$= \frac{1}{2}(1)^2 \times 5 = \mathbf{2.5\ W}$$

EXAMPLE 3.43

Find the load impedance that transfers the maximum average power to the load and determine the maximum average power transferred to the load $\mathbf{Z}_L$ shown in Fig. 3.113.

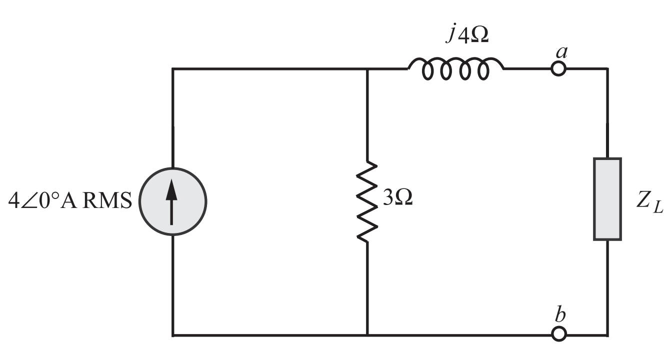


Figure 3.113

SOLUTION

The first step in the analysis is to find the Thevenin equivalent circuit by disconnecting the load $\mathbf{Z}_L$. This leads to a circuit diagram as shown in Fig. 3.114.

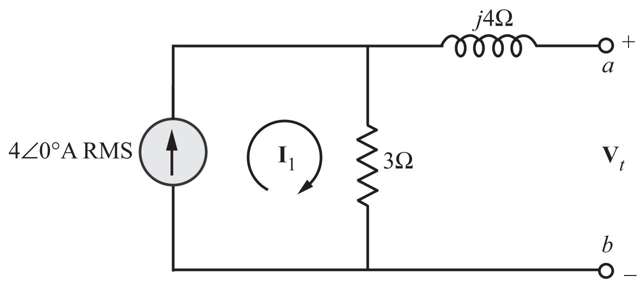


Figure 3.114

Hence
$$\mathbf{V}_t = \mathbf{V}_{oc} = 4\,\underline{/0^\circ} \times 3$$
$$= 12\,\underline{/0^\circ}\ \text{Volts(RMS)}$$

To find $\mathbf{Z}_t$, let us deactivate all the independent sources of Fig. 3.114. This leads to a circuit diagram as shown in Fig 3.114 (a):

$$\mathbf{Z}_t = 3 + j4\ \Omega$$

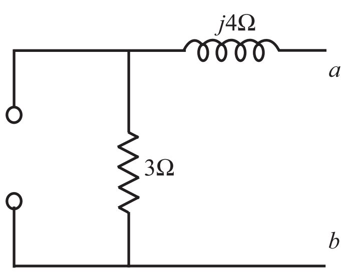


Figure 3.114 (a)

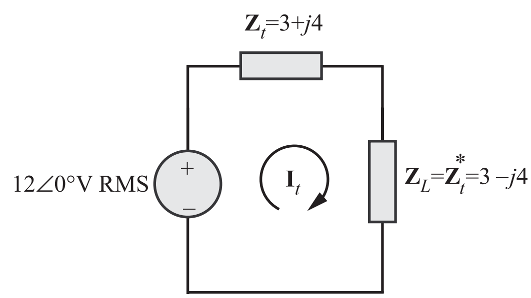


Figure 3.115

The Thevenin equivalent circuit with $\mathbf{Z}_L$ is as shown in Fig. 3.115.
For maximum average power transfer to the load, $\mathbf{Z}_L = \mathbf{Z}_t^* = 3 - j4$.

$$\mathbf{I}_t = \frac{12\,\underline{/0^\circ}}{3 + j4 + 3 - j4} = 2\,\underline{/0^\circ}\ \text{A(RMS)}$$

Hence, maximum average power delivered to the load is

$$P = |I_t|^2 R_L = 4(3) = 12\ \text{W}$$

It may be noted that the scaling factor $\frac{1}{2}$ is not taken since the phase current is already expressed by its RMS value.

EXAMPLE 3.44

Refer the circuit given in Fig. 3.116. Find the value of R_L that will absorb the maximum average power.

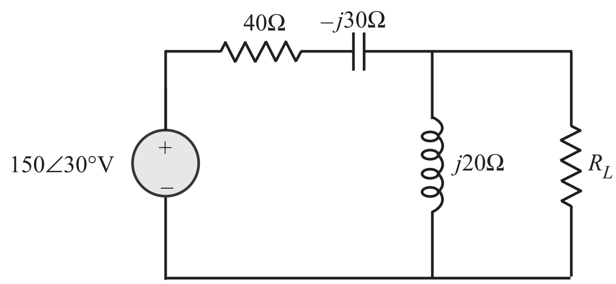


Figure 3.116

SOLUTION

Disconnecting the load resistor R_L from the original circuit diagram leads to a circuit diagram as shown in Fig. 3.117.

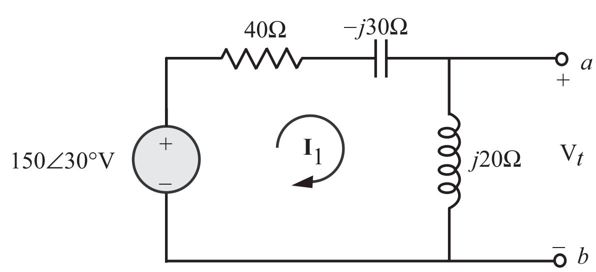


Figure 3.117

$$\begin{aligned}\mathbf{V}_t = \mathbf{V}_{oc} &= \mathbf{I}_1\,(j20)\\ &= \frac{150\,\underline{/30^\circ}\ \times j20}{(40 - j30 + j20)}\\ &= 72.76\,\underline{/134^\circ}\ \text{Volts.}\end{aligned}$$

To find $\mathbf{Z}_t$, let us deactivate all the independent sources present in Fig. 3.117 as shown in Fig 3.117 (a).

$$\begin{aligned}\mathbf{Z}_t &= (40 - j30)\,||j20\\ &= \frac{j20\,(40 - j30)}{j20 + 40 - j30} = (9.412 + j22.35)\ \Omega\end{aligned}$$

The Value of R_L that will absorb the maximum average power is

$$R_L = |\mathbf{Z}_t| = \sqrt{(9.412)^2 + (22.35)^2}$$
$$= 24.25\ \Omega$$

The Thevenin equivalent circuit with R_L inserted is as shown in Fig 3.117 (b).
Maximum average power absorbed by R_L is

$$P_{\max} = \frac{1}{2}\,|I_t|^2\, R_L$$

where $$\mathbf{I}_t = \frac{72.76\,\underline{/134^\circ}}{(9.412 + j22.35 + 24.25)}$$
$$= 1.8\,\underline{/100.2^\circ}\ \text{A}$$

$$\Rightarrow \quad P_{\max} = \frac{1}{2}(1.8)^2 \times 24.25$$
$$= \mathbf{39.29\ W}$$

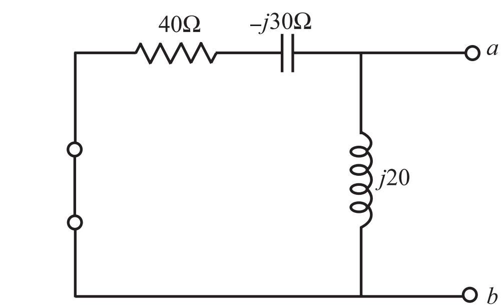


Figure 3.117 (a)

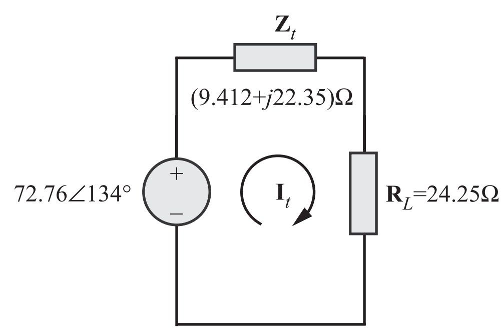


Figure 3.117 (b) Thevenin equivalent circuit

EXAMPLE 3.45

For the circuit of Fig. 3.118: (a) what is the value of Z_L that will absorb the maximum average power? (b) what is the value of maximum power?

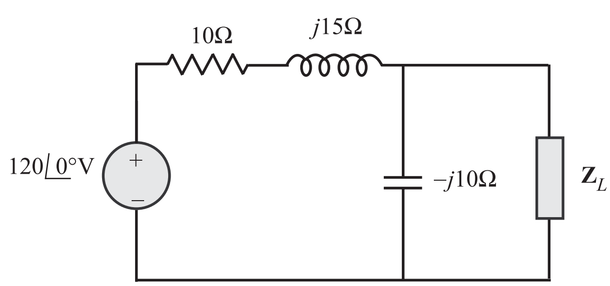


Figure 3.118

SOLUTION

Disconnecting $\mathbf{Z}_L$ from the original circuit we get the circuit as shown in Fig. 3.119. The first step is to find $\mathbf{V}_t$.

$$\mathbf{V}_t = \mathbf{V}_{oc} = \mathbf{I}_1 (-j10)$$

$$= \left[\frac{120 \underline{/0^\circ}}{10 + j15 - j10}\right](-j10)$$

$$= 107.33 \underline{/-116.57^\circ} \;\; \text{V}$$

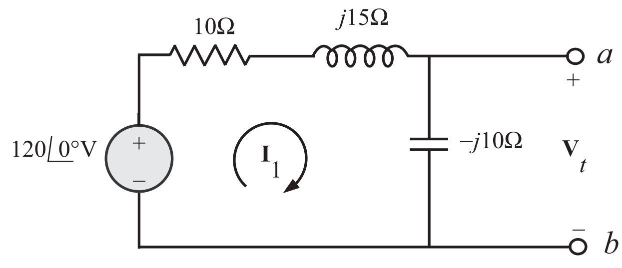
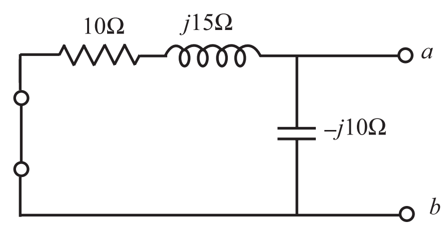


Figure 3.119

The next step is to find $\mathbf{Z}_t$. This requires deactivating the independent voltage source of Fig. 3.119.



$$\mathbf{Z}_t = (10 + j15) \,||\, (-j10)$$

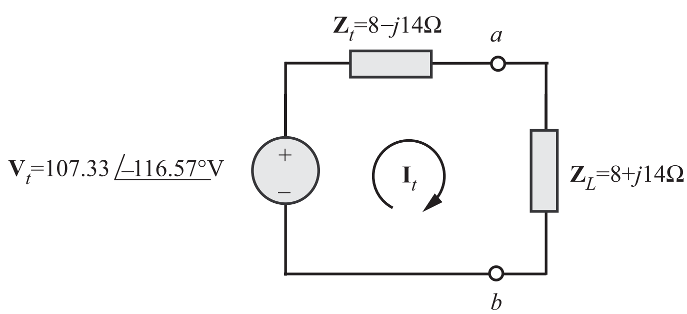
$$= \frac{-j10\,(10 + j15)}{-j10 + 10 + j15}$$

$$= 8 - j14\,\Omega$$

The value of $\mathbf{Z}_L$ for maximum average power absorbed is

$$\mathbf{Z}_t^* = 8 + j14\,\Omega$$

The Thevenin equivalent circuit along with $\mathbf{Z}_L = 8 + j14\,\Omega$ is as shown below:



$$\mathbf{I}_t = \frac{107.33 \underline{/-116.57^\circ}}{8 - j14 + 8 + j14}$$

$$= \frac{107.33}{16} \underline{/-116.57^\circ} \; A$$

Hence,

$$P_{\max} = \frac{1}{2}\,|I_t|^2\, R_L$$

$$= \frac{1}{2}\left(\frac{107.33}{16}\right)^2 \times 8$$

$$= \mathbf{180\ Walts}$$

EXAMPLE 3.46

(a) For the circuit shown in Fig. 3.120, what is the value of $\mathbf{Z}_L$ that results in maximum average power that will be transferred to $\mathbf{Z}_L$? What is the maximum power ?

(b) Assume that the load resistance can be varied between 0 and 4000 Ω and the capacitive reactance of the load can be varied between 0 and −2000 Ω. What settings of R_L and X_C transfer the most average power to the load ? What is the maximum average power that can be transferred under these conditions?

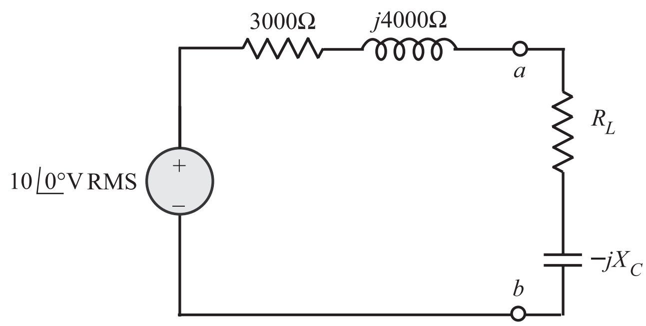


Figure 3.120

SOLUTION

(a) If there are no constraints on R_L and X_L, the load indepedance $\mathbf{Z}_L = \mathbf{Z}_t^* = (3000 - j4000)\ \Omega$.

Since the voltage source is given in terms of its RMS value, the average maximum power delivered to the load is

$$P_{\text{max}} = |\mathbf{I}_t|^2 R_L$$

where
$$\mathbf{I}_t = \frac{10\ \underline{/0^\circ}}{3000 + j4000 + 3000 - j4000}$$
$$= \frac{10}{2 \times 3000}\text{A}$$

$\Rightarrow$
$$P_{\text{max}} = |\mathbf{I}_t|^2 R_L$$
$$= \frac{100}{4 \times (3000)^2} \times 3000$$
$$= \mathbf{8.33\ mW}$$

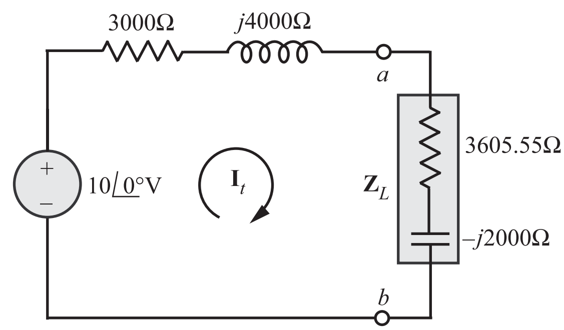
(b) Since R_L and X_C are restricted, we first set X_C as close to $-4000\ \Omega$ as possible; hence $X_C = -2000\ \Omega$. Next we set R_L as close to $\sqrt{R_t^2 + (X_C + X_L)^2}$ as possible.

Thus,
$$R_L = \sqrt{3000^2 + (-2000 + 4000)^2} = 3605.55\ \Omega$$

Since R_L can be varied between 0 to 4000 Ω, we can set R_L to 3605.55 Ω. Hence $\mathbf{Z}_L$ is adjusted to a value

$$\mathbf{Z}_L = 3605.55 - j2000\ \Omega.$$

$$\mathbf{I}_t = \frac{10\,\underline{/0^\circ}}{3000 + j4000 + 3605.55 - j2000}$$
$$= 1.4489\,\underline{/-16.85^\circ}\ \text{mA}$$



The maximum average power delivered to the load is

$$P_{\max} = |\mathbf{I}_t|^2\, R_L$$
$$= \left(1.4489 \times 10^{-3}\right)^2 \times 3605.55$$
$$= \mathbf{7.57\ mW}$$

Note that this is less than the power that can be delivered if there are no constraints on R_L and X_L.

EXAMPLE 3.47

A load impedance having a constant phase angle of -45° is connected across the load terminals a and b in the circuit shown in Fig. 3.121. The magnitude of $\mathbf{Z}_L$ is varied until the average power delivered, which is the maximum possible under the given restriction.

(a) Specify $\mathbf{Z}_L$ in rectangular form.

(b) Calculate the maximum average power delivered under this condition.

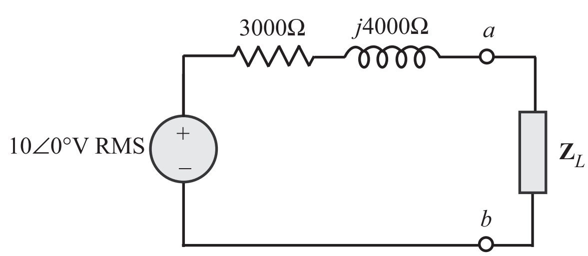


Figure 3.121

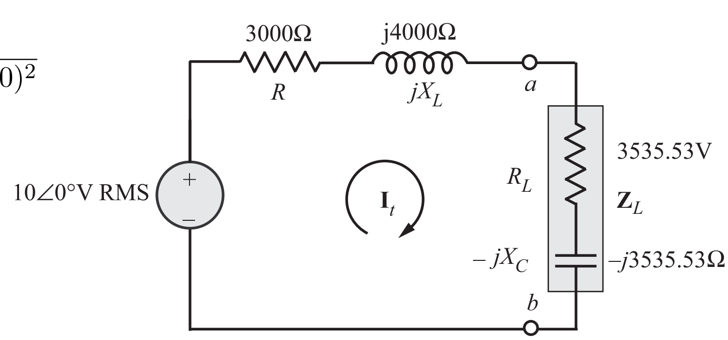
SOLUTION

Since the phase angle of $\mathbf{Z}_L$ is fixed at -45°, for maximum power transfer to $\mathbf{Z}_L$ it is mandatory that

$$|\mathbf{Z}_L| = |\mathbf{Z}_t|$$
$$= \sqrt{(3000)^2 + (4000)^2}$$
$$= 5000\ \Omega.$$

Hence, $\mathbf{Z}_L = |\mathbf{Z}_L|\,\underline{/-45^\circ}$

$$= \frac{5000}{\sqrt{2}} - j\frac{5000}{\sqrt{2}}$$



$$\begin{aligned} \mathbf{I}_t &= \frac{10\,\underline{/0^\circ}}{(3000+3535.53)+j(4000-3535.53)} \\ &= 1.526\,\underline{/-4.07^\circ}\ \text{mA} \\ P_{\max} &= |\mathbf{I}_t|^2\, R_L \\ &= \left(1.526\times10^{-3}\right)^2 \times 3535.53 \\ &= \mathbf{8.23\ mW} \end{aligned}$$

This power is the maximum average power that can be delivered by this circuit to a load impedance whose angle is constant at -45°. Again this quantity is less than the maximum power that could have been delivered if there is no restriction on $\mathbf{Z}_L$. In example 3.46 part (a), we have shown that the maximum power that can be delivered without any restrictions on $\mathbf{Z}_L$ is 8.33 mW.

3.7 Reciprocity theorem

The reciprocity theorem states that in a linear bilateral single source circuit, the ratio of excitation to response is constant when the positions of excitation and response are interchanged.

Conditions to be met for the application of reciprocity theorem :

(i) The circuit must have a single source.

(ii) Initial conditions are assumed to be absent in the circuit.

(iii) Dependent sources are excluded even if they are linear.

(iv) When the positions of source and response are interchanged, their directions should be marked same as in the original circuit.

EXAMPLE 3.48

Find the current in 2 Ω resistor and hence verify reciprocity theorem.

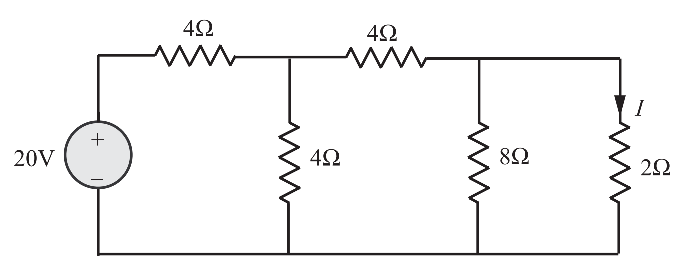


Figure 3.122

SOLUTION

The circuit is redrawn with markings as shown in Fig 3.123 (a).

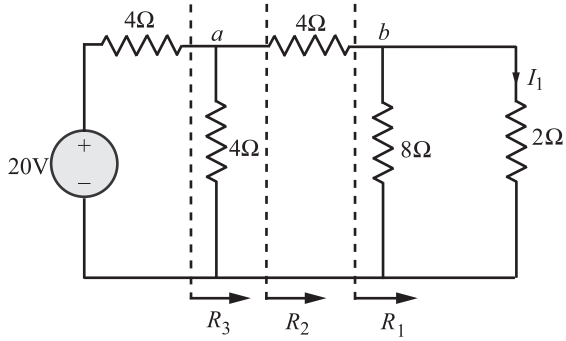


Figure 3.123 (a)

Then,
$$R_1 = (8^{-1} + 2^{-1})^{-1} = 1.6\Omega$$
$$R_2 = 1.6 + 4 = 5.6\Omega$$
$$R_3 = (5.6^{-1} + 4^{-1})^{-1} = 2.3333\Omega$$

Current supplied by the source $= \dfrac{20}{4 + 2.3333} = 3.16$ A

Current in branch $ab = I_{ab} = 3.16 \times \dfrac{4}{4 + 4 + 1.6} = 1.32$ A

Current in $2\Omega, I_1 = 1.32 \times \dfrac{8}{10} = 1.05$ A

Verification using reciprocity theorem

The circuit is redrawn by interchanging the position of excitation and response as shown in Fig 3.123 (b).

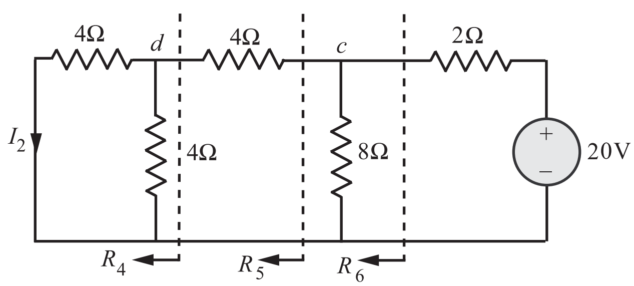


Figure 3.123 (b)

Solving the equivalent resistances,

$$R_4 = 2\Omega, \quad R_5 = 6\Omega, \quad R_6 = 3.4286\Omega$$

Now the current supplied by the source

$$= \frac{20}{3.4286 + 2} = 3.6842\text{A}$$

Therefore,

$$I_{cd} = 3.6842 \times \frac{8}{8+6} = 2.1053\text{A}$$
$$I_2 = \frac{2.1053}{2} = 1.05\text{A}$$

As $I_1 = I_2 = 1.05$ A, reciprocity theorem is verified.

EXAMPLE 3.49

In the circuit shown in Fig. 3.124, find the current through $1.375\ \Omega$ resistor and hence verify reciprocity theorem.

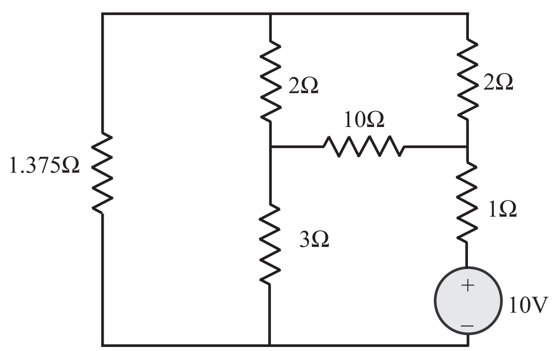


Figure 3.124

SOLUTION

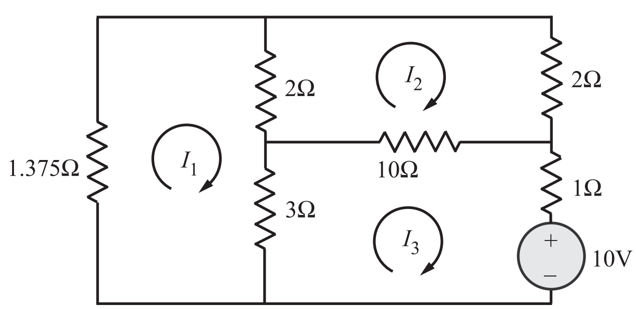


Figure 3.125

KVL clockwise for mesh 1 :

$$6.375I_1 - 2I_2 - 3I_3 = 0$$

KVL clockwise for mesh 2 :

$$-2I_1 + 14I_2 - 10I_3 = 0$$

KVL clockwise for mesh 3 :

$$-3I_1 - 10I_2 + 14I_3 = -10$$

Putting the above three mesh equations in matrix form, we get

$$\begin{bmatrix} 6.375 & -2 & -3 \\ -2 & 14 & -10 \\ -3 & -10 & 14 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -10 \end{bmatrix}$$

Using Cramer's rule, we get

$$I_1 = -2\text{A}$$

Negative sign indicates that the assumed direction of current flow should have been the other way.

Verification using reciprocity theorem :

The circuit is redrawn by interchanging the positions of excitation and response. The new circuit is shown in Fig. 3.126.

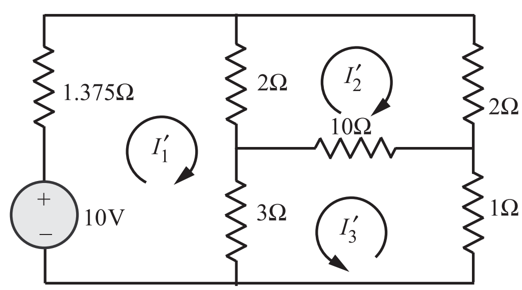


Figure 3.126

The mesh equations in matrix form for the circuit shown in Fig. 3.126 is

$$\begin{bmatrix} 6.375 & -2 & 3 \\ -2 & 14 & 10 \\ 3 & 10 & 14 \end{bmatrix} \begin{bmatrix} I_1' \\ I_2' \\ I_3' \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}$$

Using Cramer's rule, we get

$$I_3' = -2 \text{ A}$$

Since $I_1 = I_3' = -2$ A, the reciprocity theorem is verified.

EXAMPLE 3.50

Find the current $\mathbf{I}_x$ in the $j2\ \Omega$ impedance and hence verify reciprocity theorem.

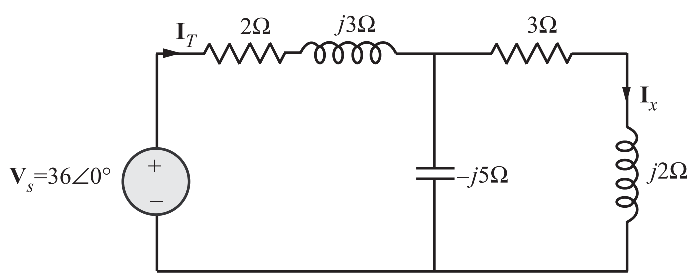


Figure 3.127

SOLUTION

With reference to the Fig. 3.127, the current through $j2\,\Omega$ impepance is found using series−parallel reduction techniques.

Total impedance of the circuit is

$$\begin{aligned}\mathbf{Z}_T &= (2+j3)+(-j5)||(3+j2)\\ &= 2+j3+\frac{(-j5)(3+j2)}{-j5+3+j2}\\ &= 6.537\,\underline{/19.36^\circ}\ \ \Omega\end{aligned}$$

The total current in the network is

$$\begin{aligned}\mathbf{I}_T &= \frac{36\,\underline{/0^\circ}}{6.537\,\underline{/19.36^\circ}}\\ &= 5.507\,\underline{/-19.36^\circ}\ \ \mathrm{A}\end{aligned}$$

Using the principle of current division, we find that

$$\begin{aligned}\mathbf{I}_x &= \frac{\mathbf{I}_T\,(-j5)}{-j5+3+j2}\\ &= 6.49\,\underline{/-64.36^\circ}\ \ \mathrm{A}\end{aligned}$$

Verification of reciprocity theorem :

The circuit is redrawn by changing the positions of excitation and response. This circuit is shown in Fig. 3.128.

Total impedance of the circuit shown in Fig. 3.128 is

$$\begin{aligned}\mathbf{Z}'_T &= (3+j2)+(2+j3)\,||\,(-j5)\\ &= (3+j2)+\frac{(2+j3)\,(-j5)}{2+j3-j5}\\ &= 9.804\,\underline{/19.36^\circ}\ \ \Omega\end{aligned}$$

The total current in the circuit is

$$\mathbf{I}'_T = \frac{36\,\underline{/0^\circ}}{Z'_T} = 3.672\,\underline{/-19.36^\circ}\ \ \mathrm{A}$$

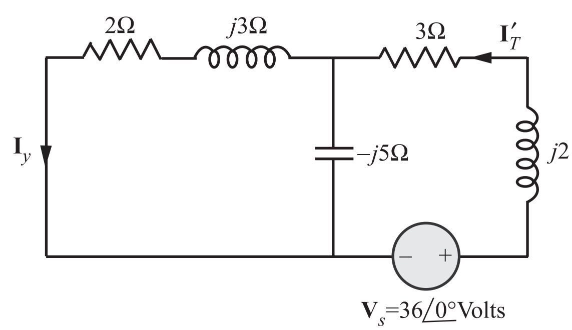


Figure 3.128

Using the principle of current division,

$$\mathbf{I}_y = \frac{\mathbf{I}'_T\,(-j5)}{-j5+2+j3} = 6.49\,\underline{/-64.36^\circ}\ \ \mathrm{A}$$

It is found that $\mathbf{I}_x = \mathbf{I}_y$, thus verifying the reciprocity theorem.

EXAMPLE 3.51

Refer the circuit shown in Fig. 3.129. Find current through the ammeter, and hence verify reciprocity theorem.

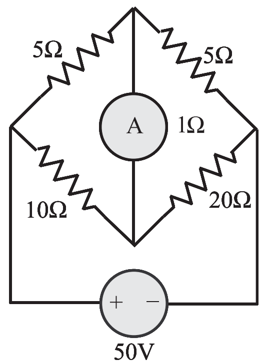


Figure 3.129

SOLUTION

To find the current through the ammeter :

By inspection the loop equations for the circuit in Fig. 3.130 can be written in the matrix form as

$$\begin{bmatrix} 16 & -1 & -10 \\ -1 & 26 & -20 \\ -10 & -20 & 30 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 50 \end{bmatrix}$$

Using Cramer's rule, we get

$$I_1 = 4.6 \text{ A}$$
$$I_2 = 5.4 \text{ A}$$

Hence current through the ammeter $= I_2 - I_1 = 5.4 - 4.6 = 0.8$A.

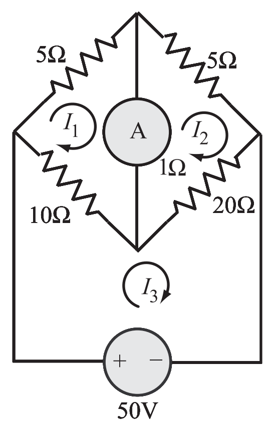


Figure 3.130

Verification of reciprocity theorem:

The circuit is redrawn by interchanging the positions of excitation and response as shown in Fig. 3.131.

By inspection the loop equations for the circuit can be written in matrix form as

$$\begin{bmatrix} 15 & 0 & -10 \\ 0 & 25 & -20 \\ -10 & -20 & 31 \end{bmatrix} \begin{bmatrix} I_1' \\ I_2' \\ I_3' \end{bmatrix} = \begin{bmatrix} -50 \\ 50 \\ 0 \end{bmatrix}$$

Using Cramer's rule we get

$$I_3' = 0.8 \text{ A}$$

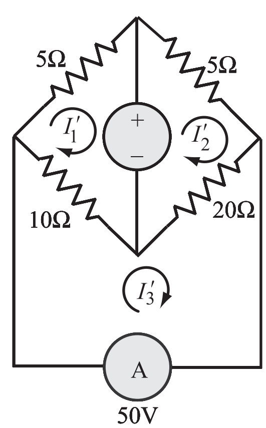


Figure 3.131

Hence, current through the Ammeter = 0.8 A.

It is found from both the cases that the response is same. Hence the reciprocity theorem is verified.

EXAMPLE 3.52

Find current through 5 ohm resistor shown in Fig. 3.132 and hence verify reciprocity theorem.

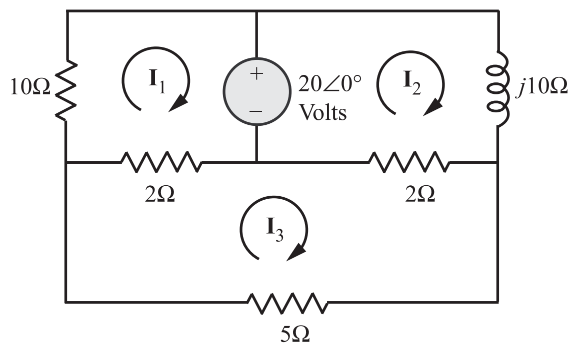


Figure 3.132

SOLUTION

By inspection, we can write

$$\begin{bmatrix} 12 & 0 & -2 \\ 0 & 2+j10 & -2 \\ -2 & -2 & 9 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \mathbf{I}_3 \end{bmatrix} = \begin{bmatrix} -20 \\ 20 \\ 0 \end{bmatrix}$$

Using Cramer's rule, we get

$$\mathbf{I}_3 = 0.5376\,\underline{/-126.25^\circ}\ \text{A}$$

Hence, current through 5 ohm resistor $= 0.5376\,\underline{/-126.25^\circ}\ \text{A}$

Verification of reciprocity theorem:

The original circuit is redrawn by interchanging the excitation and response as shown in Fig. 3.133.

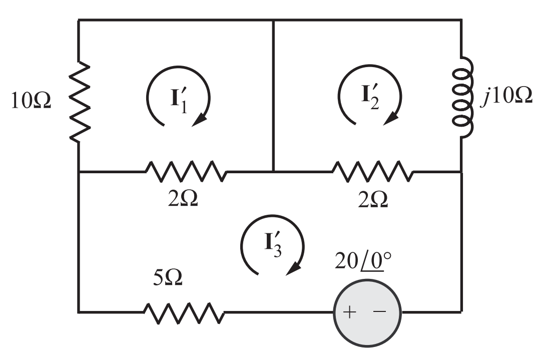


Figure 3.133

Putting the three equations in matrix form, we get

$$\begin{bmatrix} 12 & 0 & -2 \\ 0 & 2+j10 & -2 \\ -2 & -2 & 9 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1' \\ \mathbf{I}_2' \\ \mathbf{I}_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 20 \end{bmatrix}$$

Using Cramer's rule, we get

$$\mathbf{I}_1' = 0.3876 \underline{/-2.35} \text{ A}$$
$$\mathbf{I}_2' = 0.456 \underline{/-78.9^\circ} \text{ A}$$

Hence,

$$\mathbf{I}_2' - \mathbf{I}_1' = -0.3179 - j0.4335$$
$$= 0.5376 \underline{/-126.25^\circ} \text{ A}$$

The response in both cases remains the same. Thus verifying reciprocity theorem.

3.8 Millman's theorem

It is possible to combine number of voltage sources or current sources into a single equivalent voltage or current source using Millman's theorem. Hence, this theorem is quite useful in calculating the total current supplied to the load in a generating station by a number of generators connected in parallel across a busbar.

Millman's theorem states that if n number of generators having generated emfs $\mathbf{E}_1, \mathbf{E}_2, \cdots \mathbf{E}_n$ and internal impedances $\mathbf{Z}_1, \mathbf{Z}_2, \cdots \mathbf{Z}_n$ are connected in parallel, then the emfs and impedances can be combined to give a single equivalent emf of $\mathbf{E}$ with an internal impedance of equivalent value $\mathbf{Z}$.

where

$$\mathbf{E} = \frac{\mathbf{E}_1\mathbf{Y}_1 + \mathbf{E}_2\mathbf{Y}_2 + \ldots + \mathbf{E}_n\mathbf{Y}_n}{\mathbf{Y}_1 + \mathbf{Y}_2 + \ldots + \mathbf{Y}_n}$$

and

$$\mathbf{Z} = \frac{1}{\mathbf{Y}_1 + \mathbf{Y}_2 + \ldots + \mathbf{Y}_n}$$

where $\mathbf{Y}_1, \mathbf{Y}_2 \cdots \mathbf{Y}_n$ are the admittances corresponding to the internal impedances $\mathbf{Z}_1, \mathbf{Z}_2 \cdots \mathbf{Z}_n$ and are given by

$$\mathbf{Y}_1 = \frac{1}{\mathbf{Z}_1}$$
$$\mathbf{Y}_2 = \frac{1}{\mathbf{Z}_2}$$
$$\vdots$$
$$\mathbf{Y}_n = \frac{1}{\mathbf{Z}_n}$$

Fig. 3.134 shows a number of generators having emfs $\mathbf{E}_1, \mathbf{E}_2 \cdots \mathbf{E}_n$ connected in parallel across the terminals x and y. Also, $\mathbf{Z}_1, \mathbf{Z}_2 \cdots \mathbf{Z}_n$ are the respective internal impedances of the generators.

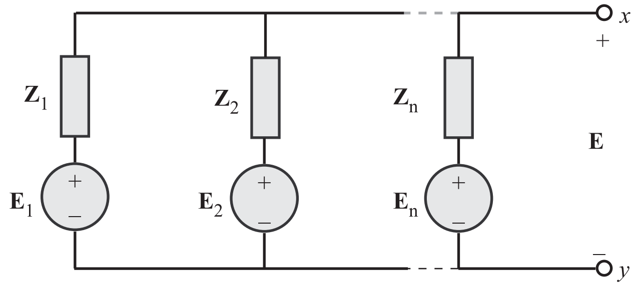


Figure 3.134

The Thevenin equivalent circuit of Fig. 3.134 using Millman's theorem is shown in Fig. 3.135. The nodal equation at x gives

$$\frac{\mathbf{E}_1-\mathbf{E}}{\mathbf{Z}_1}+\frac{\mathbf{E}_2-\mathbf{E}}{\mathbf{Z}_2}+\cdots+\frac{\mathbf{E}_n-\mathbf{E}}{\mathbf{Z}_n}=0$$

$$\Rightarrow \quad \left[\frac{\mathbf{E}_1}{\mathbf{Z}_1}+\frac{\mathbf{E}_2}{\mathbf{Z}_2}+\cdots+\frac{\mathbf{E}_n}{\mathbf{Z}_n}\right]=\mathbf{E}\left[\frac{1}{\mathbf{Z}_1}+\frac{1}{\mathbf{Z}_2}+\cdots+\frac{1}{\mathbf{Z}_n}\right]$$

$$\Rightarrow \quad \mathbf{E}_1\mathbf{Y}_1+\mathbf{E}_2\mathbf{Y}_2+\cdots+\mathbf{E}_n\mathbf{Y}_n=\mathbf{E}\left[\frac{1}{\mathbf{Z}}\right]$$

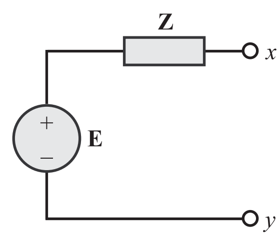


Figure 3.135

where **Z** = Equivalent internal impedance.

or $$[\mathbf{E}_1\mathbf{Y}_1+\mathbf{E}_2\mathbf{Y}_2+\cdots+\mathbf{E}_n\mathbf{Y}_n]=\mathbf{E}\mathbf{Y}$$

$$\Rightarrow \quad \mathbf{E}=\frac{\mathbf{E}_1\mathbf{Y}_1+\mathbf{E}_2\mathbf{Y}_2+\cdots+\mathbf{E}_n\mathbf{Y}_n}{\mathbf{Y}}$$

where $$\mathbf{Y}=\mathbf{Y}_1+\mathbf{Y}_2+\cdots+\mathbf{Y}_n$$

and $$\mathbf{Z}=\frac{1}{\mathbf{Y}}=\frac{1}{\mathbf{Y}_1+\mathbf{Y}_2+\cdots+\mathbf{Y}_n}$$

EXAMPLE 3.53

Refer the circuit shown in Fig. 3.136. Find the current through 10 Ω resistor using Millman's theorem.

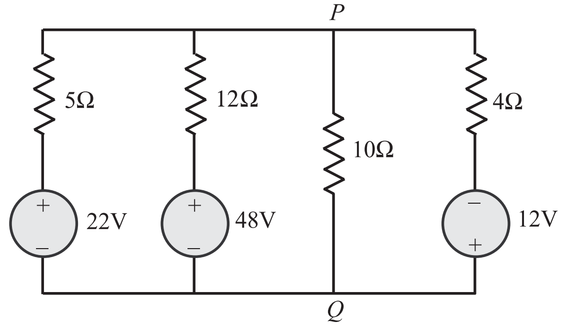


Figure 3.136

SOLUTION

Using Millman's theorem, the circuit shown in Fig. 3.136 is replaced by its Thevenin equivalent circuit across the terminals PQ as shown in Fig. 3.137.

$$\mathbf{E} = \frac{\mathbf{E}_1\mathbf{Y}_1 + \mathbf{E}_2\mathbf{Y}_2 - \mathbf{E}_3\mathbf{Y}_3}{\mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3}$$

$$= \frac{22\left(\frac{1}{5}\right) + 48\left(\frac{1}{12}\right) - 12\left(\frac{1}{4}\right)}{\frac{1}{5} + \frac{1}{12} + \frac{1}{4}}$$

$$= 10.13 \text{ Volts}$$

$$R = \frac{1}{\mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3}$$

$$= \frac{1}{0.2 + 0.083 + 0.25}$$

$$= 1.88\ \Omega$$

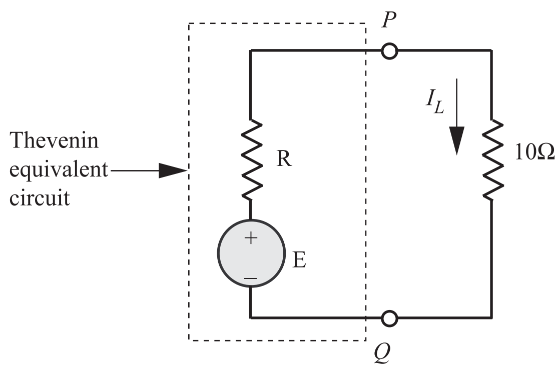


Figure 3.137

Hence, $$I_L = \frac{E}{R + 10} = 0.853 \text{ A}$$

EXAMPLE 3.54

Find the current through $(10 - j3)\Omega$ using Millman's theorem. Refer Fig. 3.138.

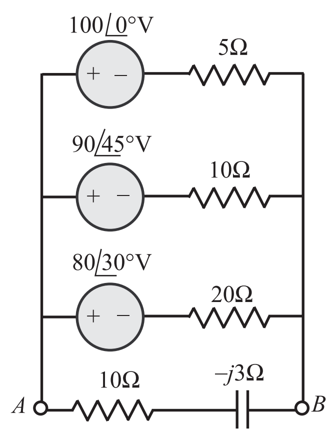


Figure 3.138

SOLUTION

The circuit shown in Fig. 3.138 is replaced by its Thevenin equivalent circuit as seen from the terminals, A and B using Millman's theorem. Fig. 3.139 shows the Thevenin equivalent circuit along with $\mathbf{Z}_L = 10 - j3\ \Omega$.

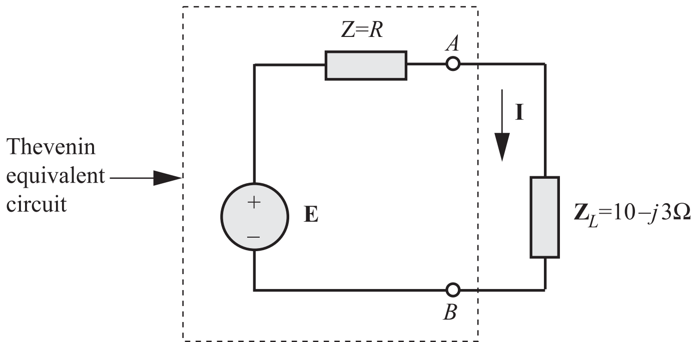


Figure 3.139

$$\mathbf{E} = \frac{\mathbf{E}_1\mathbf{Y}_1 + \mathbf{E}_2\mathbf{Y}_2 - \mathbf{E}_3\mathbf{Y}_3}{\mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3}$$

$$= \frac{100\,\underline{/0^\circ}\left(\frac{1}{5}\right) + 90\,\underline{/45^\circ}\left(\frac{1}{10}\right) + 80\,\underline{/30^\circ}\left(\frac{1}{20}\right)}{\frac{1}{5} + \frac{1}{10} + \frac{1}{20}}$$

$$= 88.49\,\underline{/15.66^\circ}\text{ V}$$

$$\mathbf{Z} = R = \frac{1}{\mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3} = \frac{1}{\frac{1}{5} + \frac{1}{10} + \frac{1}{20}} = 2.86\ \Omega$$

$$\mathbf{I} = \frac{\mathbf{E}}{\mathbf{Z} + \mathbf{Z}_L} = \frac{88.49\,\underline{/15.66}}{2.86 + 10 - j3} = 6.7\,\underline{/28.79^\circ}\ \mathbf{A}$$

Alternately,

$$\mathbf{E} = \frac{\mathbf{E}_1\mathbf{Y}_1 + \mathbf{E}_2\mathbf{Y}_2 + \mathbf{E}_3\mathbf{Y}_3 + \mathbf{E}_4\mathbf{Y}_4}{\mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3 + \mathbf{Y}_4}$$

$$= \frac{100 \times 5^{-1} + 90\,\underline{/45^\circ} \times 10^{-1} + 80\,\underline{/30^\circ} \times 20^{-1}}{5^{-1} + 10^{-1} + 20^{-1} + (10 - j3)^{-1}}$$

$$= 70\,\underline{/12^\circ}\text{ V}$$

Therefore,

$$I = \frac{70\,\underline{/12^\circ}}{10 - j3}$$

$$= 6.7\,\underline{/28.8^\circ}\text{ A}$$

EXAMPLE 3.55

Refer the circuit shown in Fig. 3.140. Use Millman's theorem to find the current through $(5+j5)\ \Omega$ impedance.

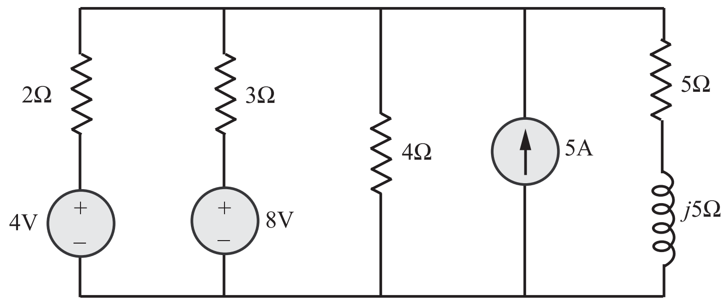


Figure 3.140

SOLUTION

The original circuit is redrawn after performing source transformation of 5 A in parallel with $4\ \Omega$ resistor into an equivalent voltage source and is shown in Fig. 3.141.

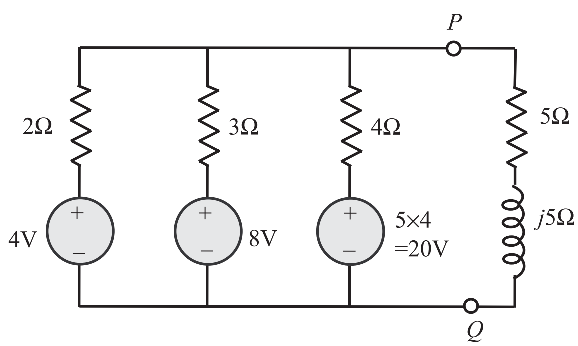


Figure 3.141

Treating the branch $5 + j5\Omega$ as a branch with $\mathbf{E}_s = 0V$,

$$\mathbf{E}_{PQ} = \frac{\mathbf{E}_1\mathbf{Y}_1 + \mathbf{E}_2\mathbf{Y}_2 + \mathbf{E}_3\mathbf{Y}_3 + \mathbf{E}_4\mathbf{Y}_4}{\mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3 + \mathbf{Y}_4}$$
$$= \frac{4 \times 2^{-1} + 8 \times 3^{-1} + 20 \times 4^{-1}}{2^{-1} + 3^{-1} + 4^{-1} + (5 - j5)^{-1}}$$
$$= 8.14 \angle 4.83^\circ \text{ V}$$

Therefore current in $(5 + j5)\Omega$ is

$$\mathbf{I} = \frac{8.14 \angle 4.83^\circ}{5 + j5} = 1.15 \angle -40.2^\circ \text{ A}$$

Alternately

$\mathbf{E}_{PQ}$ with $(5 + j5)$ open

$$\mathbf{E}_{PQ} = \frac{\mathbf{E}_1\mathbf{Y}_1 + \mathbf{E}_2\mathbf{Y}_2 + \mathbf{E}_3\mathbf{Y}_3}{\mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3}$$
$$= \frac{4 \times 2^{-1} + 8 \times 3^{-1} + 20 \times 4^{-1}}{2^{-1} + 3^{-1} + 4^{-1}}$$
$$= 8.9231\text{V}$$

Equivalent resistance $R = (2^{-1} + 3^{-1} + 4^{-1})^{-1} = 0.9231\Omega$
Therefore current in $(5 + j5)\Omega$ is

$$I = \frac{8.9231}{0.9231 + 5 + j5} = 1.15 \underline{/-40.2^\circ} \text{ A}$$

EXAMPLE 3.56

Find the current through 2 Ω resistor using Millman's theorem. Refer the circuit shown in Fig. 3.142.

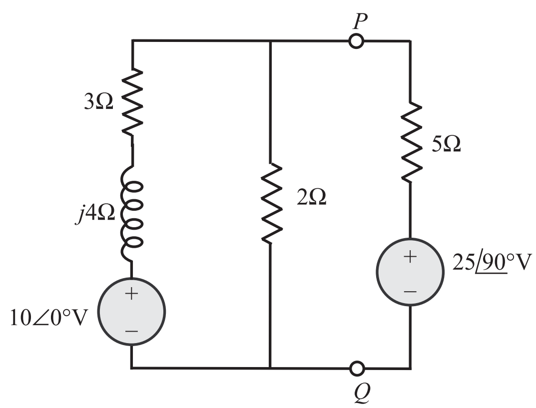


Figure 3.142

SOLUTION

The Thevenin equivalent circuit using Millman's theorem for the given problem is as shown in Fig. 3.142(a).

where
$$\mathbf{E} = \frac{\mathbf{E}_1\mathbf{Y}_1 + \mathbf{E}_2\mathbf{Y}_2}{\mathbf{Y}_1 + \mathbf{Y}_2}$$

$$= \frac{10 \underline{/10^\circ} \left[\dfrac{1}{3 + j4}\right] + 25 \underline{/90^\circ} \left[\dfrac{1}{5}\right]}{\dfrac{1}{3 + j4} + \dfrac{1}{5}}$$

$$= 10.06 \underline{/97.12^\circ} \text{ V}$$

$$\mathbf{Z} = \frac{1}{\mathbf{Y}_1 + \mathbf{Y}_2} = \frac{1}{\dfrac{1}{3 + j4} + \dfrac{1}{5}}$$

$$= 2.8 \underline{/26.56^\circ} \ \Omega$$

Hence,
$$\mathbf{I}_L = \frac{\mathbf{E}}{\mathbf{Z} + 2} = \frac{10.06 \underline{/97.12^\circ}}{2.8 \underline{/26.56^\circ} + 2}$$

$$= 2.15 \underline{/81.63^\circ} \text{ A}$$

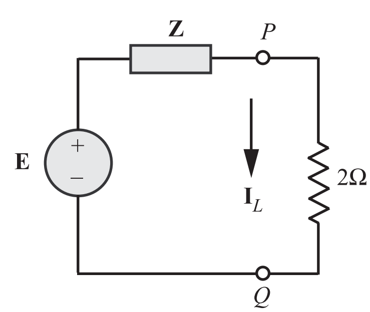


Figure 3.142(a)

Reinforcement problems

R.P 3.1

Find the current in 2 Ω resistor connected between A and B by using superposition theorem.

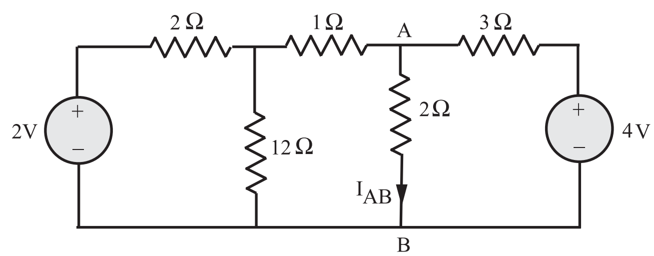


Figure R.P. 3.1

SOLUTION

Fig. R.P. 3.1(a), shows the circuit with 2V-source acting alone (4V-source is shorted).

Resistance as viewed from 2V-source is $2 + R_1\ \Omega$,

$$\text{where} \quad R_1 = \left(\frac{3 \times 2}{5} + 1\right) \Big\| 12$$

$$= \frac{(1.2 + 1) \times 12}{14.2} = 1.8592\ \Omega$$

$$\text{Hence,} \quad I_a = \frac{2}{2 + 1.8592} = 0.5182\text{ A}$$

$$\text{Then,} \quad I_b = I_a \times \frac{12}{12 + 1 + 1.2} = 0.438\text{ A}$$

$$\therefore \quad I_1 = 0.438 \times \frac{3}{5} = 0.2628\text{ A}$$

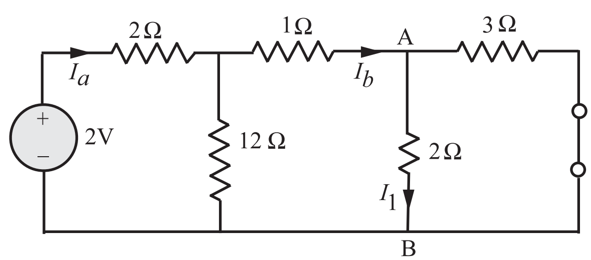


Figure R.P. 3.1(a)

With 4V-source acting alone, the circuit is as shown in Fig. R.P. 3.1(b).

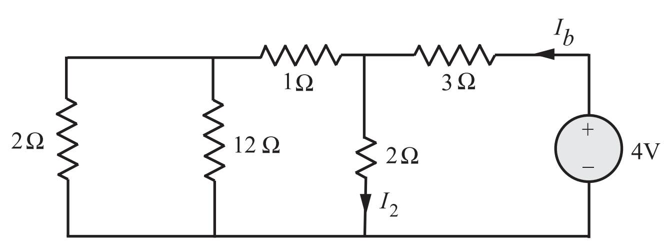


Figure R.P.3.1(b)

The resistance as seen by 4V-source is $3 + R_2$ where

$$R_2 = \left(\frac{2 \times 12}{14} + 1\right) \bigg\| \, 2$$

$$= \frac{2.7143 \times 2}{4.7143} = 1.1551\ \Omega$$

Hence, $$\mathbf{I}_b = \frac{4}{3 + 1.1551} = 0.9635\ \text{A}$$

Thus, $$\mathbf{I}_2 = \frac{\mathbf{I}_b \times 2.7143}{4.7143} = 0.555\ \text{A}$$

Finally, applying the principle of superposition,

we get, $$\mathbf{I}_{AB} = \mathbf{I}_1 + \mathbf{I}_2$$

$$= 0.2628 + 0.555$$

$$= 0.818\ \text{A}$$

R.P 3.2

For the network shown in Fig. R.P. 3.2, apply superposition theorem and find the current **I**.

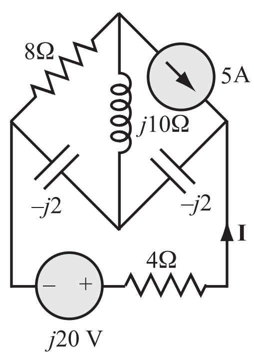


Figure R.P. 3.2

SOLUTION

Open the 5A-current source and retain the voltage source. The resulting network is as shown in Fig. R.P. 3.2(a).

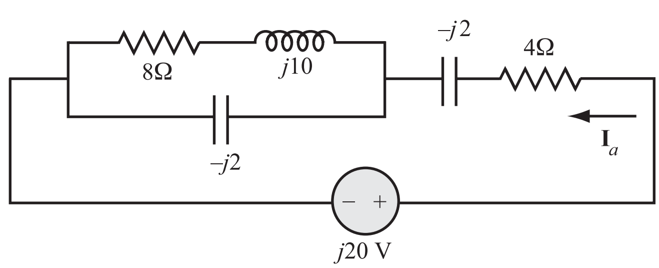


Figure R.P. 3.2(a)

The impedance as seen from the voltage source is

$$\mathbf{Z} = (4 - j2) + \frac{(8 + j10)\,(-j2)}{8 + j8} = 6.01\,\underline{/-45^\circ} \quad \Omega$$

Hence,
$$\mathbf{I}_a = \frac{j20}{\mathbf{Z}} = 3.328\,\underline{/135^\circ}\ \text{A}$$

Next, short the voltage source and retain the current source. The resulting network is as shown in Fig. R.P. 3.2 (b).

Here, $\mathbf{I}_3 = 5\text{A}$. Applying *KVL* for mesh 1 and mesh 2, we get

$$8\mathbf{I}_1 + (\mathbf{I}_1 - 5)\,j10 + (\mathbf{I}_1 - \mathbf{I}_2)\,(-j2) = 0$$

and
$$(\mathbf{I}_2 - \mathbf{I}_1)\,(-j2) + (\mathbf{I}_2 - 5)\,(-j2) + 4\mathbf{I}_2 = 0$$

Simplifying, we get

$$(8 + j8)\mathbf{I}_1 + j2\mathbf{I}_2 = j50$$

and
$$j2\mathbf{I}_1 + (4 - j4)\mathbf{I}_2 = -j10$$

Solving, we get

$$\mathbf{I}_b = \mathbf{I}_2 = \frac{\begin{vmatrix} 8 + j8 & j50 \\ j2 & -j10 \end{vmatrix}}{\begin{vmatrix} 8 + j8 & j2 \\ j2 & 4 - j4 \end{vmatrix}}$$
$$= 2.897\,\underline{/-23.96^\circ}\ \text{A}$$

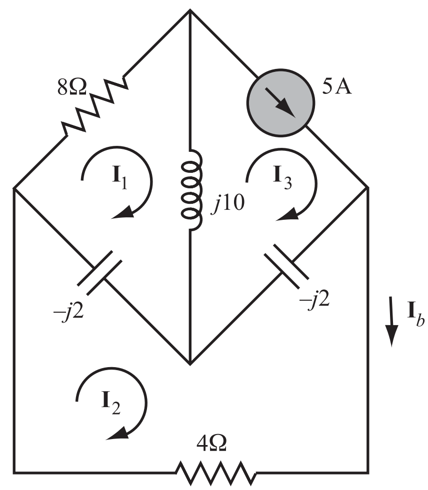


Figure R.P. 3.2(b)

Since, $\mathbf{I}_a$ and $\mathbf{I}_b$ are flowing in opposite directions, we have

$$\mathbf{I} = \mathbf{I}_a - \mathbf{I}_b = 6.1121\,\underline{/144.78^\circ}\ \text{A}$$

R.P 3.3

Apply superposition theorem and find the voltage across 1 Ω resistor. Refer the circuit shown in Fig. R.P. 3.3. Take $v_1(t) = 5\cos(t + 10^\circ)$ and $i_2(t) = 3\sin 2t$ A.

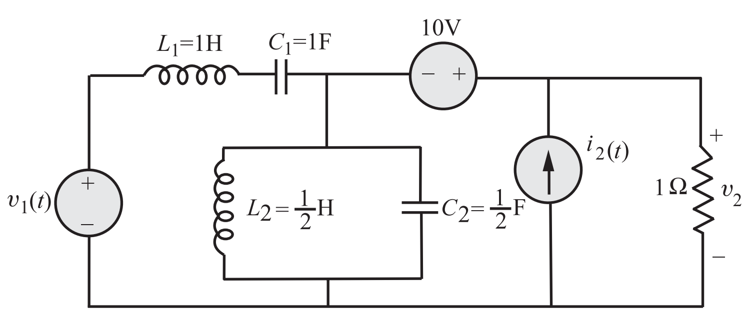


Figure R.P. 3.3

SOLUTION

To begin with let us assume $v_1(t)$ alone is acting. Accordingly, short 10V - source and open $i_2(t)$. The resulting phasor network is shown in Fig. R.P. 3.3(a).

$$\begin{aligned}
\omega &= 1\text{rad/sec} \\
5\cos(t+10^\circ) &\rightarrow 5\,\underline{/10^\circ}\ \text{V} \\
L_1 = 1\text{H} &\rightarrow j\,\omega L_1 = j1\,\Omega \\
C_1 = 1\text{F} &\rightarrow \frac{1}{j\,\omega C_1} = -j1\,\Omega \\
L_2 = \frac{1}{2}\text{H} &\rightarrow j\,\omega L_2 = j\frac{1}{2}\,\Omega \\
C_2 = \frac{1}{2}\text{F} &\rightarrow \frac{1}{j\,\omega C_2} = -j2\,\Omega
\end{aligned}$$

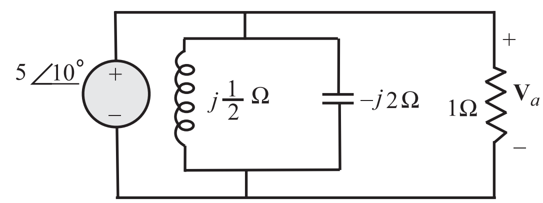


Figure R.P. 3.3(a)

$$\therefore \qquad \mathbf{V}_a = 5\,\underline{/10^\circ}\ \text{V}$$

$$\Rightarrow \qquad v_a(t) = 5\cos[t+10^\circ]$$

Let us next assume that $i_2(t)$ alone is acting. The resulting network is shown in Fig. R.P. 3.3(b).

$$\begin{aligned}
\omega &= 2\ \text{rad/sec} \\
3\sin 2t &\rightarrow 3\,\underline{/0^\circ}\ \text{A} \\
C_1 = 1\text{F} &\rightarrow \frac{1}{j\,\omega C_1} = -j\frac{1}{2}\,\Omega \\
L_1 = 1\text{H} &\rightarrow j\,\omega L_1 = j2\,\Omega \\
C_2 = \frac{1}{2}\text{F} &\rightarrow \frac{1}{j\,\omega C_2} = -j1\,\Omega \\
L_2 = \frac{1}{2}\text{H} &\rightarrow j\,\omega L_2 = j1\,\Omega
\end{aligned}$$

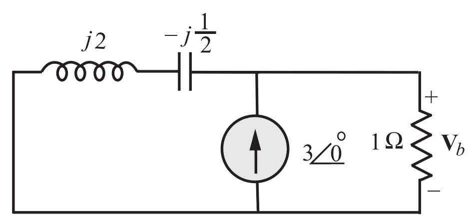


Figure R.P. 3.3(b)

$$\mathbf{V}_b = 3\,\underline{/0^\circ} \times \frac{j1.5}{1+j1.5} = 2.5\,\underline{/33.7^\circ}\ \text{A}$$

$$\Rightarrow \qquad v_b(t) = 2.5\sin[2t+33.7^\circ]\,\text{A}$$

Finally with 10V-source acting alone, the network is as shown in Fig. R.P. 3.3(c). Since $\omega = 0$, inductors are shorted and capacitors are opened.

Hence, $\mathbf{V}_c = 10$ V

Applying principle of superposition, we get.

$$\begin{aligned}
v_2(t) &= v_a(t) = v_b(t) + \mathbf{V}_c \\
&= 5\cos(t+10^\circ) + 2.5\sin(2t+33.7^\circ) + 10\text{Volts}
\end{aligned}$$

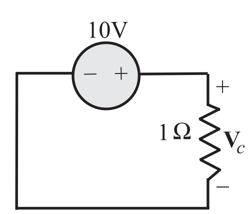


Figure R.P. 3.3(c)

R.P 3.4

Calculate the current through the galvanometer for the Kelvin double bridge shown in Fig. R.P. 3.4. Use Thevenin's theorem. Take the resistance of the galvanometer as 30 Ω.

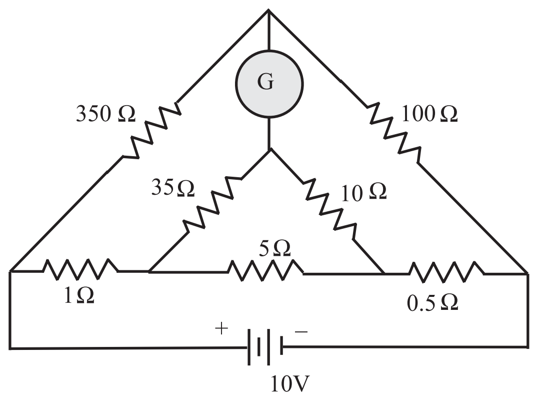


Figure R.P. 3.4

SOLUTION

With G being open, the resulting network is as shown in Fig. R.P. 3.4(a).

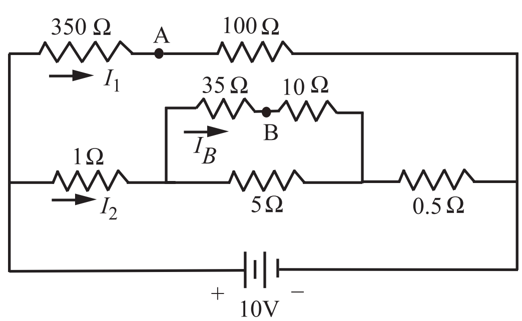


Figure 3.4(a)

$$V_A = I_1 \times 100 = \frac{10}{450} \times 100 = \frac{20}{9} \text{ V}$$

$$I_2 = \frac{10}{1.5 + \dfrac{45 \times 5}{50}} = 1.66, \quad I_B = \frac{I_2 \times 5}{45 + 5} = 0.1 I_2$$

Hence,
$$V_B = I_2 \times 0.5 + I_B \times 10$$
$$= 2.5 \text{ V}$$

Thus,
$$V_{AB} = V_t = V_A - V_B = \frac{20}{9} - 2.5 = \frac{-5}{18} \text{ Volts}$$

To find R_t, short circuit the voltage source. The resulting network is as shown in Fig. R.P. 3.4(b).

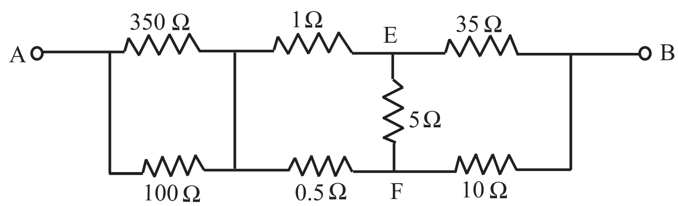


Figure R.P. 3.4 (b)

Transforming the Δ between B, E and F into an equivalent Y, we get

$$R_B = \frac{35 \times 10}{50} = 7\ \Omega, \quad R_E = \frac{35 \times 5}{50} = 3.5\ \Omega, \quad R_F = \frac{5 \times 10}{50} = 1\ \Omega$$

The reduced network after transformation is as shown in Fig. R.P. 3.4(c).

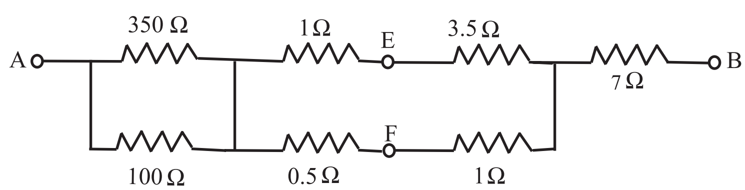


Figure R.P. 3.4(c)

Hence,
$$R_{AB} = R_t = \frac{350 \times 100}{450} + \frac{4.5 \times 1.5}{6} + 7$$
$$= 85.903\ \Omega$$

The Thevenin's equivalent circuit as seen from A and B with 30 Ω connected between A and B is as shown in Fig. R.P. 3.4(d).

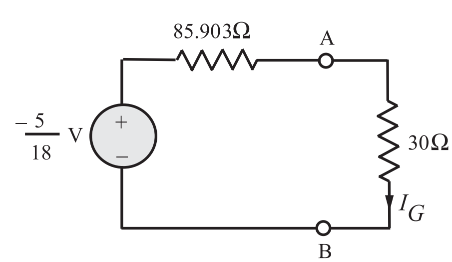


Figure R.P. 3.4(d)

$$I_G = \frac{-\frac{5}{18}}{85.903 + 30} = -2.4\text{mA}$$

Negative sign implies that the current flows from B to A.

R.P 3.5

Find I_s and R so that the networks N_1 and N_2 shown in Fig. R.P. 3.5 are equivalent.

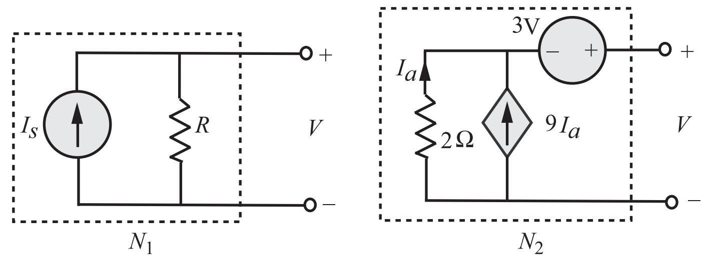


Figure R.P. 3.5

SOLUTION

Transforming the current source in N_1 into an equivalent voltage source, we get N_3 as shown in Fig. R.P. 3.5(a).

From N_3, we can write,
$$V - IR = I_S R \tag{3.28}$$

From N_2 we can write,
$$I = -10 I_a$$

Also from N_2,
$$V - 3 = -2I_a$$

$$\Rightarrow \quad V - 3 = -2\left(\frac{-I}{10}\right)$$

$$\Rightarrow \quad V - 3 = \frac{I}{5}$$

$$\Rightarrow \quad V - \frac{I}{5} = 3 \tag{3.29}$$

For equivalence of N_1 and N_2, it is requirred that equations (3.28) and (3.29) must be same. Comparing these equations, we get

$$IR = \frac{I}{5} \quad \text{and} \quad I_S R = 3$$

$$R = 0.2\,\Omega \quad \text{and} \quad I_S = \frac{3}{0.2} = 15\text{A}$$

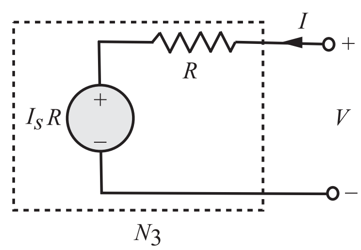


Figure R.P. 3.5(a)

R.P 3.6

Obtain the Norton's equivalent of the network shown in Fig. R.P. 3.6.

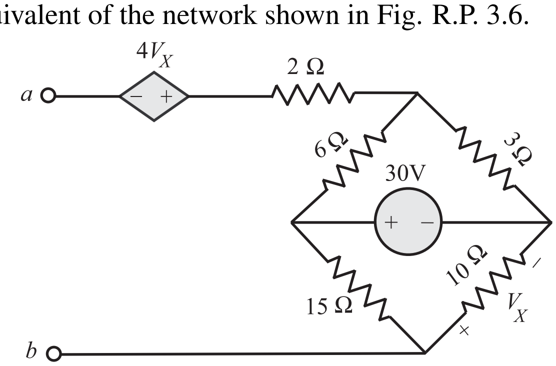


Figure R.P. 3.6

SOLUTION

Terminals a and b are shorted. This results in a network as shown in Fig. R.P. 3.6(a)

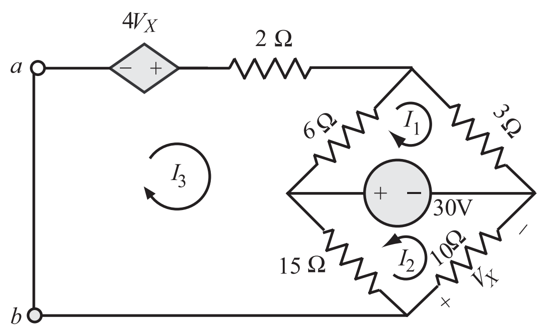


Figure R.P. 3.6(a)

The mesh equations are

(i) $$9I_1 + 0I_2 - 6I_3 = 30 \quad (3.30)$$

(ii) $$0I_1 + 25I_2 + 15I_3 = 30 \quad (3.31)$$

(iii) $$-6I_1 + 15I_2 + 23I_3 = 4V_X = 4\,(10I_2)$$

$$\Rightarrow \quad -6I_1 - 25I_2 + 23I_3 = 0 \quad (3.32)$$

Solving equations (3.30), (3.31) and (3.32), we get

$$I_N = I_{sc} = I_3 = 1.4706\text{A}$$

With terminals ab open, $I_3 = 0$. The corresponding equations are

$$9I_1 = 30 \qquad \text{and} \quad 25I_2 = 50$$

Hence, $$I_1 = \frac{30}{9}\text{A} \quad \text{and} \quad I_2 = \frac{30}{25}\text{ A}$$

Then, $$V_X = 10I_2 = 10 \times \frac{30}{25} = 12\text{ V}$$

Hence, $$V_t = V_{oc} = 15I_2 - 6I_1 - 4V_X$$
$$= -50\text{ V}$$

Thus, $$R_t = \frac{V_{oc}}{I_{sc}} = \frac{-50}{1.4706} = -34\ \Omega$$

Hence, Norton's equivalent circuit is as shown in Fig. R.P. 3.6(b).

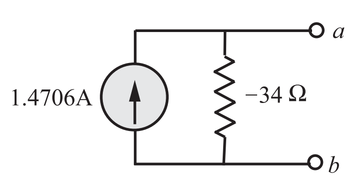


Figure R.P. 3.6(b)

R.P 3.7

For the network shown in Fig. R.P. 3.7, find the Thevenin's equivalent to show that

$$V_t = \frac{V_1}{2}(1 + a + b - ab)$$

and $$Z_t = \frac{3 - b}{2}$$

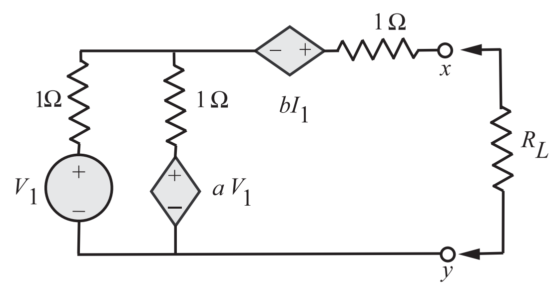


Figure R.P. 3.7

SOLUTION

With xy open, $I_1 = \dfrac{V_1 - aV_1}{2}$

Hence,

$$\begin{aligned} V_{oc} = V_t &= aV_1 + I_1 + bI_1 \\ &= aV_1 + \frac{V_1 - aV_1}{2} + b\left(\frac{V_1 - aV_1}{2}\right) \\ &= \frac{V_1}{2}[1 + a + b - ab] \end{aligned}$$

With xy shorted, the resulting network is as shown in Fig. R.P. 3.7(a).

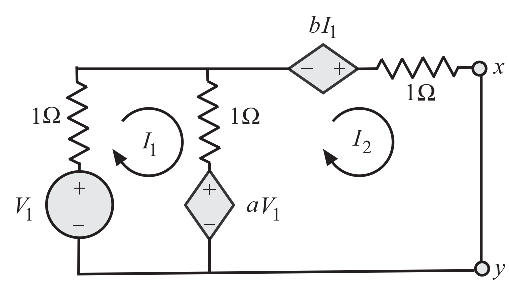


Figure R.P. 3.7(a)

Applying *KVL* equations, we get

(i) $$I_1 + (I_1 - I_2) = V_1 - aV_1$$

$$\Rightarrow \quad 2I_1 - I_2 = V_1 - aV_1 \tag{3.33}$$

(ii) $$(I_2 - I_1) + I_2 = aV_1 + bI_1$$

$$\Rightarrow \quad -(1 + b)\, I_1 + 2I_2 = aV_1 \tag{3.34}$$

Solving equations (3.33) and (3.34), we get

$$I_{sc} = I_2 = \frac{V_1(1 + a + b - ab)}{3 - b}$$

Hence,
$$Z_t = \frac{V_{oc}}{I_{sc}} = \frac{V_1}{2}\frac{(1+a+b-ab)}{V_1\,(1+a+b-ab)}(3-b)$$
$$= \frac{3-b}{2}$$

R.P 3.8

Use Norton's theorem to determine I in the network shown in Fig. R.P. 3.8. Resistance Values are in ohms.

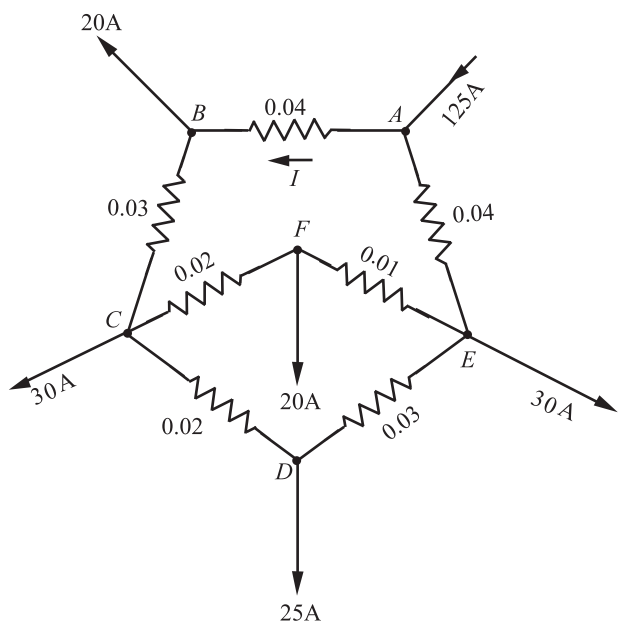


Figure R.P. 3.8

SOLUTION

Let $I_{AE} = x$ and $I_{EF} = y$. Then by applying *KCL* at various junctions, the branch currents are marked as shown in Fig. R.P. 3.8(a). $I_{sc} = 125 - x = I_{AB}$ on shorting A and B.

Applying *KVL* to the loop $ABCFEA$, we get

$$0.04x + 0.01y + 0.02\,(y-20) + 0.03\,(x-105) = 0$$
$$\Rightarrow \qquad 0.07x + 0.03y = 3.55 \tag{3.35}$$

Applying *KVL* to the loop $EDCEF$, we get

$$(x-y-30)\,0.03 + (x-y-55)\,0.02 - (y-20)\,0.02 - 0.01y = 0$$
$$\Rightarrow \qquad 0.05x - 0.08y = 1.6 \tag{3.36}$$

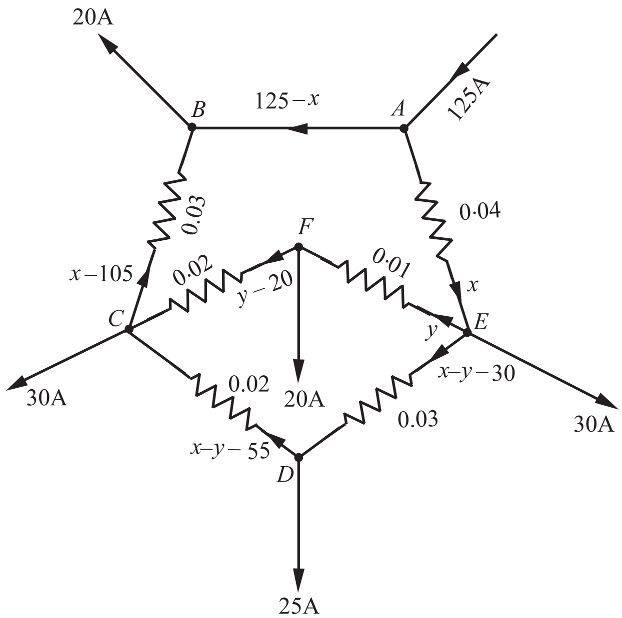


Figure R.P. 3.8(a)

Solving equations (3.35) and (3.36), we get

$$x = 46.76 \text{ A}$$

Hence,

$$I_{sc} = I_N = 120 - x$$
$$= 78.24 \text{ A}$$

The circuit to calculate R_t is as shown in Fig. R.P. 3.8(b). All injected currents have been opened.

$$R_t = 0.03 + 0.04 + \frac{0.03 \times 0.05}{0.08}$$
$$= 0.08875\ \Omega$$

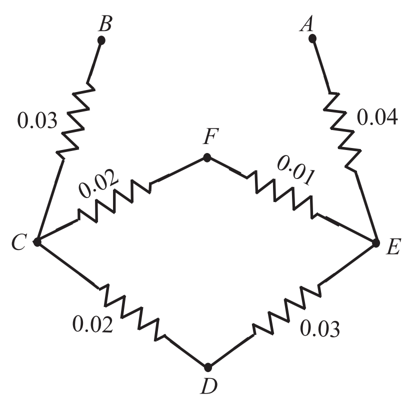


Figure R.P. 3.8(b)

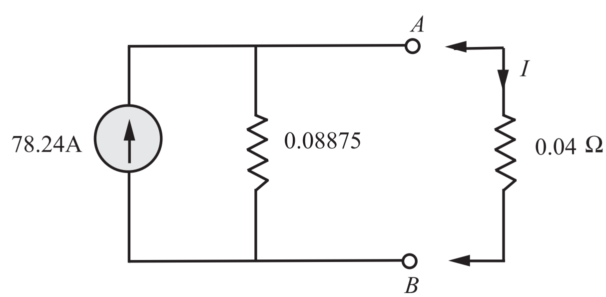


Figure R.P. 3.8(c)

The Norton's equivalent network is as shown in Fig. R.P. 3.8(c).

$$\begin{aligned} I &= 78.24 \times \frac{0.08875}{0.08875 + 0.04} \\ &= 53.9\text{A} \end{aligned}$$

R.P 3.9

For the circuit shown in Fig. R.P. 3.9, find R such that the maximum power delivered to the load is 3 mW.

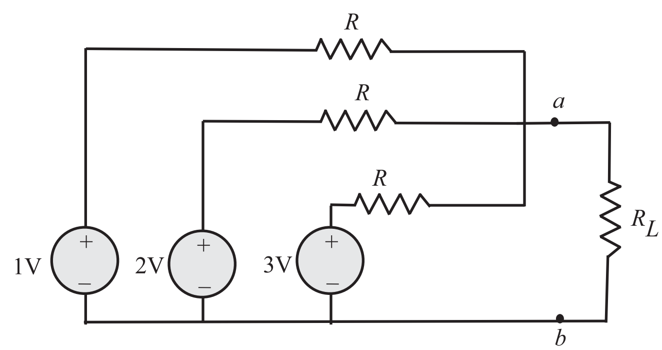


Figure R.P. 3.9

SOLUTION

For a resistive network, the maximum power delivered to the load is

$$P_{\max} = \frac{V_t^2}{4R_t}$$

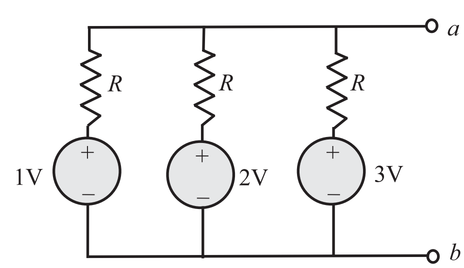


Figure R.P. 3.9(a)

The network with R_L removed is as shown in Fig. R.P. 3.9(a).

Let the opent circuit voltage between the terminals a and b be V_t.

Then, applying KCL at node a, we get

$$\frac{V_t - 1}{R} + \frac{V_t - 2}{R} + \frac{V_t - 3}{R} = 0$$

Simplifying we get $\quad V_t = 2$ Volts

With all voltage sources shorted, the resistance, R_t as viewed from the terminals, a and b is found as follows:

$$\frac{1}{R_t} = \frac{1}{R} + \frac{1}{R} + \frac{1}{R} = \frac{3}{R}$$

$$\Rightarrow \qquad R_t = \frac{R}{3}\,\Omega$$

Hence, $$P_{\max} = \frac{2^2}{4 \times \frac{R}{3}} = \frac{3}{R} = 3 \times 10^{-3}$$

$$\Rightarrow \quad R = 1\ \text{k}\Omega$$

R.P 3.10

Refer Fig. R.P. 3.10, find X_1 and X_2 interms of R_1 and R_2 to give maximum power dissipation in R_2.

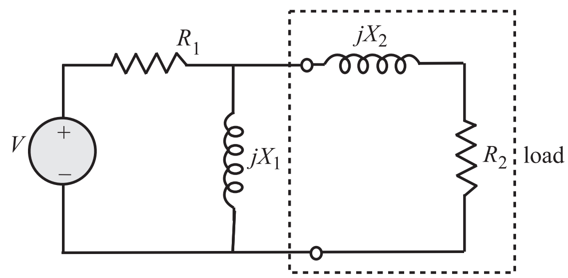


Figure R.P. 3.10

SOLUTION

The circuit for finding $\mathbf{Z}_t$ is as shown in Figure R.P. 3.10(a).

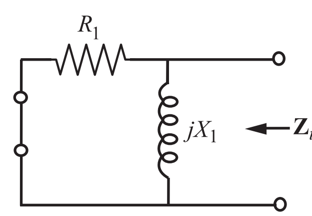


Figure R.P. 3.10(a)

$$\mathbf{Z}_t = \frac{R_1\,(jX_1)}{R_1 + jX_1}$$

$$= \frac{R_1 X_1^2 + jR_1^2 X_1}{R_1^2 + X_1^2}$$

For maximum power transfer,

$$\mathbf{Z}_L = \mathbf{Z}_t^*$$

$$\Rightarrow \quad R_2 + jX_2 = \frac{R_1 X_1^2}{R_1^2 + X_1^2} - j\frac{R_1^2 X_1}{R_1^2 + X_1^2}$$

Hence, $$R_2 = \frac{R_1 X_1^2}{R_1^2 + X_1^2}$$

$$\Rightarrow \quad X_1 = \pm R_1 \sqrt{\frac{R_2}{R_1 - R_2}} \tag{3.37}$$

$$X_2 = -\frac{R_1^2 X_1}{R_1^2 + X_1^2} \tag{3.38}$$

Substituting equation (3.37) in equation (3.38) and simplifying, we get

$$X_2 = \sqrt{R_2\,(R_1 - R_2)}$$

Exercise Problems

E.P 3.1

Find i_x for the circuit shown in Fig. E.P. 3.1 by using principle of superposition.

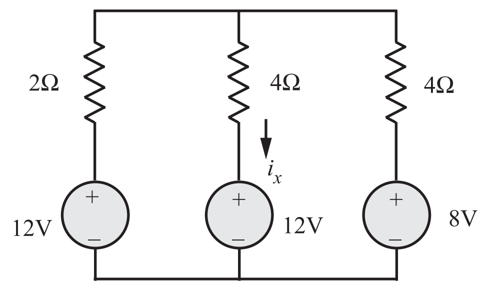


Figure E.P. 3.1

Ans : $\boldsymbol{i_x = -\frac{1}{4}}$ **A**

E.P 3.2

Find the current through branch PQ using superposition theorem.

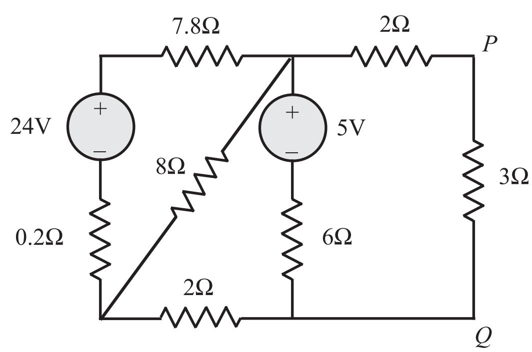


Figure E.P. 3.2

Ans : 1.0625 A

E.P 3.3

Find the current through 15 ohm resistor using superposition theorem.

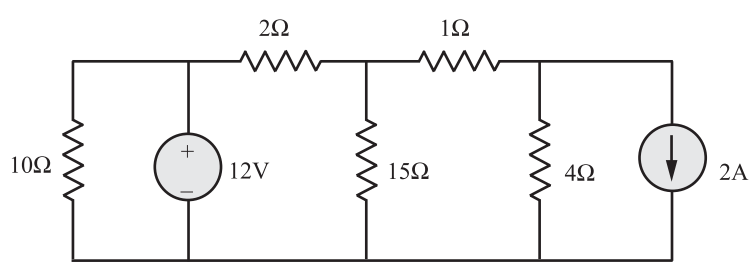


Figure E.P. 3.3

Ans : 0.3826 A

E.P 3.4

Find the current through $3 + j4\ \Omega$ using superposition theorem.

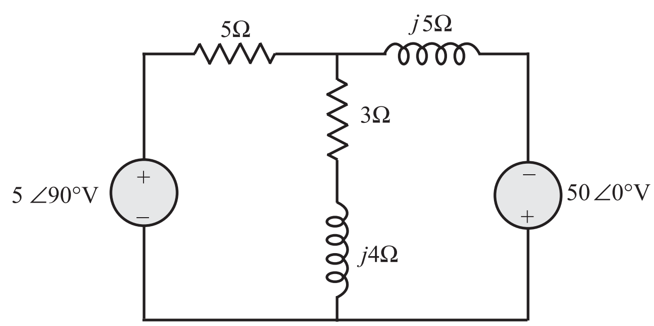


Figure E.P. 3.4

Ans : 8.3 $\underline{/85.3^\circ}$ A

E.P 3.5

Find the current through $\mathbf{I}_x$ using superposition theorem.

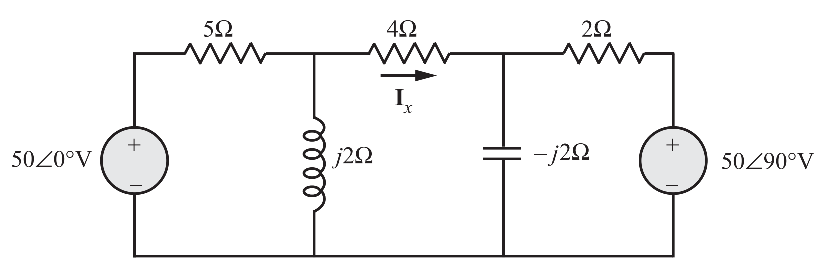


Figure E.P. 3.5

Ans : 3.07 $\underline{/-163.12^\circ}$ A

E.P 3.6

Determine the current through $1\ \Omega$ resistor using superposition theorem.

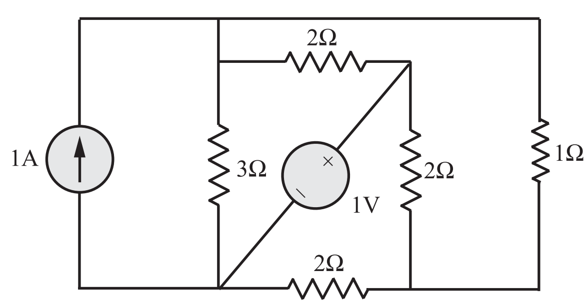


Figure E.P. 3.6

Ans : 0.406 A

E.P 3.7

Obtain the Thevenin equivalent circuit at terminals $a - b$ of the network shown in Fig. E.P. 3.7.

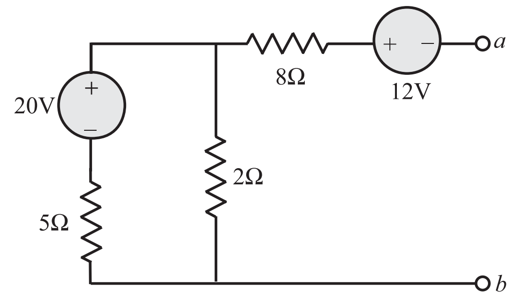


Figure E.P. 3.7

Ans : $\boldsymbol{V_t = 6.29\ \text{V}, R_t = 9.43\ \Omega}$

E.P 3.8

Find the Thevenin equivalent circuit at terminals $x - y$ of the circuit shown in Fig. E.P. 3.8.

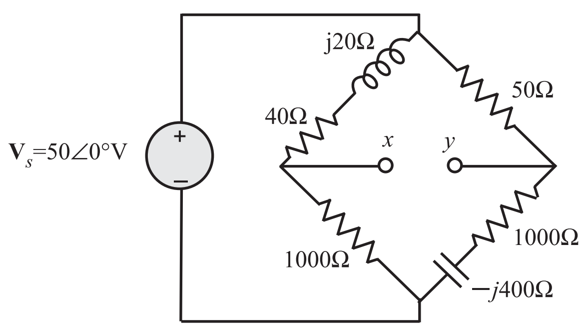


Figure E.P. 3.8

Ans : $\boldsymbol{V_t = 0.192\,\underline{/-43.4^\circ}\ \text{V},\ Z_t = 88.7\,\underline{/11.55^\circ}\ \Omega}$

E.P 3.9

Find the Thevenin equivalent of the network shown in Fig. E.P. 3.9.

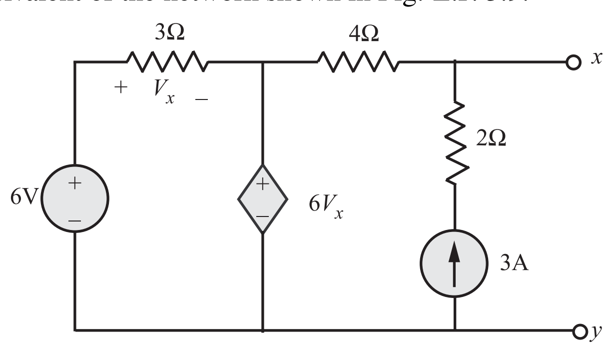


Figure E.P. 3.9

Ans : $\boldsymbol{V_t = 17.14\ \text{volts}, R_t = 4\ \Omega}$

E.P 3.10

Find the Thevenin equivalent circuit across $a - b$. Refer Fig. E.P. 3.10.

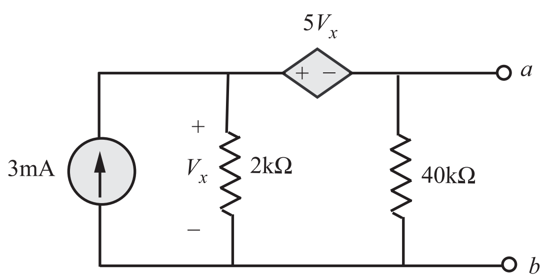


Figure E.P. 3.10

Ans : $\boldsymbol{V_t = -30\ \text{V}, R_t = 10\ \text{k}\Omega}$

E.P 3.11

Find the Thevenin equivalent circuit across $a - b$ for the network shown in Fig. E.P. 3.11.

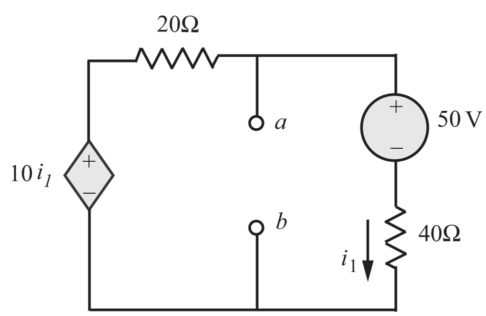


Figure E.P. 3.11

Ans : Verify your result with other methods.

E.P 3.12

Find the current through 20 ohm resistor using Norton equivalent.

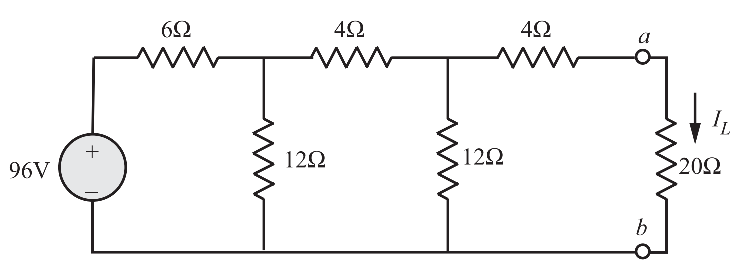


Figure E.P. 3.12

Ans : $\boldsymbol{I_N = 4.36\ \text{A},\ R_N = R_t = 8.8\ \Omega,\ I_L = 1.33\ \text{A}}$

E.P 3.13

Find the current in 10 ohm resistor using Norton's theorem.

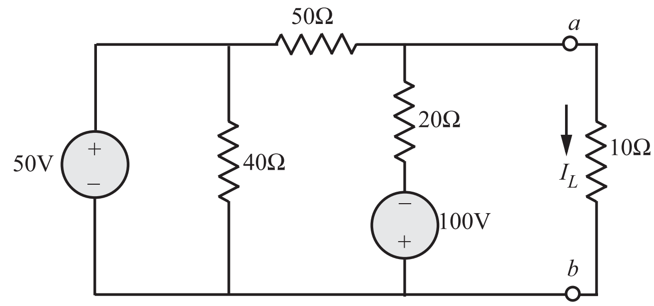


Figure E.P. 3.13

Ans : $\boldsymbol{I_N = -4\text{ A},\ R_t = R_N = \dfrac{100}{7}\ \Omega,\ I_L = -0.5\text{ A}}$

E.P 3.14

Find the Norton equivalent circuit between the terminals $a-b$ for the network shown in Fig. E.P. 3.14.

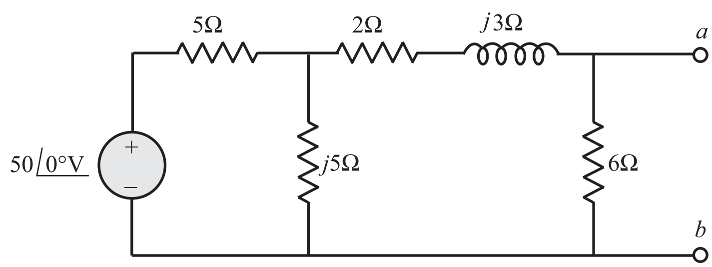


Figure E.P. 3.14

Ans : $\boldsymbol{I_N = 4.98310\,\angle{-5.71^\circ}\text{ A},\ \mathbf{Z}_t = \mathbf{Z}_N = 3.6\,\angle{23.1^\circ}\ \Omega}$

E.P 3.15

Determine the Norton equivalent circuit across the terminals $P - Q$ for the network shown in Fig. E.P. 3.15.

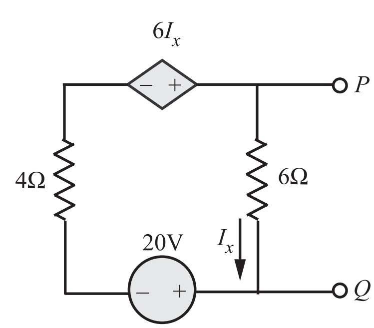


Figure E.P. 3.15

Ans : $\boldsymbol{I_N = 5\text{ A},\ R_N = R_t = 6\ \Omega}$

E.P 3.16

Find the Norton equivalent of the network shown in Fig. E.P. 3.16.

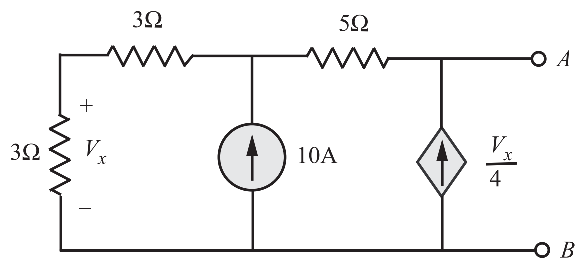


Figure E.P. 3.16

Ans : $\boldsymbol{I_N = 8.87\ \text{A},\ R_N = R_t = 43.89\ \Omega}$

E.P 3.17

Determine the value of R_L for maximum power transfer and also find the maximum power transferred.

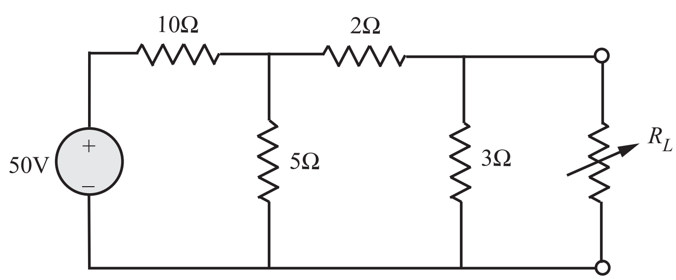


Figure E.P. 3.17

Ans : $\boldsymbol{R_L = 1.92\ \Omega,\ P_{\max} = 4.67\ \text{W}}$

E.P 3.18

Calculate the value of Z_L for maximum power transfer and also calculate the maximum power.

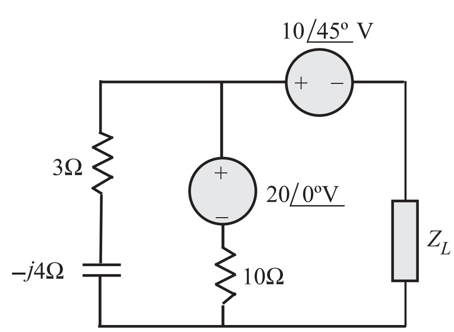


Figure E.P. 3.18

Ans : $\boldsymbol{Z_L = (7.97 + j2.16)\Omega,\ P_{\max} = 0.36\ \text{W}}$

E.P 3.19

Determine the value of R_L for maximum power transfer and also calculate the value of maximum power.

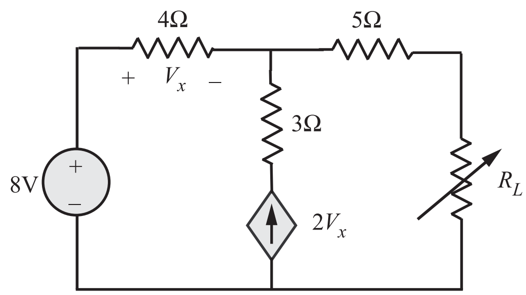


Figure E.P. 3.19

Ans : $\boldsymbol{R_L = 5.44\ \Omega,\ P_{\max} = 2.94\ \mathrm{W}}$

E.P 3.20

Determine the value of Z_L for maximum power transfer. What is the value of maximum power?

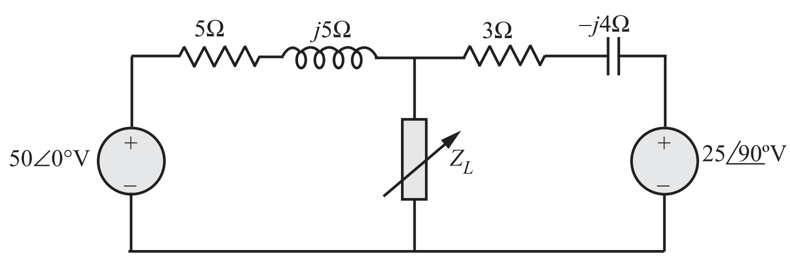


Figure E.P. 3.20

Ans : $\boldsymbol{Z_L = 4.23 + j1.15\ \Omega,\ P_{\max} = 5.68\ \mathrm{Watts}}$

E.P 3.21

Obtain the Norton equivalent across $x - y$.

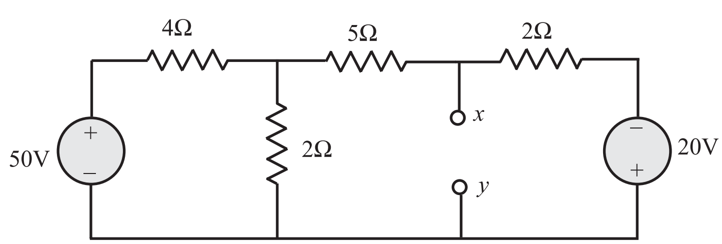


Figure E.P. 3.21

Ans : $\boldsymbol{I_N = I_{SC} = 7.35\mathrm{A},\ R_t = R_N = 1.52\ \Omega}$

E.P 3.22

Find the Norton equivalent circuit at terminals $a - b$ of the network shown in Fig. E.P. 3.22.

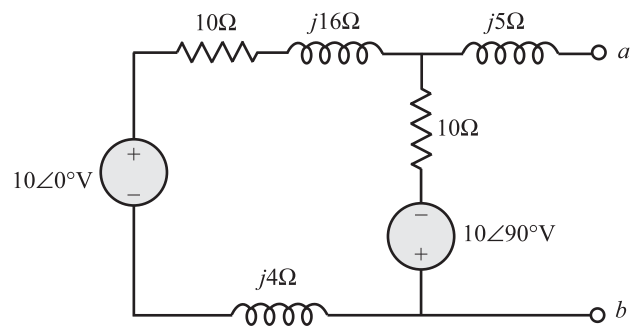


Figure E.P. 3.22

Ans : $\mathbf{I_N = 1.05 \underline{/251.6^\circ}\ A,\ Z_t = Z_N = 10.6 \underline{/45^\circ}\ \Omega}$

E.P 3.23

Find the Norton equivalent across the terminals $X - Y$ of the network shown in Fig. E.P. 3.23.

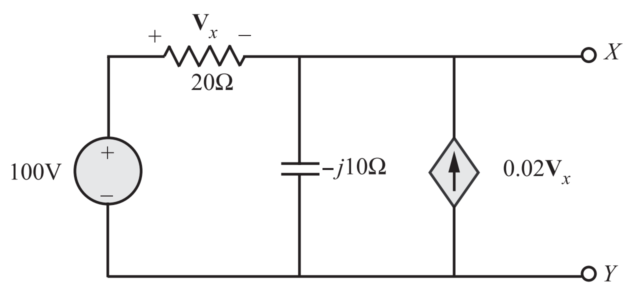


Figure E.P. 3.23

Ans : $\mathbf{I_N = 7A,\ Z_t = 8.19 \underline{/-55^\circ}\ \Omega}$

E.P 3.24

Determine the current through 10 ohm resistor using Norton's theorem.

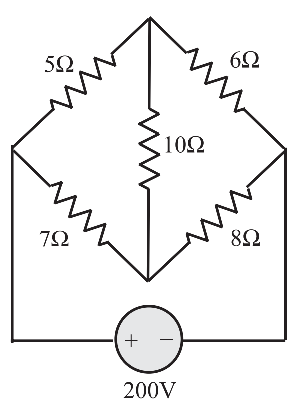


Figure E.P. 3.24

Ans : **0.15A**

E.P 3.25

Determine the current I using Norton's theorem.

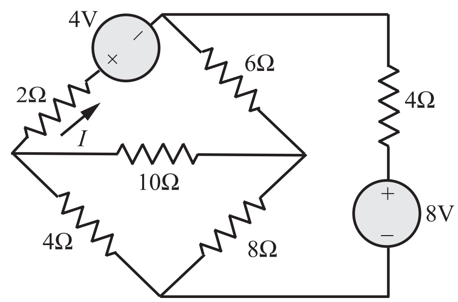


Figure E.P. 3.25

Ans : Verify your result with other methods.

E.P 3.26

Find $\mathbf{V}_x$ in the circuit shown in Fig. E.P. 3.26 and hence verify reciprocity theorem.

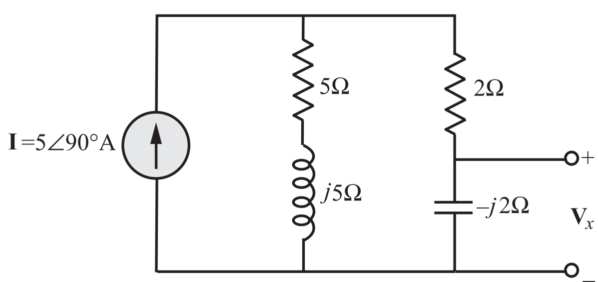


Figure E.P. 3.26

Ans : $\mathbf{V_x = 9.28\,\underline{/21.81^\circ}\ V}$

E.P 3.27

Find V_x in the circuit shown in Fig. E.P. 3.27 and hence verify reciprocity theorem.

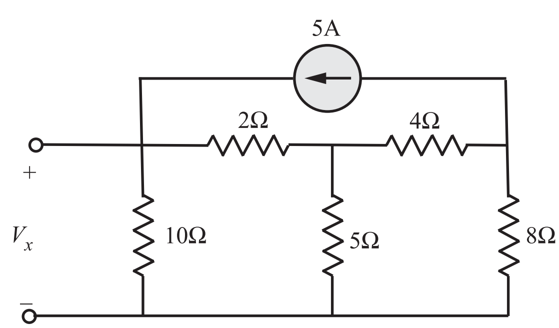


Figure E.P. 3.27

Ans : $\boldsymbol{V_x = 10.23}$ **Volts**

E.P 3.28

Find the current i_x in the bridge circuit and hence verify reciprocity theorem.

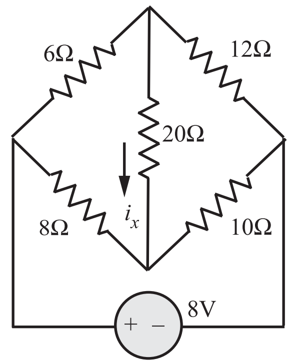


Figure E.P. 3.28

Ans : $\boldsymbol{i_x = 0.031}$ **A**

E.P 3.29

Find the current through 4 ohm resistor using Millman's theorem.

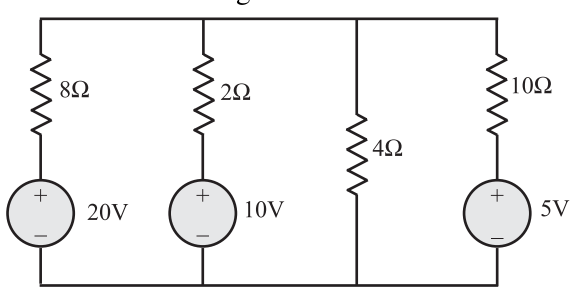


Figure E.P. 3.29

Ans : $\boldsymbol{I = 2.05}$ **A**

E.P 3.30

Find the current through the impedance of $(10 + j10)\ \Omega$ using Millman's theorem.

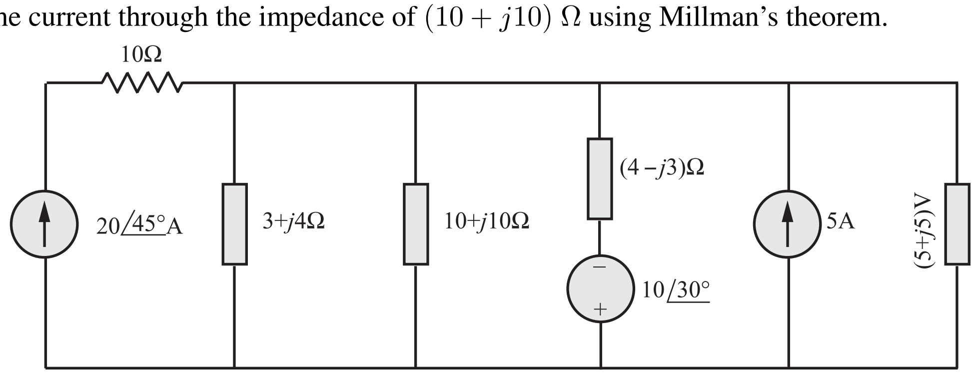


Figure E.P. 3.30

Ans : $\boldsymbol{3.384\,\angle 12.6^\circ}$ **A**

E.P 3.31

Using Millman's theorem, find the current flowing through the impedance of $(4 + j3)\ \Omega$.

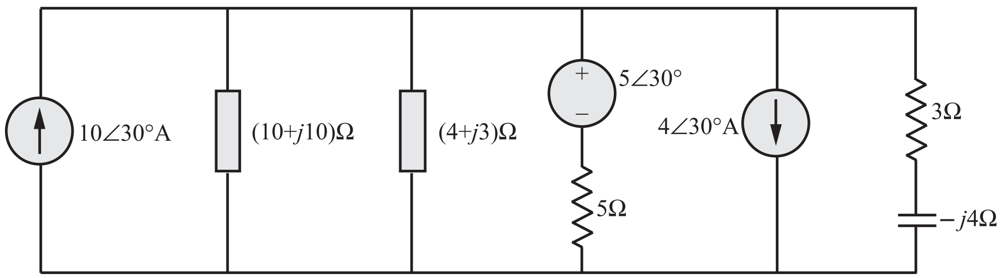


Figure E.P. 3.31

Ans : $\mathbf{3.64\,\underline{/\,15.23^\circ}\ A}$

Chapter 6

Resonance

6.1 Introduction

A.C Circuits made up of resistors, inductors and capacitors are said to be resonant circuits when the current drawn from the supply is in phase with the impressed sinusoidal voltage. Then

1. the resultant reactance or susceptance is zero.
2. the circuit behaves as a resistive circuit.
3. the power factor is unity.

A second order series resonant circuit consists of R, L and C in series. At resonance, voltages across C and L are equal and opposite and these voltages are many times greater than the applied voltage. They may present a dangerous shock hazard.

A second order parallel resonant circuit consists of R, L and C in parallel. At resonance, currents in L and C are circulating currents and they are considerably larger than the input current. Unless proper consideration is given to the magnitude of these currents, they may become very large enough to destroy the circuit elements.

Resonance is the phenomenon which finds its applications in communication circuits: The ability of a radio or Television receiver (1) to select a particular frequency or a narrow band of frequencies transmitted by broad casting stations or (2) to suppress a band of frequencies from other broad casting stations, is based on resonance.

Thus resonance is desired in tuned circuits, design of filters, signal processing and control engineering. But it is to be avoided in other circuits. It is to be noted that if $R = 0$ in a series RLC circuit, the circuits acts as a short circuit at resonance and if $R = \infty$ in parallel RLC circuit, the circuit acts as an open circuit at resonance.

6.2 Transfer Functions

As ω is varied to achieve resonance, electrical quantities are expressed as functions of ω, normally denoted by $F(j\omega)$ and are called transfer functions. Accordingly the following notations are used.

$$Z(j\omega) = \frac{V(j\omega)}{I(j\omega)} = \text{Impedance function}$$
$$Y(j\omega) = \frac{I(j\omega)}{V(j\omega)} = \text{Admittance function}$$
$$G(j\omega) = \frac{V_2(j\omega)}{V_1(j\omega)} = \text{Voltage ratio transfer function}$$
$$\alpha(j\omega) = \frac{I_2(j\omega)}{I_1(j\omega)} = \text{Current ratio transfer function}$$

If we put $j\omega = s$ then the above quantities will be $Z(s), Y(s), G(s), \alpha(s)$ respectively. These are treated later in this book.

6.3 Series Resonance

Fig. 6.1 represents a series resonant circuit.

Resonance can be achieved by

1. varying frequency ω
2. varying the inductance L
3. varying the capacitance C

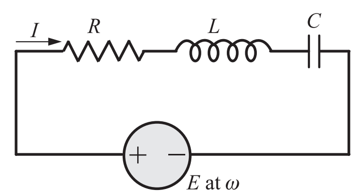


Figure 6.1 Series Resonant Circuit

The current in the circuit is

$$I = \frac{E}{R + j(X_L - X_C)} = \frac{E}{R + jX}$$

At resonance, X is zero. If ω_0 is the frequency at which resonance occurs, then $\omega_0 L = \frac{1}{\omega_0 C}$ or $\omega_0 = \frac{1}{\sqrt{LC}}$ = resonant frequency.

The current at resonance is $I_m = \frac{V}{R}$ = maximum current.

The phasor diagram for this condition is shown in Fig. 6.2.

The variation of current with frequency is shown in Fig. 6.3.

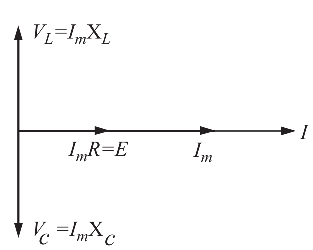


Figure 6.2

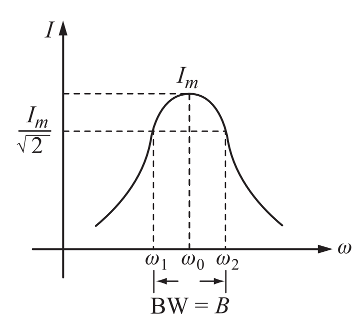


Figure 6.3

It is observed that there are two frequencies, one above and the other below the resonant frequency, ω_0 at which current is same.

Fig. 6.4 represents the variations of $X_L = \omega L; X_C = \dfrac{1}{\omega C}$ and $|Z|$ with ω.

From the equation $\omega_0 = \dfrac{1}{\sqrt{LC}}$ we see that any constant product of L and C give a particular resonant frequency even if the ratio $\dfrac{L}{C}$ is different. The frequency of a constant frequency source can also be a resonant frequency for a number of L and C combinations. Fig. 6.5 shows how the sharpness of tuning is affected by different $\dfrac{L}{C}$ ratios, but the product LC remaining constant.

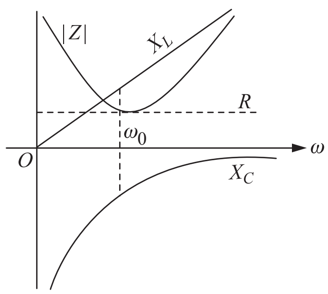


Figure 6.4

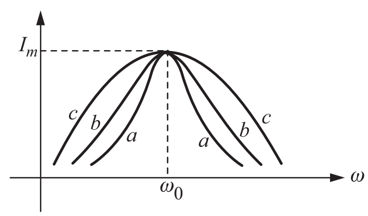


Figure 6.5

For larger $\dfrac{L}{C}$ ratio, current varies more abruptly in the region of ω_0. Many applications call for narrow band that pass the signal at one frequency and tend to reject signals at other frequencies.

6.4 Bandwidth, Quality Factor and Half Power Frequencies

At resonance $I = I_m$ and the power dissipated is

$$P_m = I_m^2 R \text{ watts.}$$

When the current is $I = \dfrac{I_m}{\sqrt{2}}$ power dissipated is

$$\frac{P_m}{2} = \frac{I_m^2 R}{2} \text{ watts.}$$

From $\omega - I$ characteristic shown in Fig. 6.3, it is observed that there are two frequencies ω_1 and ω_2 at which the current is $I = \dfrac{I_m}{\sqrt{2}}$. As at these frequencies the power is only one half of that at ω_0, these are called half power frequencies or cut off frequencies.

The ratio, $$\frac{\text{current at half power frequencies}}{\text{Maximum current}} = \frac{I_m}{\sqrt{2} I_m} = \frac{1}{\sqrt{2}}$$

When expressed in dB it is $20 \log \dfrac{1}{\sqrt{2}} = -3\text{dB}.$

Therefore ω_1 and ω_2 are also called -3 dB frequencies.

As $\dfrac{I_m}{\sqrt{2}} = \dfrac{E}{\sqrt{2}R}$, the magnitude of the impedance at half-power frequencies is $\sqrt{2}R = |R + j(X_L - X_C)|$

Therefore, the resultant reactance, $X = X_L - X_C = R$.

The frequency range between half - power frequencies is $\omega_2 - \omega_1$, and it is referred to as passband or band width.

$$\text{BW} = \omega_2 - \omega_1 = B.$$

The sharpness of tuning depends on the ratio $\dfrac{R}{L}$, a small ratio indicating a high degree of selectivity. The quality factor of a circuit can be expressed in terms of R and L of the inductor.

$$\text{Quality factor} = Q = \frac{\omega_0 L}{R}$$

Writing $\omega_0 = 2\pi f_0$ and multiplying numerator and denominator by $\dfrac{1}{2}I_m^2$, we get,

$$Q = 2\pi f_0 \frac{\frac{1}{2}L{I_m}^2}{\frac{1}{2}{I_m}^2 R} = 2\pi \times \frac{\frac{1}{2}L{I_m}^2}{\frac{1}{2}{I_m}^2 RT}$$

$$= 2\pi \times \frac{\text{Maximum energy stored}}{\text{total energy lost in a period}}$$

Selectivity is the reciprocal of Q.

As $\quad Q = \dfrac{\omega_0 L}{R}$ and $\omega_0 L = \dfrac{1}{\omega_0 C}$,

$$Q = \frac{1}{\omega_0 CR}$$

and since $\omega_0 = \dfrac{1}{\sqrt{LC}}$, we have

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}}$$

6.5 Expressions for ω_1 and ω_2, and Bandwidth

At half power frequencies ω_1 and ω_2,

$$I = \frac{E}{\sqrt{2}R} = \frac{E}{\{R^2 + (X_L - X_C)^2\}^{\frac{1}{2}}}$$

$$\therefore \qquad |X_L - X_C| = R \quad \text{i.e.,} \quad \left|\omega L - \frac{1}{\omega C}\right| = R$$

At $\omega = \omega_2$,

$$R = \omega_2 L - \frac{1}{\omega_2 C}$$

Simplifying, $\quad \omega_2^2 LC - \omega_2 CR - 1 = 0$

Solving, we get

$$\omega_2 = \frac{RC + \sqrt{R^2C^2 + 4LC}}{2LC} = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \tag{6.1}$$

Note that only + sign is taken before the square root. This is done to ensure that ω_2 is always positive.

At $\omega = \omega_1$,

$$R = \frac{1}{\omega_1 C} - \omega_1 L$$

$$\Rightarrow \quad \omega_1^2 LC + \omega_1 CR - 1 = 0$$

Solving,

$$\omega_1 = \frac{-RC + \sqrt{R^2C^2 + 4LC}}{2LC}$$

$$= \frac{-R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \tag{6.2}$$

While determining ω_1, only positive value is considered.

Subtracting equation(6.1) from equation (6.2), we get

$$\omega_2 - \omega_1 = \frac{R}{L} = \text{Band width.}$$

Since $Q = \dfrac{\omega_0 L}{R}$, Band width is expressed as

$$B = \omega_2 - \omega_1 = \frac{R}{L} = \frac{\omega_0}{Q}.$$

and therefore

$$Q = \frac{\omega_0}{\omega_2 - \omega_1} = \frac{\omega_0}{B}$$

Multiplying equations (6.1) and (6.2), we get

$$\omega_1\omega_2 = \frac{R^2}{4L^2} + \frac{1}{LC} - \frac{R^2}{4L^2} = \frac{1}{LC} = \omega_0^2$$

or

$$\omega_0 = \sqrt{\omega_1\omega_2}$$

The resonance frequency is the geometric mean of half power frequencies.

Normally $\dfrac{R}{2L} << \dfrac{1}{\sqrt{LC}}$, in which case $Q \geq 5$

Then,

$$\omega_1 \simeq -\frac{R}{2L} + \sqrt{\frac{1}{LC}} \quad \text{and} \quad \omega_2 \simeq \frac{R}{2L} + \frac{1}{\sqrt{LC}}$$

$$= \frac{R}{2L} + \omega_0 \quad \text{and} \quad \omega_2 = \frac{R}{2L} + \omega_0$$

$\therefore$

$$\omega_0 = \frac{\omega_1 + \omega_2}{2} = \text{Arithmetic mean of } \omega_1 \text{ and } \omega_2$$

Since $\dfrac{R}{L} = \dfrac{\omega_0}{Q}$, Equations for ω_1 and ω_2 as given by equations (6.1) and (6.2) can be expressed in terms of Q as

$$\omega_2 = \frac{\omega_0}{2Q} + \sqrt{\left(\frac{\omega_0}{2Q}\right)^2 + \omega_0^2}$$

$$= \omega_0 \left[\frac{1}{2Q} + \sqrt{1 + \left(\frac{1}{2Q}\right)^2} \right]$$

Similarly
$$\omega_1 = \omega_0 \left[-\frac{1}{2Q} + \sqrt{1 + \left(\frac{1}{2Q}\right)^2} \right]$$

Normally, $\frac{R}{2L} << \frac{1}{\sqrt{LC}}$ and then $Q > 5$.

Consequently ω_1 and ω_2 can be approximated as

$$\omega_1 \simeq -\frac{R}{2L} + \sqrt{\frac{1}{LC}} = -\frac{R}{2L} + \omega_0 = -\frac{B}{2} + \omega_0$$
$$\omega_2 \simeq \frac{R}{2L} + \sqrt{\frac{1}{LC}} = +\frac{R}{2L} + \omega_0 = \frac{B}{2} + \omega_0$$

so that
$$\omega_0 = \frac{\omega_1 + \omega_2}{2}.$$

6.6 Frequency Response of Voltage across L and C

As frequency is varied, both the voltages across L and C increase with frequency upto ω_0 and they are equal at ω_0. But their maximum values do not occur at ω_0. V_c reaches its maximum at $\omega < \omega_0$ and V_L reaches its maximum at $\omega > \omega_0$. This can be verified by calculating the frequency at which each occurs.

6.7 Expression for ω at which V_L is Maximum

Current in the circuit shown in Figure 6.1 is

$$I = \frac{E}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

Voltage across L is

$$V_L = \omega L I = \frac{E\omega L}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

Squaring

$$V_L^2 = \frac{E^2\omega^2 L^2}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

This is maximum when $\frac{d{V_L}^2}{d\omega} = 0$

That is,

$$E^2L^2\left[\left\{R^2+\left(\omega C-\frac{1}{\omega C}\right)^2\right\}2\omega-\omega^2\left\{2\left(\omega L-\frac{1}{\omega C}\right)\left(L+\frac{1}{\omega^2 C}\right)\right\}\right]=0$$

$$R^2+\left(\omega L-\frac{1}{\omega C}\right)^2=\left(\omega L-\frac{1}{\omega C}\right)\left(\omega L+\frac{1}{\omega C}\right)$$

$$R^2+\omega^2\cancel{L^2}+\frac{1}{\omega^2C^2}-2\frac{L}{C}=\omega^2\cancel{L^2}-\frac{1}{\omega^2C^2}$$

$$R^2\omega^2C^2+1-2\omega^2LC=-1$$

or
$$\omega^2(2LC-R^2C^2)=2$$

$$\omega^2=\frac{2}{2LC-R^2C^2}$$

$$=\frac{1}{LC\left(1-\frac{R^2C}{2L}\right)}$$

Let this frequency be ω_L.

Then,
$$\omega_L^2=\omega_0^2\frac{1}{1-\frac{1}{2Q^2}}$$

$$\omega_L=\omega_0\sqrt{\frac{1}{1-\frac{1}{2Q^2}}}.$$

That is, $\omega_L>\omega_0$.

6.8 Expression for ω at which V_C is Maximum

Now
$$V_C=\frac{E}{\omega C\sqrt{R^2+\left(\omega^2L-\frac{1}{\omega C}\right)^2}}$$

$$V_C^2=\frac{E^2}{\omega^2C^2\left(R^2+\left(\omega L-\frac{1}{\omega C}\right)^2\right)}$$

This is maximum when $\dfrac{d}{d\omega}(V_C^2)=0$.
That is,

$$-\frac{E^2}{C^2}\left[\omega^2\left\{2\left(\omega L-\frac{1}{\omega C}\right)\left(L-\frac{1}{\omega^2C}\right)+2\omega\left\{R^2+\left(\omega L-\frac{1}{\omega C}\right)^2\right\}\right]=0$$

$$R^2+\left(\omega L-\frac{1}{\omega C}\right)^2=-\left(\omega L-\frac{1}{\omega C}\right)\left(\omega L+\frac{1}{\omega C}\right)$$

$$
\begin{aligned}
R^2 + \omega^2 L^2 + \cancel{\frac{1}{\omega^2 C^2}} - 2\frac{L}{C} &= \cancel{\frac{1}{\omega^2 C^2}} - \omega^2 L^2 \\
2\omega^2 L^2 + R^2 &= 2\frac{L}{C} \\
\omega^2 &= \frac{2\frac{L}{C} - R^2}{2L^2} = \frac{1}{LC} - \frac{R^2}{2L^2} \\
&= \frac{1}{LC}\left(1 - \frac{R^2}{2}\frac{C}{L}\right) = \omega_0^2\left(1 - \frac{1}{2Q^2}\right)
\end{aligned}
$$

Let this frequency be ω_C

$$\omega_C = \omega_0\sqrt{1 - \frac{1}{2Q^2}}$$

i.e., $\qquad \omega_C < \omega_0$

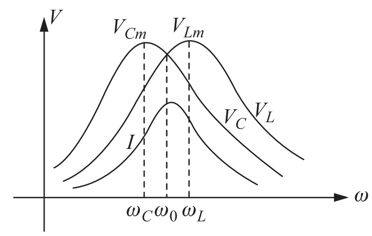


Figure 6.6

Variations of V_C and V_L as functions of ω are shown in Fig. 6.6.

We know that $$V_C = \frac{E}{\sqrt{\omega^2 C^2\left\{R^2 + \frac{(\omega^2 LC - 1)^2}{\omega^2 C^2}\right\}}} = \frac{E}{\sqrt{\{R^2\omega^2 C^2 + (\omega^2 LC - 1)^2\}}} \tag{6.3}$$

Consider $\omega^2 C^2 R^2 + (\omega^2 LC - 1)^2$ and at $\omega = \omega_C$. Then equation(6.3) becomes

$$
\begin{aligned}
\omega_C^2 C^2 R^2 + (\omega_C^2 LC - 1)^2 &= \omega_0^2\left(1 - \frac{1}{2Q^2}\right)C^2R^2 + \left\{\omega_0^2\left(1 - \frac{1}{2Q^2}\right)LC - 1\right\}^2 \\
&= \frac{1}{Q^2}\left(1 - \frac{1}{2Q^2}\right) + \left\{\cancel{\omega_0^2}\left(1 - \frac{1}{2Q^2}\right)\frac{1}{\cancel{\omega_0^2}} - 1\right\}^2 \\
&= \frac{1}{Q^2}\left(1 - \frac{1}{2Q^2}\right) + \left(\frac{1}{4Q^4}\right) = \frac{1}{Q^2} - \frac{1}{2Q^4} + \frac{1}{4Q^4} = \frac{1}{Q^2}\left[1 - \frac{1}{4Q^2}\right]
\end{aligned}
$$

since $\qquad \dfrac{1}{LC} = \omega_0^2$ and $\omega_0 CR = \dfrac{1}{Q}$

Substituting the above expression in the denominator of equation (6.3), we get

$$V_{cm} = \frac{EQ}{\sqrt{1 - \frac{1}{4Q^2}}}$$

6.9 Selectivity with Variable L

In a series resonant circuit connected to a constant voltage, with a constant frequency, when L is varied to achieve resonance, the following conditions prevail:

1. X_C is constant and $I = \dfrac{E}{\sqrt{R^2 + X_C^2}}$ when $L = 0$.
2. With increase in L, X_L increases and $I_m = \dfrac{V}{R}$ at $X_L = X_C$
3. With further increase in L, I proceeds to fall.

All these conditions are depicted in Fig. 6.7 $V_{C\,\max}$ occurs at ω_0 but $V_{L\,\max}$ occurs at a point beyond ω_0.
L at which V_L becomes a maximum is obtained in terms of other constants.

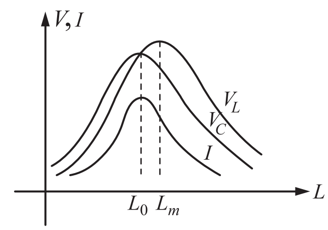


Figure 6.7

$$V_L = \frac{EX_L}{\{R^2 + (X_L - X_C)^2\}^{\frac{1}{2}}}$$

$$V_L^2 = \frac{E^2 X_L^2}{R^2 + (X_L - X_C)^2}$$

This is maximum when $\dfrac{dV_L^2}{dX_L} = 0$.

Therefore, $$\left\{R^2 + (X_L - X_C)^2\right\} 2X_L = X_L^2 \{2(X_L - X_C)\}$$

$$R^2 + X_L^2 + X_C^2 - 2X_L X_C = X_L^2 - X_L X_C$$

Therefore, $$X_L = \frac{R^2 + X_C^2}{X_C}$$

Let the corresponding value of L is L_m.

Then, $$L_m = C(R^2 + X_C^2)$$

and L_0 = value of L at ω_0 such that

$$\omega_0 L = \frac{1}{\omega_0 C}.$$

6.10 Selectivity with Variable C

In a series resonant circuit connected to a constant voltage, constant frequency supply, if C is varied to achive resonance, the following conditions prevail:

1. X_L is constant.
2. X_C varies as inversely as C

   when $C = 0$, $\quad I = 0$.

   when $\omega C = \dfrac{1}{\omega L}$, $\quad I = I_m = \dfrac{V}{R}$.
3. with further increase in C, I starts decreasing as shown in Fig. 6.8, where C_m is the value of capacitance at maximum voltage across C and C_0 is the value of the capacitance at ω_0.

C at which V_C becomes maximum can be determined in terms of other circuit constants as follows.

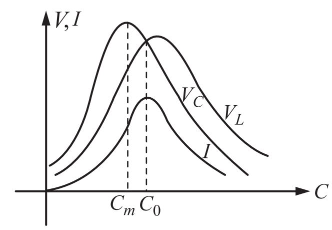


Figure 6.8

$$V_C = \frac{EX_C}{\sqrt{R^2 + (X_L - X_C)^2}}$$
$$V_C^2 = \frac{E^2X_C^2}{R^2 + (X_L - X_C)^2}$$

For maximim V_C, $$\frac{dV_C^2}{dX_C} = 0$$

Then, $$\left\{R^2 + (X_L - X_C)^2\right\} 2X_C - X_C^2 \left\{2(X_L - X_C)(-1)\right\} = 0$$
$$R^2 + X_L^2 + \cancel{X_C^2} - 2X_LX_C = -X_LX_C + \cancel{X_C^2}$$
$$X_C = \frac{R^2 + X_L^2}{X_L}$$

Let the correrponding value of C be C_m.

Then, $$C_m = \frac{L}{R^2 + X_L^2}.$$

6.11 Transfer Functions

6.11.1 Voltage ratio transfer function of a series resonant circuit and frequency response

For the circuit shown in Fig. 6.9, we can write

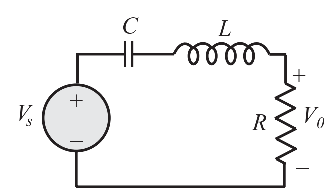


Figure 6.9

$$H(j\omega) = \frac{V_0(j\omega)}{V_s(j\omega)} = \frac{R}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$
$$= \frac{1}{1 + j\left\{\frac{\omega L}{R} - \frac{1}{\omega CR}\right\}}$$
$$= \frac{1}{1 + j\left\{\frac{\omega_0 L}{\omega_0 R}\omega - \frac{\omega_0}{\omega\omega_0 CR}\right\}}$$
$$= \frac{1}{1 + jQ\left[\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right]}$$
$$= \frac{1}{\left[1 + Q^2\left[\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right]^2\right]^{\frac{1}{2}}} \angle \tan^{-1}\left[Q\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right]$$

Let δ be a measure of the deviation in ω from ω_0. It is defined as

$$\delta = \frac{\omega - \omega_0}{\omega_0} = \frac{\omega}{\omega_0} - 1$$

Then $$\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} = (\delta + 1) - \frac{1}{\delta + 1} = \frac{(\delta + 1)^2 - 1}{\delta + 1} = \frac{\delta^2 + 2\delta}{\delta + 1}$$

For small deviations from $\omega_0, \delta << 1$. Then,

$$\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \simeq 2\delta$$

Then, $$H(j\omega) = \frac{1}{1 + j2Q\delta} = \frac{1}{\sqrt{1 + 4Q^2\delta^2}} \angle - \tan^{-1} 2Q\delta$$

The amplitfude and phase response curves are as shown in Fig. 6.10.

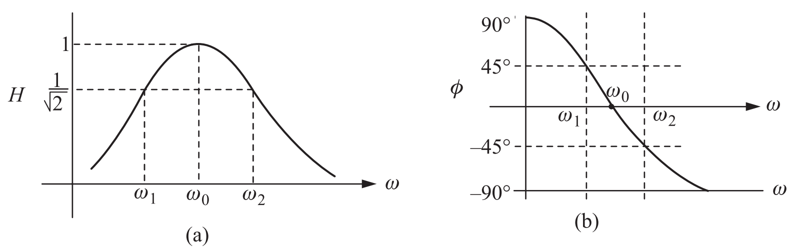


Figure 6.10 (a) and (b): Amplitude and Phase response of a series resonance circuit

6.11.2 Impedance function

The Impedance as a function of $j\omega$ is given by

$$\begin{aligned} Z(j\omega) &= R + j\left(\omega L - \frac{1}{\omega C}\right) \\ &= R\left[1 + j\left(\frac{\omega L}{R} - \frac{1}{\omega C R}\right)\right] \\ &= R\left[1 + jQ\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right] \\ &= R\sqrt{1 + Q^2\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2} \angle \tan^{-1} Q\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) \end{aligned}$$

For small deviations from ω_0, we can write

$$Z(j\omega) \simeq R[1 + j2Q\delta] = R\sqrt{1 + 4Q^2\delta^2} \angle \tan^{-1} 2Q\delta$$

6.12 Parallel Resonance

The dual of a series resonant circuit is often considered as a parallel resonant circuit and it is as shown in Fig. 6.11.

The phasor diagram for resonance is shown in Fig. 6.12.

The admittance as seen by the current source is

$$\begin{aligned} Y(j\omega) &= Y_R + Y_L + Y_C \\ &= \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right) = G + jB \end{aligned}$$

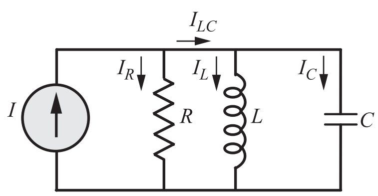


Figure 6.11 Parallel Resonance Circuit

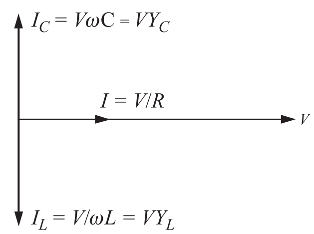


Figure 6.12 Phasor Diagram

If the resonance occurs at ω_0, then the susceptance B is zero. That is,

$$\omega_0 C = \frac{1}{\omega_0 L}$$

or

$$\omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/ sec}.$$

At resonance,

$$I_{C0} = -I_{L0} = j\omega_0 CRI$$

and

$$I_{LC} = I_{C0} + I_{L0} = 0$$

The quality factor, as in the case of series resonant circuit is defined as

$$\begin{aligned} Q &= 2\pi \frac{\text{Maximum energy stored}}{\text{Energy dissipated in a period}} \\ &= 2\pi \frac{\frac{1}{2} C V_m^2}{\frac{1}{2} \frac{V_m^2}{R} T} \\ &= 2\pi f_0 CR = \omega_0 CR. \end{aligned}$$

Since

$$\omega_0 C = \frac{1}{\omega_0 L},$$

$$Q = \frac{R}{\omega_0 L}.$$

On either side of ω_0 there are two frequencies at which the voltage is same. At resonance, the voltage is maximum and is given by $V_m = IR$ and is evident from the response curve as shown in Fig. 6.13. At this frequency, $p = p_m = \dfrac{V_m^2}{R}$ watts. The frequencies at which the voltage is $\dfrac{1}{\sqrt{2}}$ times the maximum voltage are called half power frequencies or cut off frequencies, since at these frequencies,

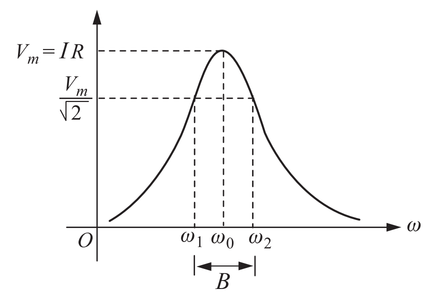


Figure 6.13

$$p = \frac{\left(\frac{V_m}{\sqrt{2}}\right)^2}{R} = \frac{V_m^2}{2R} = \text{half of the maximum power.}$$

At any ω,

$$Y = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)$$

At ω_1 and ω_2,

$$|Y| = \frac{1}{\sqrt{2}R} = \sqrt{\left(\frac{1}{R}\right)^2 + \left(\omega C - \frac{1}{\omega L}\right)^2}$$

Squaring,

$$\frac{1}{2R^2} = \frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2$$

Therefore, $$\left(\omega C - \frac{1}{\omega L}\right) = \frac{1}{R}$$

At $\omega = \omega_2$,

$$\omega_2 C - \frac{1}{\omega_2 L} = \frac{1}{R}$$

$$\omega_2^2 LC - 1 = \frac{\omega_2 L}{R}$$

$$\omega_2^2 LCR - R - \omega_2 L = 0$$

Hence, $$\omega_2 = \frac{L + \sqrt{L^2 + 4LCR^2}}{2LCR}$$

Note that only positive sign is used before the square root to ensure that ω_2 is positive.

Thus, $$\omega_2 = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

Similarly, $$\omega_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

So that, bandwidth

$$B = \omega_2 - \omega_1 = \frac{1}{RC}$$

and $$\omega_1\omega_2 = \left(\frac{1}{2RC}\right)^2 + \frac{1}{LC} - \left(\frac{1}{2RC}\right)^2$$
$$= \frac{1}{LC} = \omega_0^2$$

Thus, $$\omega_0 = \sqrt{\omega_1\omega_2}$$

As $\omega_0 = \dfrac{1}{\sqrt{LC}}$ and $Q = \omega_0 RC = \dfrac{R}{\omega_0 L}$

$$Q = \frac{R}{L}\sqrt{LC} = R\sqrt{\frac{C}{L}}$$

Since $\dfrac{1}{2RC} = \dfrac{B}{2}$

$$\omega_2 = \frac{B}{2} + \sqrt{\left(\frac{B}{2}\right)^2 + \omega_0^2}$$

and $$\omega_1 = -\frac{B}{2} + \sqrt{\left(\frac{B}{2}\right)^2 + \omega_0^2}$$

Using $B = \dfrac{\omega_0}{Q}$,

$$\omega_2 = \omega_0\left[\frac{1}{2Q} + \sqrt{1 + \left(\frac{1}{2Q}\right)^2}\right]$$

and $$\omega_1 = \omega_0\left[-\frac{1}{2Q} + \sqrt{1 + \left(\frac{1}{2Q}\right)^2}\right]$$

6.13 Transfer Function and Frequency Response

The transfer function for a parallel RLC circuit shown in Fig. 6.14. is $H(j\omega)$, the current ratio transfer function.

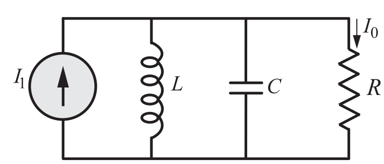


Figure 6.14 Parallel RLC Circuit

$$H(j\omega) = \frac{I_0(j\omega)}{I_1(j\omega)} = \frac{1}{RY(j\omega)}$$
$$= \frac{1}{R}\frac{1}{\frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)} = \frac{1}{1 + jR\left(\omega C - \frac{1}{\omega L}\right)}$$
$$= \frac{1}{1 + j\left(\frac{\omega\omega_0 CR}{\omega_0} - \frac{\omega_0 R}{\omega_0 \omega L}\right)} = \frac{1}{1 + jQ\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)}$$

As in the case of series resonance, here also let

$$\delta = \frac{\omega - \omega_0}{\omega_0} = \frac{\omega}{\omega_0} - 1$$

then,

$$\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} = \frac{\delta^2 + 2\delta}{\delta + 1}$$

For $\delta << 1$, for small deviations from ω_0

$$\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \simeq 2\delta$$

Therefore,

$$H(j\omega) = \frac{1}{1 + j2Q\delta}$$

6.14 Resonance in a Two Branch $RL - RC$ Parallel Circuit

Consider the two branch parallel circuit shown in Fig. 6.15. Let E be the voltage across each of the parallel circuit shown in the figure. The vector diagram at resonance is shown in Figure 6.1.

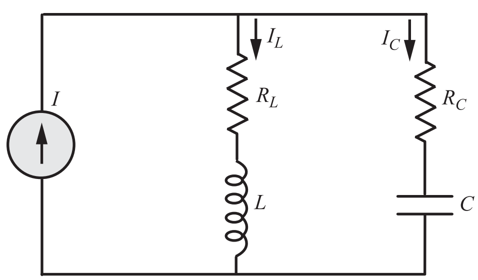


Figure 6.15 Two branch Parallel Circuit

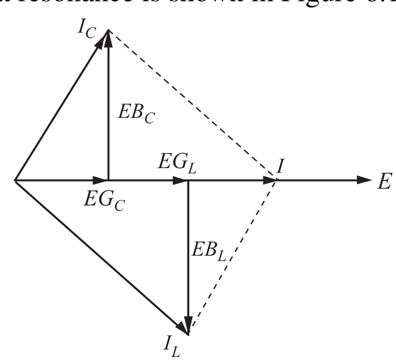


Figure 6.16

The admittance of the circuit is $Y(j\omega) = G_L - jB_L + G_C + jB_C$
For resonance,

$$B_L = B_C$$

If this occurs at $\omega = \omega_0$,

then
$$\frac{\omega_0 L}{RL^2 + \omega_0^2\, L^2} = \frac{\frac{1}{\omega_0 C}}{R_C^2 + \frac{1}{\omega_0^2\, C^2}}$$

$$= \frac{\omega_0 C}{R_C^2 \omega_0^2\, C^2 + 1}$$

$$L(1 + \omega_0^2 C^2\, R_C^2) = C(R_L^2 + \omega_0^2\, L^2)$$

$$\omega_0^2(LC^2\, R_C^2 - L^2 C) = R_L^2\, C - L$$

$$\omega_0^2 = \frac{R_L^2\,C - 1}{LC^2\,R_C^2 - L^2C}$$

$$= \frac{1}{LC}\frac{R_L^2\,C - L}{(R_C^2\,C - L)} = \frac{1}{LC}\frac{R_L^2 - \frac{L}{C}}{(R_C^2 - \frac{L}{C})}$$

Therefore,
$$\omega_0 = \frac{1}{\sqrt{LC}}\sqrt{\frac{R_L^2 - \frac{L}{C}}{R_C^2 - \frac{L}{C}}}$$

This is the expression for resonant frequency. It is to be noted that

1. resonance is not possible for certain combination of circuit elements unlike in a series circuit where resonance is always possible.
2. resonance is also possible by varying of R_L or R_C.

Consider the case where

$$R_C^2 < \frac{L}{C} < R_L^2$$

or

$$R_L^2 < \frac{L}{C} < R_C^2$$

In both these cases, the quantity under radical is negative and therefore resonance is not possible.

The admittance at resonance of the above parallel circuit is

$$Y_0 = \left(\frac{R_L}{R_L^2 + X_{L_0}^2} + \frac{RC}{R_C^2 + X_{C_0}^2}\right)\,\mathrm{S}$$

where X_{L_0} and X_{C_0} are the inductive and capacitive reactances respectively at resonance.

If
$$R_L = R_C \neq \sqrt{\frac{L}{C}}$$

then
$$\omega_0 = \frac{1}{\sqrt{LC}}$$

as in R, L, C series circuit.

If
$$R_L = R_C = \sqrt{\frac{L}{C}}$$

which means

$$R_L^2 = R_C^2 = R^2 = \frac{L}{C} = X_L X_C.$$

Then,
$$B_L - B_C = \frac{X_L}{R_L^2 + X_L^2} - \frac{X_C}{R_C^2 + X_C^2}$$

$$= \frac{1}{X_L + X_C} - \frac{1}{X_L + X_C} = 0$$

In this case, the circuit acts as a pure resistive circuit irrespective of frequency. That is, the circuit is resonant for all frequencies.

In this case the circuit admittance is

$$Y = \frac{R_L}{R_L^2 + X_L^2} + \frac{R_C}{R_C^2 + X_C^2}$$

$$= R\left[\frac{R^2 + X_L^2 + R^2 + X_C^2}{R^4 + R^2(X_L^2 + X_C^2) + X_L^2 X_C^2}\right]$$

$$= R\frac{2R^2 + X_L^2 + X_C^2}{2R^4 + R^2(X_L^2 + X_C^2)}$$

$$= \frac{R}{R^2}\left[\frac{2R^2 + X_L^2 + X_C^2}{2R^2 + X_L^2 + X_C^2}\right]$$

$$= \frac{1}{R} = \sqrt{\frac{C}{L}}$$

or

$$Z = R = \sqrt{\frac{L}{C}}$$

6.14.1 Resonance by varying inductance

If resonance is achieved by varying only L in the circuit shown in Figure 6.15 but with constant current constant frequency source, then the condition for resonance is

$$B_L = B_C$$

$$\Rightarrow \qquad \frac{X_L}{R_L^2 + X_L^2} = \frac{X_C}{R_C^2 + X_C^2} = \frac{X_C}{Z_C^2} \quad \text{where} \quad Z_C^2 = R_C^2 + X_C^2$$

Then,

$$X_L^2 X_C - X_L Z_C^2 + X_C R_L^2 = 0$$

Solving, for X_L we get

$$X_L = \frac{Z_C^2 \pm \sqrt{Z_C^4 - 4X_C^2 R_L^2}}{2X_C}$$

Therefore,

$$L = \frac{C}{2}\left[Z_C^2 \pm \sqrt{Z_C^4 - 4X_C^2 R_L^2}\right] \quad \left(\text{since } X_L X_C = \frac{L}{C}\right)$$

The following conditions arise:

1. If $Z_C^4 > 4X_C^2\, R_L^2, L$ has two values for the circuit to resonate.
2. For $Z_C^4 = 4X_C^2\, R_L^2, L = \frac{1}{2}CZ_C^2$ for reasonance.
3. For $Z_C^4 < 4X_C^2\, R_L^2$, No value of L makes the circuit to resonate.

6.14.2 Resonance by varying capacitance

As in the previous case, we have at resonance,,

$$B_L = B_C$$

$$\Rightarrow \qquad \frac{X_C}{R_C^2 + X_C^2} = \frac{X_L}{Z_L^2}, \text{ where } Z_L^2 = R_L^2 + X_L^2$$

Simplifying we get,

$$X_C^2 X_L - X_C Z_L^2 + R_C^2 X_L = 0$$

$$X_C = \frac{Z_L^2 \pm \sqrt{Z_L^4 - 4X_L^2 R_C^2}}{2X_L}$$

Therefore,

$$C = \frac{2L}{Z_L^2 \pm \sqrt{Z_L^4 - 4X_L^2 R_C^2}}$$

The following conditions arise:

1. For $Z_L^4 > 4X_L^2\ R_C^2$, there are two values for C to resonante.
2. For $Z_4^4 = 4X_L^2\ R_C^2$, resonance occurs at $C = \frac{2L}{Z_L^2}$.
3. For $Z_L^4 < 4X_L^2\ R_C^2$, no value of C makes the circuit to resonate.

6.14.3 Resonance by varying R_L or R_C

It is often possible to adjust a two branch parallel combination to resonate by varying either R_L or R_C. This is because, when the supply is of constant current and, constant frequency, these resistors control inphase and quadratare components of the currents in the two parallel paths.

From the condition $B_L = B_C$, we get

$$\frac{X_L}{R_L^2 + X_L^2} = \frac{X_C}{R_C^2 + X_C^2}$$

$$R_L^2 = \frac{X_L}{X_C} R_C^2 + X_L X_C - X_L^2$$

$$R_L = \sqrt{\frac{X_L}{X_C} R_C^2 + X_L X_C - X_L^2} \tag{6.4}$$

This equation gives the value of R_L for resonance when all other quantities are constant and the term under radical is positive.

Similarly if only R_C is variable, keeping all other quantities constant, the value of R_C for resonanace is given by

$$R_C = \sqrt{\frac{X_C}{X_L} R_L^2 + X_L X_C - X_C^2}$$

provided the term under radical is positive.

6.15 Practical Parallel and Series Resonant Circuits

A practical resonant parallel circuit contains an inductive coil of resistance R and inductance L in parallel with a capacitor C as shown in Fig. 6.17. It is called a tank circuit because it stores energy in the magnetic field of the coil and in the electric field of the capacitor. Note that resistance R_C of the capacitor is negligibly small.

Condition for parallel resonance is shown by the phasor diagram of Fig. 6.18.

$$I_C = I_L \sin\phi$$

That is,
$$B_C = B_L$$
$$\Rightarrow \qquad \frac{\omega L}{R^2 + \omega^2 L^2} = \omega C.$$

Let the value of ω which satisfy this condition be ω_0.

Then,
$$R^2 + \omega_0^2 L^2 = \frac{L}{C} \tag{6.5}$$
$$\omega_0^2 = \left(\frac{L}{C} - R^2\right)\frac{1}{L^2} = \frac{1}{LC}\left(1 - \frac{R^2C}{L}\right)$$
$$\omega_0 = \frac{1}{\sqrt{LC}}\sqrt{1 - \frac{R^2C}{L}} \tag{6.6}$$

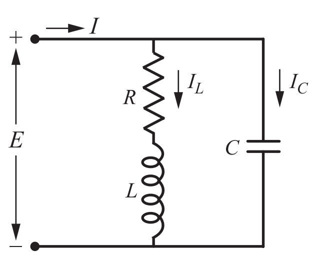


Figure 6.17

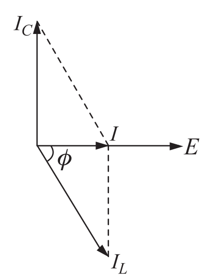


Figure 6.18

Admittance of the circuit shown in figure 6.17 is

$$Y(j\omega) = \frac{1}{R + j\omega L} + j\omega C$$
$$= \frac{R}{R^2 + \omega^2 L^2} - \frac{j\omega L}{R^2 + \omega^2 L^2} + j\omega C$$

At $\omega = \omega_0$, $Y(j\omega)$ is purely real.

Hence,
$$Y(j\omega_0) = \frac{R}{R^2 + \omega_0^2 L^2} \tag{6.7}$$

Substituting for ω_0 in equation (6.7),

$$Y(j\omega_0) = \frac{R}{R^2 + \omega_0^2 L^2} = \frac{R}{\frac{L}{C}} = \frac{RC}{L} \tag{6.8}$$

and the circuit is a pure resistive with $R_0 = \dfrac{L}{CR}$, which is called the dynamic resistance of the circuit. This is greater than R if there is resonance. However, note that if $\dfrac{R^2C}{L} > 1$, there is no resonance.

Fig. 6.19 shows a practical series resonant circuit. The input impedance as a function of ω is

$$Z(j\omega) = j\omega L + \frac{G}{G^2 + \omega^2 C^2} - j\frac{\omega C}{G^2 + \omega^2 C^2}$$

Condition for resonance is

$$\omega L = \frac{\omega C}{G^2 + \omega^2 C^2}$$

$$\omega^2 = \left(\frac{C}{L} - G^2\right)\frac{1}{C^2} = \frac{1}{LC} - \frac{1}{C^2R^2}$$

$$= \frac{1}{LC}\left(1 - \frac{L}{CR^2}\right)$$

$$\omega = \frac{1}{\sqrt{LC}}\left(1 - \frac{L}{CR^2}\right)$$

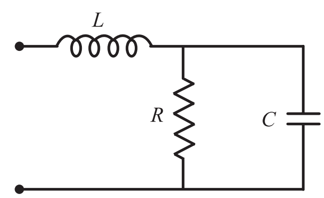


Figure 6.19

Impedance at resonance is

$$Z_0 = \frac{G}{G^2 + \omega C^2} = \frac{G}{\frac{C}{L}} = \frac{L}{CR}$$

The circuit at resonance is a purely resistive, and $Z_0 = R_0 = \dfrac{L}{CR}$. However, note that here also resonance is not possible for $\dfrac{L}{CR^2} > 1$.

In both the circuits, shown in Figs 6.18 and 6.19, resonance is achieved by varying either C or L until the input impedance or admittance is real and this process is called tuning. For this reason these circuits are called tuned circuits.

Series circuits

EXAMPLE 6.1

Two coils, one of $R_1 = 0.51\ \Omega$, $L_1 = 32$ mH, the other of $R_2 = 1.3\ \Omega$ and $L_2 = 15$ mH and two capacitors of 25 μF and 62 μF are all in series with a resistance of $0.24\ \Omega$. Determine the following for this circuit

(i) Resonance frequency

(ii) Q of each coil

(iii) Q of the circuit

(iv) Cut off frequencies

(v) Power dissipated at resonance if $E = 10$ V.

SOLUTION

From the given values, we find that

$$R_s = 0.51 + 1.3 + 0.24 = 2.05\Omega$$
$$L_s = 32 + 15 = 47 \text{ mH}$$
$$C_s = \frac{25 \times 62}{87}\ \mu\text{F} = 17.816\ \mu\text{F}$$

(i) Resonant frequency:

$$\omega_0 = \frac{1}{\sqrt{L_s C_s}}$$
$$= \frac{1}{\sqrt{47 \times 10^{-3} \times 17.816 \times 10^{-6}}}$$
$$= 1092.8 \text{ rad/sec}$$

(ii) Q of coils:

For Coil 1,
$$Q_1 = \frac{\omega_0 L_1}{R_1}$$
$$= \frac{1092.8 \times 32 \times 10^{-3}}{0.51} = 68.57$$

For Coil 2,
$$Q_2 = \frac{\omega_0 L_2}{R_2}$$
$$= \frac{1092.8 \times 15 \times 10^{-3}}{1.3} = 12.6$$

(iii) Q of the circuit:

$$Q = \frac{\omega_0 L_s}{R_s}$$
$$= \frac{1092.8 \times 47 \times 10^{-3}}{2.05} = 25$$

(iv) Cut off frequencies: Band width is,

$$B = \frac{\omega_0}{Q} = \frac{1092.8}{25} = 43.72$$

Considering $Q > 5$, the cut off frequencies,

$$\omega_{2,1} = \omega_0 \pm \frac{B}{2} = 1092.8 \pm 21.856$$

Therefore,
$$\omega_2 = 1115 \text{ rad/sec} \quad \text{and } \omega_1 = 1071 \text{ rad/sec}.$$

(v) Power dissipated at resonance:

Given $E = 10$ V

We know that at resonance, only the resistance portion will come in to effect. Therefore

$$P = \frac{E^2}{R} = \frac{10^2}{2.05} = 48.78 \text{ W}$$

EXAMPLE 6.2

For the circuit shown in Fig. 6.20, find the out put voltages at

(i) $\omega = \omega_0$

(ii) $\omega = \omega_1$

(iii) $\omega = \omega_2$

when $v_s(t) = 800 \cos \omega t$ mV.

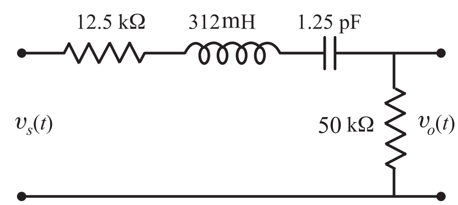


Figure 6.20

SOLUTION

For the circuit, using the values given, we can find that resonant frequency

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$= \frac{1}{\sqrt{312 \times 10^{-3} \times 1.25 \times 10^{-12}}} = 1.6 \times 10^6 \text{ rad/ sec}$$

Quality factor:

$$Q = \frac{\omega_0 L}{R}$$

$$= \frac{1.6 \times 10^6 \times 312 \times 10^{-3}}{62.5 \times 10^3} = 8$$

Band width:

$$B = \frac{\omega_0}{Q}$$

$$= \frac{1.6 \times 10^6}{8} = 0.2 \times 10^6 \text{ rad/ sec}$$

As $Q > 5$,

$$\omega_{2,1} = \omega_0 \pm \frac{B}{2}$$

$$= (1.6 \pm 0.1)10^6 \text{ rad/ sec}$$

Hence, $\omega_2 = 1.7 \times 10^6$ rad/ sec

and $\omega_1 = 1.5 \times 10^6$ rad/ sec

(i) Output voltage at ω_0:

Using the relationship of transfer function, we get

$$H(j\omega)|_{\omega=\omega_0} = \frac{50I_m}{62.5I_m}$$
$$= 0.8 \angle 0^\circ$$

Since the current is maximum at resonance and is same in both resistors,

$$v_o(t) = 0.8 \times 800 \cos(1.6 \times 10^6 t) \text{ mV}$$
$$= 640 \cos(1.6 \times 10^6 t) \text{ mV}$$

At ω_1 and ω_2, $Z_{\text{in}} = \sqrt{2}R_s \angle \pm 45^\circ$. Therefore,

$$H(j\omega)|_{\omega=\omega_1} = \frac{R_{\text{out}}}{Z_{\text{in}}} = \frac{50}{\sqrt{2} \times 62.5} \angle 45^\circ$$
$$= 0.5657 \angle 45^\circ$$

and $$H(j\omega)|_{\omega=\omega_2} = 0.5657 \angle 45^\circ$$

(ii) Out put voltage at $\omega = \omega_1$

$$v_o(t) = 0.5657 \times 800 \cos(1.6 \times 10^6 t + 45^\circ) \text{ mv}$$
$$= 452.55 \cos(1.6 \times 10^6 t + 45^\circ) \text{ mV}$$

(iii) Out put voltage at $\omega = \omega_2$

$$v_o(t) = 452.55 \cos(1.6 \times 10^6 t - 45^\circ) \text{ mV}$$

EXAMPLE 6.3

In a series circuit $R = 6\ \Omega$, $\omega_0 = 4.1 \times 10^6$ rad/sec, band width = 10^5 rad/sec. Compute L, C, half power frequencies and Q.

SOLUTION

We know that Quality factor,

$$Q = \frac{\omega_0}{B} = \frac{4.1 \times 10^6}{10^5} = 41$$

Also, $$Q = \frac{\omega_0 L}{R}$$

Therefore, $$L = \frac{QR}{\omega_0} = \frac{41 \times 6}{4.1 \times 10^6} = 60\ \mu\text{H}$$

and $$Q = \frac{1}{\omega_0 CR}$$

Hence, $$C = \frac{1}{\omega_0 QR}$$
$$= \frac{1}{4.1 \times 10^6 \times 41 \times 6} = 991.5 \text{ pF}$$

As $Q > 5$,

$$\omega_{2,1} = \omega_0 \pm \frac{B}{2} = 4.1 \times 10^6 \pm \frac{10^5}{2}$$

That is,
$$\omega_2 = 4.15 \times 10^6 \text{ rad/sec}$$
and
$$\omega_1 = 4.05 \times 10^6 \text{ rad/sec}$$

EXAMPLE 6.4

In a series resonant circuit, the current is maximum when $C = 500$ pF and frequency is 1 MHz. If C is changed to 600 pF, the current decreases by 50%. Find the resistance, inductance and quality factor.

SOLUTION

☐ Case 1

Given,

$$C = 500 \text{ pF}$$
$$I = I_m$$
$$f = 1 \times 10^6 \text{ Hz}$$
$$\Rightarrow \quad \omega_0 = 2\pi \times 10^6 \text{ rad/sec}$$

We know that

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Therefore, Inductance,

$$L = \frac{1}{\omega_0^2 C} = \frac{10^{12}}{(2\pi \times 10^6)^2 \times 500}$$
$$= 0.0507 \text{ mH}$$

☐ Case 2

When C = 600 pF,

$$I = \frac{I_m}{2} = \frac{E}{2R} \quad \Rightarrow \quad |Z| = 2R$$
$$\sqrt{R^2 + X^2} = 2R \quad \Rightarrow \quad X = \sqrt{3}R$$
$$X = X_L - X_C$$
$$= 2\pi \times 10^6 \times 0.0507 \times 10^{-3} - \frac{10^{12}}{2\pi \times 10^6 \times 600}$$
$$= 318.56 - 265.26$$
$$= 53.3\ \Omega = \sqrt{3}R$$

Therefore resistance,

$$R = \frac{53.3}{\sqrt{3}} = 30.77\Omega$$

Quality factor,

$$Q = \frac{\omega_0 L}{R} = \frac{318.56}{30.77} = 10.35$$

EXAMPLE 6.5

In a series circuit with $R = 50\ \Omega$, L = 0.05 H and $C = 20\ \mu$F, frequency is varied till the voltage across C is maximum. If the applied voltage is 100 V, find the maximum voltage across the capacitor and the frequency at which it occurs. Repeat the problem for $R = 10\ \Omega$.

SOLUTION

□ Case 1

Given $R = 50\ \Omega$, L = 0.05 H, $C = 20\ \mu$F

We know that

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{10^3}{\sqrt{0.05 \times 20}} = 10^3 \text{ rad/sec}$$

$$Q = \frac{\omega_0 L}{R} = \frac{10^3 \times 0.05}{20} = 1$$

Using the given value of E = 100 V in the relationship

$$V_{Cm} = \frac{QE}{\sqrt{1 - \frac{1}{4Q^2}}}$$

we get

$$V_{Cm} = \frac{100}{\sqrt{1 - \frac{1}{4}}} = 115.5 \text{ V}$$

and the corresponding frequency at this voltage is

$$\omega_C = \omega_0 \sqrt{1 - \frac{1}{2Q^2}}$$

$$= 10^3 \sqrt{\frac{1}{2}} = 707 \text{ rad/ sec}$$

□ Case 2

When $R = 10\ \Omega$,

$$Q = \frac{10^3 \times 0.05}{10} = 5$$

$$V_{Cm} = \frac{5 \times 100}{\sqrt{1 - \frac{1}{4 \times 25}}} = 502.5 \text{ V}$$

$$\omega_C = 10^3 \sqrt{1 - \frac{1}{50}} = 990 \text{ rad/ sec}$$

EXAMPLE 6.6

(i) A series resonant circuit is tuned to 1 MHz. The quality factor of the coil is 100. What is the ratio of current at a frequency 20 kHz below resonance to the maximum current?
(ii) Find the frequency above resonance when the current is reduced to 90% of the maximum current.

SOLUTION

(i) Let ω_a be the frequency 20 kHz below the resonance, I_a be the current and Z_a be the impedance at this frequency.

Then
$$\omega_a = 10^6 - 20 \times 10^3 = 980 \text{ kHz}$$
$$\frac{\omega_a}{\omega_0} - \frac{\omega_0}{\omega_a} = \frac{980}{10^3} - \frac{10^3}{980}$$
$$= -40.408 \times 10^{-3} = 2\delta$$

Now the ratio of current,
$$\frac{I_a}{I_m} = \frac{R}{Z_a} = \frac{1}{1 + j(2\delta)Q}$$
$$= \frac{1}{1 - j100(40.408 \times 10^{-3})}$$
$$= \frac{1}{1 - j4.0408}$$
$$= 0.2402 \underline{/76^\circ}$$

(ii) Let ω_b be the frequency at which $I_b = 0.9 I_m$

Then
$$\left|\frac{I_b}{I_m}\right| = \frac{1}{1 + j(2\delta)Q} = 0.9$$

or
$$\sqrt{1 + x^2} = \frac{1}{0.9}$$

where
$$x = (2\delta)100$$

Then,
$$1 + x^2 = \frac{1}{0.81} = 1.2346$$

or
$$x^2 = 0.2346$$

and
$$x = 0.4843$$

We know that
$$\delta = \frac{\omega_b}{\omega_0} - 1 = \frac{0.4843}{200}$$

Hence
$$\omega_b = \left(1 + \frac{0.4843}{200}\right)\omega_0$$
$$= 1.00242 \text{ MHz}$$

EXAMPLE 6.7

For the circuit shown in Fig. 6.21, obtain the values of ω_0 and v_C at ω_0.

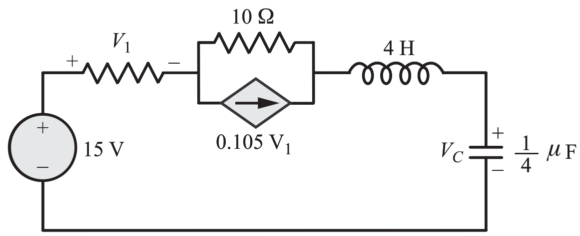


Figure 6.21

SOLUTION

For the series circuit,

$$\omega_0 = \frac{1}{\sqrt{LC}}$$
$$= \frac{1}{\sqrt{4 \times \frac{1}{4} \times 10^{-6}}} = 10^3 \text{ rad/sec}$$

At this $\omega_0, I = I_m$. Therefore,

$$V_1 = 125 I_m$$

and the circuit equation is

$$1.5 = V_1 + (I_m - 0 \cdot 105 V_1)10 + jV_L - jV_C$$

Since $V_L = V_C$, the above equation can be modified as

$$1.5 = 125 I_m + 10 I_m - 1.05 \times 125 I_m$$

Hence, $$I_m = \frac{1.5}{3.75} \text{ A}$$

and $$V_c = \frac{1.5}{3.75} \times \frac{4 \times 10^6}{10^3}$$
$$= 1600 \text{ V}$$

EXAMPLE 6.8

For the circuit shown in Fig. 6.22(a), obtain Z_{in} and then find ω_0 and Q.

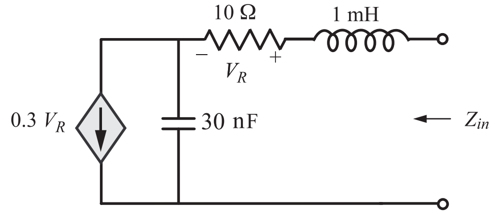


Figure 6.22(a)

SOLUTION

Taking I as the input current, we get

$$V_R = 10I$$

and the controlled current source,

$$0.3V_R = 0.3 \times 10I$$
$$= 3I$$

The input impedance can be obtained using the standard formula

$$Z_{in}(j\omega) = \frac{\text{Applied voltage}}{\text{Input current}} = \frac{V}{I} \tag{6.6}$$

For futher analysis, the circuit is redrawn as shown in Fig. 6.22(b). It may be noted that the controlled current source is transformed to its equivalent voltage source.

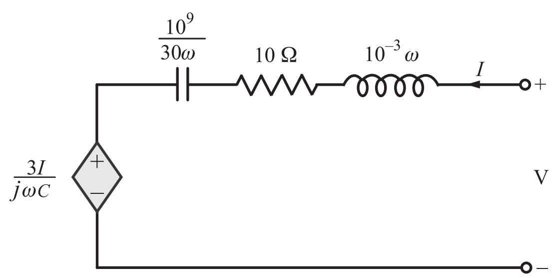


Figure 6.22(b)

Referring Fig. 6.22(b), the circuit equation may be obtained as

$$V = \left(10 + j10^{-3}\omega - \frac{j10^9}{30\omega} - \frac{j3}{30\omega \times 10^{-9}}\right) I \tag{6.7}$$

Substituting equation (6.7) in equation (6.6), we get

$$Z_{\text{in}} = 10 + j\left(10^{-3}\omega - \frac{4 \times 10^9}{30\omega}\right) \Omega$$

For resonance, Z_{in} should be purely real. This gives

$$10^{-3}\omega = \frac{4 \times 10^9}{30\omega}$$

Rearranging,

$$\omega^2 = \frac{4 \times 10^9}{30 \times 10^3}$$
$$= 0.133 \times 10^{12}$$

Solving we get

$$\omega = \omega_0 = \sqrt{0.133 \times 10^{12}}$$
$$= 365 \times 10^3 \text{ rad/ sec}$$

Quality factor

$$Q = \frac{\omega_0 L}{R}$$
$$= \frac{365 \times 10^3 \times 10^{-3}}{10}$$
$$= 36.5$$

Parallel circuits

EXAMPLE 6.9

For the circuit shown in Fig. 6.23(a), find ω_0, Q, BW and half power frequencies and the out put voltage V at ω_0.

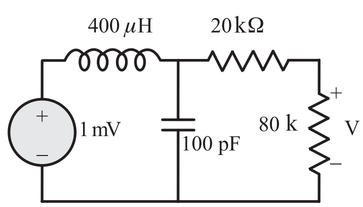


Figure 6.23(a)

SOLUTION

Transforming the voltage source into current source, the circuit in Fig. 6.28(a) can be redrawn as in Fig. 6.23(b).

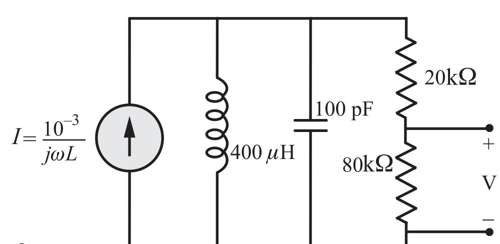


Figure 6.23(b)

Then, $$\omega_0 = \frac{1}{\sqrt{LC}}$$
$$= \frac{109}{\sqrt{400 \times 100}}$$
$$= 5 \times 10^6 \text{ rad/ sec}$$
$$Q = \omega_0 CR$$
$$= 5 \times 10^6 \times 100 \times 10^{-12} \times 100 \times 10^3 = 50$$
$$B = \frac{\omega_0}{Q} = \frac{5 \times 10^6}{50} = 10^5 \text{ rad/ sec}$$

As $Q > 10$,

$$\omega_{2,1} = \pm\frac{B}{2} + \omega_0$$
$$= 5 \times 10^6 \pm \frac{10^5}{2}$$

Hence, $\omega_2 = 5.05 \text{ M rad/ sec}$ and $\omega_1 = 4.95 \text{ M rad/ sec}$

Output voltage,

$$V = I \times 80 \text{ k}\Omega$$
$$= \frac{10^{-3} \times 80 \times 10^3}{j5 \times 10^6 \times 400 \times 10^{-6}}$$
$$= 0.04 \angle{-90^\circ} \text{ V}$$

EXAMPLE 6.10

In a parallel RLC circuit, $C = 50\ \mu$F. Determine BW, Q, R and L for the following cases.

(i) $\omega_0 = 100, \omega_2 = 120$

(ii) $\omega_0 = 100, \omega_1 = 80$

SOLUTION

(i) $\omega_0 = 100, \omega_2 = 120$

We know that

$$\omega_0 = \sqrt{\omega_1 \omega_2}$$

Rearraging we get

$$\omega_1 = \frac{\omega_0^2}{\omega_2}$$
$$= \frac{100^2}{120} = 83.33 \text{ rad/sec}$$

Band width

$$B = \omega_2 - \omega_1$$
$$= 120 - 83.33 = 36.67 \text{ rad/ sec}$$

Quality factor,

$$Q = \frac{\omega_0}{B}$$
$$= \frac{100}{36.67} = 2.73$$

We know that

$$Q = \frac{R}{\omega_0 L} = \omega_0 RC \tag{6.8}$$

Rearraging equation (6.8),

$$R = \frac{Q}{\omega_0 C}$$
$$= \frac{2.73 \times 10^6}{100 \times 50} = 546\ \Omega$$

Similarly
$$L = \frac{1}{\omega_0^2 C}$$
$$= \frac{10^6}{100^2 \times 50} = 2\ \text{H}$$

(ii) $\omega_0 = 100, \omega_1 = 80$: Solving the same way as in case (i), we get

$$\omega_2 = \frac{100^2}{80} = 125$$
$$\text{BW} = B = 125 - 80 = 45\ \text{rad/sec}$$
$$Q = \frac{100}{45} = 2.22$$

EXAMPLE 6.11

In the circuit shown in Fig. 6.24(a), $v_s(t) = 100 \cos \omega t$ volts. Find resonance frequency, quality factor and obtain i_1, i_2, i_3. What is the average power loss in 10 kΩ. What is the maximum stored energy in the inductors?

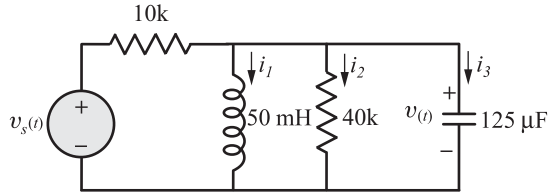


Figure 6.24(a)

SOLUTION

The circuit in Fig. 6.24(a) is redrawn by replacing its voltage source by equivalent current source as shown in Fig. 6.24(b).

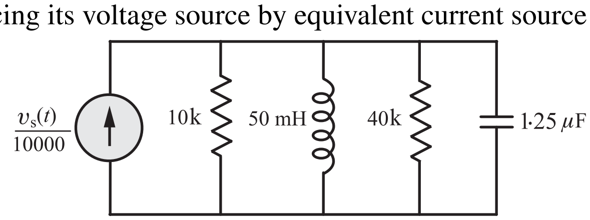


Figure 6.24(b)

Resonance frequency,

$$\omega_0 = \frac{1}{\sqrt{LC}}$$
$$= \frac{1}{\sqrt{50 \times 10^{-3} \times 1.25 \times 10^{-6}}}$$
$$= 4000\ \text{rad/sec}$$

Quality factor,

$$Q = \omega_0 C\ R_{\text{eq}}$$
$$= 4000 \times 1.25 \times 10^{-6} \times 8 \times 10^3$$
$$= 40$$

At resonance, the current source will branch into resistors only. Hence,

$$\begin{aligned} v(t) &= (10\text{k}\Omega || 40\text{k}\Omega) \times \frac{v_s(t)}{10000} \\ &= 80\cos 4000t \text{ volts} \end{aligned}$$

$i_1(t)$ lags $v(t)$ by 90°. Therefore,

$$\begin{aligned} i_1(t) &= \frac{80}{50 \times 10^{-3} \times 4000} \sin 40000t \\ &= 400 \sin 4000t \text{ mA} \\ i_2(t) &= \frac{80}{40 \times 1000} \cos 4000t \\ &= 2\cos 4000t \text{ mA} \\ i_3(t) &= -i_1(t) \\ &= -400 \sin 4000t \text{ mA} \end{aligned}$$

Average power in 10 kΩ:

$$\begin{aligned} P_{\text{av}} &= \frac{\frac{80^2}{\sqrt{2}}}{10 \times 10^3} \\ &= 0.32 \text{ W} \end{aligned}$$

Maximum stored energy in the inductance:

$$\begin{aligned} E &= \frac{1}{2} L I_m^2 \\ &= \frac{1}{2} \times 50 \times 10^{-3} \times (400 \times 10^{-3})^2 \\ &= 4 \text{ mJ} \end{aligned}$$

EXAMPLE 6.12

For the network shown in Fig. 6.25(a), obtain Y_{in} and then use it to determine the resonance frequency and quality factor.

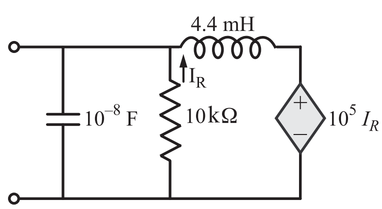


Figure 6.25(a)

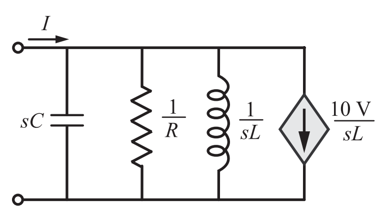


Figure 6.25(b)

SOLUTION

Considering V as the input voltage and I as the input current, it can be found that

$$10\text{k}\Omega \times I_R = -V \Rightarrow 10^4\ I_R = -V$$

The circuit in Fig. 6.25(a) is redrawn by replacing the controlled voltage source in to its equivalent current source by taking $s = j\omega$ and is shown in Fig. 6.25(b). Referring Fig. 6.25(b),

$$I - \frac{10V}{sL} = V\left(sC + \frac{1}{R} + \frac{1}{sL}\right)$$
$$\Rightarrow \qquad I = V\left(sC + \frac{1}{R} + \frac{11}{sL}\right)$$

Input admittance, with s is being replaced by $j\omega$ is

$$Y_{\text{in}} = \frac{I}{V} = \frac{1}{10^4} + j\omega 1 \times 10^{-8} - \frac{j11 \times 10^3}{\omega \times 4.4}$$
$$= 10^{-4} + j\omega \times 10^{-8} - \frac{j2500}{\omega}$$

At resonance, Y_{in} should be purely real. This enforces that

$$10^{-8}\omega = \frac{2500}{\omega}$$

Therefore,
$$\omega_0 = \sqrt{10^8 \times 2500}$$
$$= 500 \text{ K rad/ sec}$$

Quality factor:

$$Q = \omega_0 RC$$
$$= 500 \times 10^3 \times 10^4 \times 10^{-8}$$
$$= 50$$

EXAMPLE 6.13

In a parallel RLC circuit, cut off frequencies are 103 and 118 rad/sec. $|Z|$ at $\omega = 105$ rad/sec is 10 Ω. Find R, L and C.

SOLUTION

Given

$$\omega_1 = 103 \text{ rad/ sec}$$
$$\omega_2 = 118 \text{ rad/ sec}$$

Therefore

$$B = 118 - 103 = 15 \text{ rad/ sec}$$

Resonant frequency,

$$\omega_0 = \sqrt{\omega_1 \omega_2}$$
$$= \sqrt{118 \times 103} = 110.245 \text{ rad/ sec}$$

Quality factor

$$Q = \frac{\omega_0}{B}$$
$$= \frac{110.245}{15} = 7.35$$

Admittance,

$$Y = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)$$
$$= \frac{1}{R}\left[1 + j\left(\omega CR - \frac{R}{\omega L}\right)\right]$$
$$= \frac{1}{R}\left[1 + j\left(\frac{\omega_0 \omega CR}{\omega_0} - \frac{R\omega_0}{\omega\omega_0 L}\right)\right]$$

Since $$Q = \omega_0 RC = \frac{R}{\omega_0 L},$$

we get $$Y = \frac{1}{R}\left[1 + jQ\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right]$$

Note that, $$\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} = \frac{105}{110.245} - \frac{110.245}{105} = -0.0975$$

Therefore, $$Y = \frac{1}{R}\left(1 + j7.35(-0.0975)\right)$$
$$= \frac{1}{R}(1 - j0.7168) \tag{6.12}$$
$$\Rightarrow |Y| = \frac{1}{R}\sqrt{1 + (0.7168)^2} = \frac{1.23}{R}$$

It is given that $|Z| = 10$ and therefore $|Y| = \frac{1}{10}$. Putting this value of Y in equation (6.9), we get

$$\frac{1}{10} = 1.23\frac{1}{R} \quad \Rightarrow \quad R = 12.3\Omega$$

From the relationship $Q = \omega_0 CR$, we get

$$\omega_0 CR = 7.35$$

Therefore, $$C = \frac{7.35}{12.3} \times \frac{1}{110.245}$$
$$= 5.42\ \mu\text{F}$$

Inductance,

$$L = \frac{1}{\omega_0^2 C}$$
$$= \frac{1}{110.245^2 \times 5.42 \times 10^{-3}}$$
$$= 15.18 \text{ mH}$$

EXAMPLE 6.14

For the circuit shown in Fig. 6.26(a), find ω_0, V_1 at ω_0, and V_1 at a frequency 15 k rad/sec above ω_0.

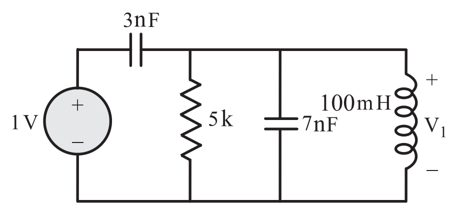


Figure 6.26(a)

SOLUTION

Changing voltage source of Fig. 6.26(a) into its equivalent current source, the circuit is redrawn as shown in Fig. 6.26(b).

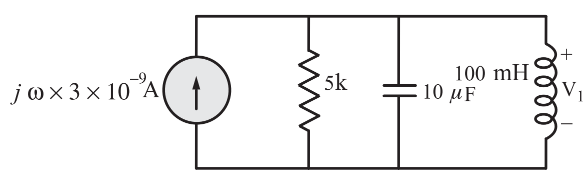


Figure 6.26(b)

Referring Fig. 6.26(b),

$$\omega_0 = \frac{1}{\sqrt{LC}}$$
$$= \frac{1}{\sqrt{100 \times 10^{-6} \times 10 \times 10^{-9}}}$$
$$= 10^6 \text{ rad/sec}$$

Voltage across the inductor at ω_0 is,

$$V_1 = j10^6 \times 3 \times 10^{-9} \times 5 \times 10^3$$
$$= j15 \text{ V}$$

Quality factor,

$$Q = \omega_0 CR$$
$$= 10^6 \times 10 \times 10^{-9} \times 5 \times 10^3$$
$$= 50$$

Given $\qquad \omega_a = \omega_0 + 15 \text{ k rad/sec}$

$$= 15 \times 10^3 + 10^6$$
$$= 1.015 \times 10^6 \text{ rad/sec}$$

Now, $\qquad \dfrac{\omega_a}{\omega_0} - \dfrac{\omega_0}{\omega_a} = 1.015 - \dfrac{1}{1.015} = 0.03$

Using this relation in the equation,

$$Y = \frac{1}{R}\left[1 + jQ\left(\frac{\omega_a}{\omega_0} - \frac{\omega_0}{\omega_a}\right)\right]$$

we get

$$Y = \frac{1}{5000}(1 + j50 \times 0.03)$$
$$= 3.6 \times 10^{-4} \angle 56.31^\circ$$

The corresponding value of V_1 is

$$V_1 = I\,Y^{-1}$$
$$= j\omega_a \times 3 \times 10^{-9} \times Y^{-1}$$
$$= \frac{j\,1.015 \times 10^6 \times 3 \times 10^{-9}}{3.6 \times 10^{-4} \angle 56.31^\circ}$$
$$= 8.444 \angle 33.69^\circ \text{ V}$$

EXAMPLE 6.15

A parallel RLC circuit has a quality factor of 100 at unity power factor and operates at 1 kHz and dissipates 1 Watt when driven by 1 A at 1 kHz. Find Bandwidth and the numerical values of R, L and C.

SOLUTION

Given $f = 1$ kHz, $P = 1$ W, $I = 1$ A, $Q = 100$, $\cos\phi = 1$

$$B = \frac{\omega_0}{Q} = \frac{10^3 \times 2\pi}{100} = 20\pi \text{ rad/sec}$$
$$P = I^2 R$$

Therefore

$$R = 1\,\Omega$$
$$L = \frac{R}{\omega_0 Q}$$
$$= \frac{1}{20\pi \times 100}$$
$$= 159\,\mu\text{H}$$
$$C = \frac{1}{\omega_0^2 L}$$
$$= \frac{10}{(20\pi)^2 159}$$
$$= 16.9\,\mu\text{F}$$

EXAMPLE 6.16

For the circuit shown in Fig. 6.27, determine resonance frequency and the input impedance.

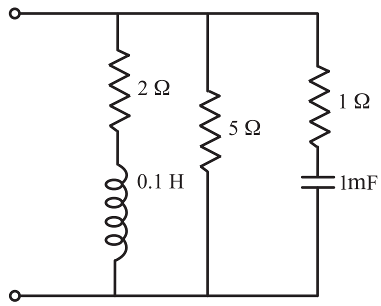


Figure 6.27

SOLUTION

Equation for resonance frequency is

$$\omega_L = \sqrt{\frac{1}{LC}\left(\frac{R_L^2 - \frac{L}{C}}{R_C^2 - \frac{L}{C}}\right)}$$

$$= \sqrt{\frac{1}{0.1 \times 10^{-3}}\left(\frac{2^2 - 100}{1 - 100}\right)}$$

$$= 98.47 \text{ rad/sec}$$

We know that

$$X_L = \omega_0 L$$

$$= 98.47 \times 0.1$$

$$= 9.847\ \Omega$$

and

$$X_C = \frac{1}{\omega_0 C}$$

$$= \frac{1}{98.47 \times 10^{-3}}$$

$$= 10.16\ \Omega$$

Admittance Y at resonance is purely real and is given by

$$\begin{aligned} Y &= G_1 + G_2 + G_3 \\ &= \frac{2}{2+(0.1\omega_0)^2} + \frac{1}{5} + \frac{1}{1+\left(\frac{10^3}{\omega_0}\right)^2} \\ Y &= \frac{2}{2^2+9.847^2} + \frac{1}{5} + \frac{1}{1+10.16^2} \\ &= 0.23\ \mathrm{S} \end{aligned}$$

and the input impedance,

$$Z = \frac{1}{Y} = 4.35\ \Omega$$

EXAMPLE 6.17

The impedance of a parallel RLC circuit as a function of ω is depicted in the diagram shown in Fig. 6.28. Determine R, L and C of the circuit. What are the new values of ω_0 and bandwidth if C is increased by 4 times?

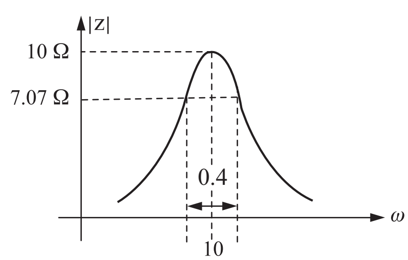


Figure 6.28

SOLUTION

It can be seen from the figure that

$$\begin{aligned} \omega_0 &= 10\ \mathrm{rad/sec} \\ B &= 0.4\ \mathrm{rad/sec} \\ R &= 10\ \Omega \end{aligned}$$

Then Quality factor

$$Q = \frac{\omega_0}{\mathrm{BW}} = \frac{10}{0.4} = 25$$

We know that

$$L = \frac{R}{\omega_0 Q} = \frac{10}{10 \times 25} = 0.04\ \mathrm{H}$$

As $Q = \omega_0 CR$,

$$C = \frac{25}{10 \times 10} = 0.25 \text{ F}$$

If C is increased by 4 times, the new value of C is 1 Farad. Therefore,

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.04}} = 5$$

and the corresponding bandwith

$$B = \frac{1}{RC} = 0.1$$

EXAMPLE 6.18

In a two branch $RL - RC$ parallel resonant circuit, $L = 0.4$ H and $C = 40$ μF. Obtain resonant frequency for the following values of R_L and R_C.

(i) $R_L = 120; R_C = 80$

(ii) $R_L = R_C = 80$

(iii) $R_L = 80; R_C = 0$

(iv) $R_L = R_C = 100$

(v) $R_L = R_C = 120$

SOLUTION

As R_L and R_C are given separately, we can use the following formula to calculate the resonant frequency.

$$\omega_0 = \frac{1}{\sqrt{LC}} \sqrt{\left(\frac{R_L^2 - \frac{L}{C}}{R_C^2 - \frac{L}{C}} \right)} \tag{6.10}$$

Let us compute the following values

$$\begin{aligned} LC &= 0.4 \times 40 \times 10^{-6} \\ &= 16 \times 10^{6} \\ \frac{1}{\sqrt{LC}} &= 250 \\ \frac{L}{C} &= 10^4 \end{aligned}$$

(i) $R_L = 120; R_C = 80$
Using equation (6.10),

$$\omega_0 = 250 \sqrt{\frac{120^2 - 10^4}{50^2 - 10^4}}$$

As the result is an imaginary number resonance is not possible in this case.

(ii) $R_L = R_C = 80$

$$\begin{aligned}\omega_0 &= 250\sqrt{\frac{80^2 - 10^4}{80^2 - 10^4}}\\ &= 250 \text{ rad/sec}\end{aligned}$$

(iii) $R_L = 80$; $R_C = 0$

$$\begin{aligned}\omega_0 &= 250\sqrt{\frac{80^2 - 10^4}{-10^4}}\\ &= 150 \text{ rad/sec}\end{aligned}$$

(iv) $R_L = R_C = 100$

$$\omega_0 = 250\sqrt{\frac{100^2 - 10^4}{100^2 - 10^4}}$$

As the result is indeterminate, the circuit resonates at all frequencies.

(v) $R_L = R_C = 120$

$$\begin{aligned}\omega_0 &= 250\sqrt{\frac{120^2 - 10^4}{120^2 - 10^4}}\\ &= 250 \text{ rad/sec}\end{aligned}$$

EXAMPLE 6.19

The following information is given in connection with a two branch parallel circuit: $R_L = 10\ \Omega$, $R_C = 20\ \Omega$, $X_C = 40\ \Omega$, $E = 120$ V and frequency = 60 Hz. What are the values of L for resonance and what currents are drawn from the supply under this condition?

SOLUTION

As the frequency is constant, the condition for resonance is

$$\frac{X_L}{R_L^2 + X_L^2} = \frac{X_C}{R_C^2 + X_C^2}$$

$$\Rightarrow \qquad \frac{X_L}{10^2 + X_L^2} = \frac{40}{20^2 + 40^2} = \frac{1}{50}$$

$$\Rightarrow \qquad X_L^2 - 50X_L + 100 = 0$$

Solving we get

$$X_L = 47.913\ \Omega \quad \text{or} \quad 2.087\ \Omega$$

Then the corresponding values of inductances are

$$L = \frac{X_L}{\omega} = 0.127 \text{ H} \quad \text{or} \quad 5.536 \text{ mH}$$

The supply current is

$$I = EG = E\left(G_L + G_C\right)$$

Thus,
$$I = 120\left(\frac{10}{10^2 + 47.913^2} + 0.02\right) = 1.7\ \text{A} \quad \text{for } X_L = 47.913\ \Omega$$

or
$$I = 120\left(\frac{10}{10^2 + 2.087^2} + 0.02\right) = 12.7\ \text{A} \quad \text{for } X_L = 2.087\ \Omega$$

Exercise Problems

E.P 6.1

Refer the circuit shown in Fig. E.P. 6.1, where R_i is the source resistance

(a) Determine the transfer function of the circuit.

(b) Sketch the magnitude plot with $R_i \neq 0$ and $R_i = 0$.

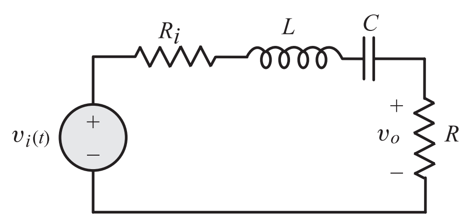


Figure E.P. 6.1

Ans: $$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{\frac{R}{L}s}{s^2 + \left(\frac{R+R_i}{L}\right)s + \frac{1}{LC}}$$

E.P 6.2

For the circuit shown in Fig. E.P. 6.2, calculate the following:

(a) $\mathbf{f_0}$, (b) **Q**, (c) $\mathbf{f_{c_1}}$, (d) $\mathbf{f_{c_2}}$ **and** (e) **B**

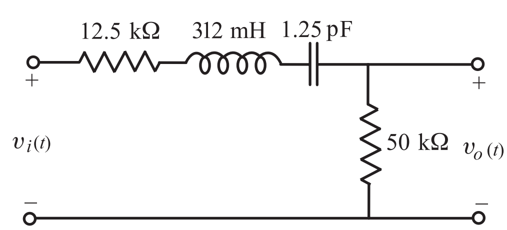


Figure E.P. 6.2

Ans: (a) **254.65 kHz** (b) **8** (c) **239.23 kHz** (d) **271.06 kHz** (e) **31.83 kHz**

E.P 6.3

Refer the circuit shown in Fig. E.P. 6.3, find the output voltage, when (a) $\omega = \omega_0$ (b) $\omega = \omega_1$, and (c) $\omega = \omega_{c2}$.

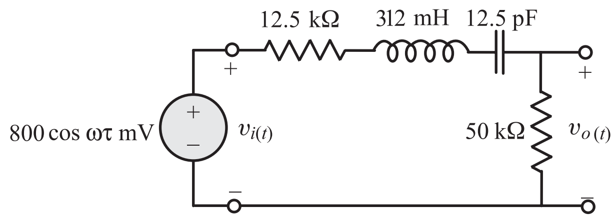


Figure E.P. 6.3

Ans: (a) $\mathbf{640\cos(1.6\times 10^6\ t)mV}$
(b) $\mathbf{452.55\cos(1.5\times 10^6\ t+45^\circ)mV}$
(c) $\mathbf{452.55\cos(1.7\times 10^6\ t-45^\circ)mV}$

E.P 6.4

Refer the circuit shown in Fig. E.P. 6.4. Calculate $Z_i(s)$ and then find (a) ω_0 and (b) Q.

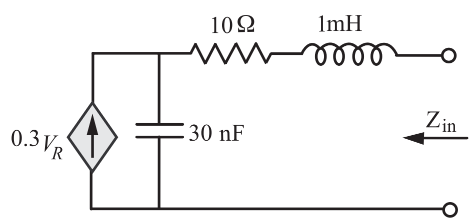


Figure E.P. 6.4

Ans: (a) **364.69 krad/sec**, (b) **36**

E.P 6.5

Refer the circuit shown in Fig. E.P. 6.5. Show that at resonance, $|V_o|_{\max} = \frac{Q|V_s|}{\sqrt{1-\frac{1}{4Q^2}}}$.

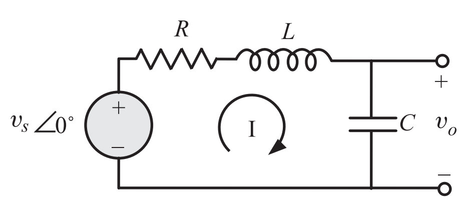


Figure E.P. 6.5

E.P 6.6

Refer the circuit given in Fig. E.P. 6.6, calculate ω_0, Q and $|V_o|_{\max}$

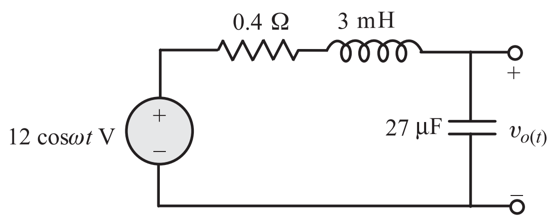


Figure E.P. 6.6

Ans: **3513.64 rad/sec**, **26.35**, **316 volts**.

E.P 6.7

A parallel network, which is driven by a variable frequency of 4 A current source has the following values: $R = 1$ kΩ , $L = 10$ mH, $C = 100\ \mu$F. Find the band width of the network, the half power frequenies and the voltage across the network at half-power frequencies.

Ans: **10 rad/sec**, **995 rad/sec**, **10005 rad/sec**

E.P 6.8

For the circuit shown in Fig. E.P. 6.8, determine the expression for the magnitude response, $|Z_{in}|$ versus ω and Z_{in} at $\omega_0 = \frac{1}{\sqrt{LC}}$.

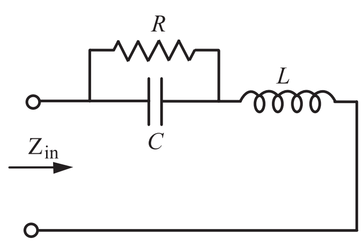


Figure E.P. 6.8

Ans: (a) $|\boldsymbol{Z_{\text{in}}}| = \sqrt{\frac{(R-\omega^2 RLC)^2+(\omega L)^2}{1+(\omega RC)^2}}$, (b) $|\boldsymbol{Z_{\text{in}}}| = \frac{1}{\sqrt{\frac{C}{L}\left(1+R^2\frac{C}{L}\right)}}$

E.P 6.9

A coil under test may be represented by the model of L in series with R. The coil is connected in series with a variable capacitor. A voltage source $v(t) = 10 \cos 1000\ t$ volts is connected to the coil. The capacitor is varied and it is found that the current is maximum when $C = 10\mu$F. Also, when $C = 12.5\mu$F, the current is 0.707 of the maximum value. Find Q of the coil at $\omega = 1000$ rad/sec.

Ans: **5**

E.P 6.10

A fresher in the devices lab for sake of curiosity sets up a series RLC network as shown in Fig. E.P.6.10. The capacitor can withstand very high voltages. Is it safe to touch the capacitor at resonance? Find the voltage across the capacitor.

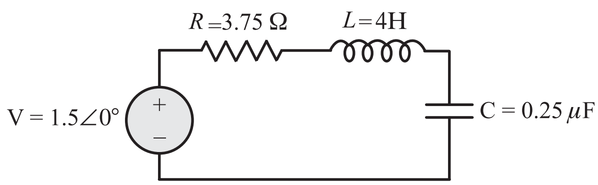


Figure E.P. 6.10

Ans: *Not safe*, $|\mathbf{V_c}|_{\mathbf{max}} = \mathbf{1600\,V}$

Chapter 4 Initial Conditions in Networks

4.1 Introduction

There are many reasons for studying initial and final conditions. The most important reason is that the initial and final conditions evaluate the arbitrary constants that appear in the general solution of a differential equation.

In this chapter, we concentrate on finding the change in selected variables in a circuit when a switch is thrown open from closed position or vice versa. The time of throwing the switch is considered to be $t = 0$, and we want to determine the value of the variable at $t = 0^-$ and at $t = 0^+$, immediately before and after throwing the switch. Thus a switched circuit is an electrical circuit with one or more switches that open or close at time $t = 0$. We are very much interested in the change in currents and voltages of energy storing elements after the switch is thrown since these variables along with the sources will dictate the circuit behaviour for $t > 0$.

Initial conditions in a network depend on the past history of the circuit (before $t = 0^-$) and structure of the network at $t = 0^+$, (after switching). Past history will show up in the form of capacitor voltages and inductor currents. The computation of all voltages and currents and their derivatives at $t = 0^+$ is the main aim of this chapter.

4.2 Initial and final conditions in elements

4.2.1 The inductor

The switch is closed at $t = 0$. Hence $t = 0^-$ corresponds to the instant when the switch is just open and $t = 0^+$ corresponds to the instant when the switch is just closed.

The expression for current through the inductor is given by

$$i = \frac{1}{L}\int_{-\infty}^{t} v\,d\tau$$

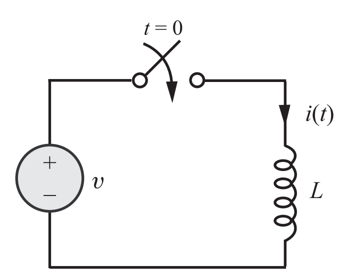


Figure 4.1 Circuit for explaining switching action of an inductor

$$\Rightarrow \qquad i = \frac{1}{L}\int_{-\infty}^{0^-} v\,d\tau + \frac{1}{L}\int_{0^-}^{t} v\,d\tau$$

$$\Rightarrow \qquad i(t) = i(0^-) + \frac{1}{L}\int_{0^-}^{t} v\,d\tau$$

Putting $t = 0^+$ on both sides, we get

$$i(0^+) = i(0^-) + \frac{1}{L}\int_{0^-}^{0^+} v\,d\tau$$

$$\Rightarrow \qquad i(0^+) = i(0^-)$$

The above equation means that the current in an inductor cannot change instantaneously. Consequently, if $i(0^-) = 0$, we get $i(0^+) = 0$. This means that at $t = 0^+$, inductor will act as an open circuit, irrespective of the voltage across the terminals. If $i(0^-) = I_o$, then $i(0^+) = I_o$. In this case at $t = 0^+$, the inductor can be thought of as a current source of I_o A. The equivalent circuits of an inductor at $t = 0^+$ is shown in Fig. 4.2.

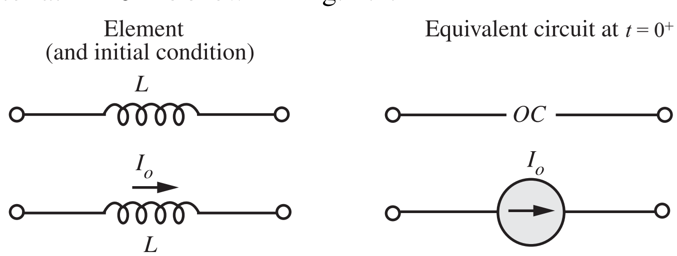


Figure 4.2 The initial-condition equivalent circuits of an inductor

The final-condition equivalent circuit of an inductor is derived from the basic relationship

$$v = L\frac{di}{dt}$$

Under steady condition, $\frac{di}{dt} = 0$. This means, $v = 0$ and hence L acts as short at $t = \infty$ (final or steady state). The final-condition equivalent circuits of an inductor is shown in Fig.4.3.

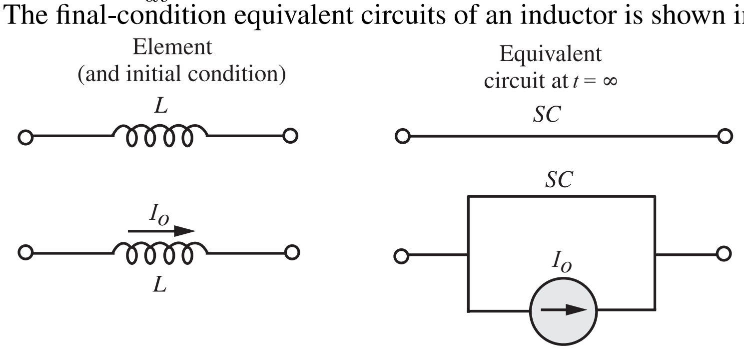


Figure 4.3 The final-condition equivalent circuit of an inductor

4.2.2 The capacitor

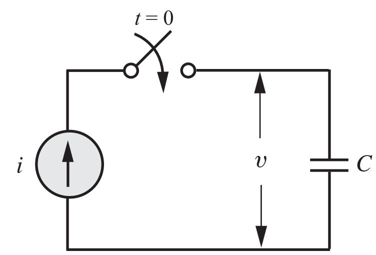


Figure 4.4 Circuit for explaining switching action of a Capacitor

The switch is closed at $t = 0$. Hence, $t = 0^-$ corresponds to the instant when the switch is just open and $t = 0^+$ corresponds to the instant when the switch is just closed. The expression for voltage across the capacitor is given by

$$v = \frac{1}{C}\int_{-\infty}^{t} i d\tau$$

$$\Rightarrow \qquad v(t) = \frac{1}{C}\int_{-\infty}^{0^-} i d\tau + \frac{1}{C}\int_{0^-}^{t} i d\tau$$

$$\Rightarrow \qquad v(t) = v(0^-) + \frac{1}{C}\int_{0^-}^{t} i d\tau$$

Evaluating the expression at $t = 0^+$, we get

$$v(0^+) = v(0^-) + \frac{1}{C}\int_{0^-}^{0^+} i d\tau \quad \Rightarrow \quad v(0^+) = v(0^-)$$

Thus the voltage across a capacitor cannot change instantaneously.

If $v(0^-) = 0$, then $v(0^+) = 0$. This means that at $t = 0^+$, capacitor C acts as short circuit. Conversely, if $v(0^-) = \frac{q_0}{C}$ then $v(0^+) = \frac{q_0}{C}$. These conclusions are summarized in Fig. 4.5.

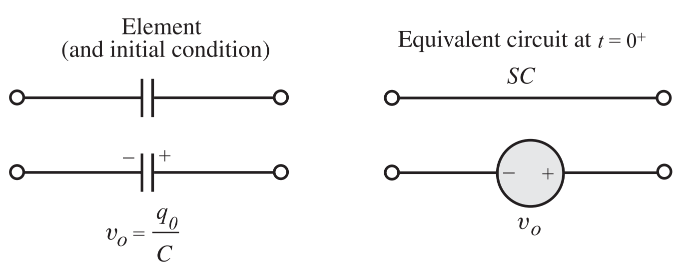


Figure 4.5 Initial-condition equivalent circuits of a capacitor

The final–condition equivalent network is derived from the basic relationship

$$i = C\frac{dv}{dt}$$

Under steady state condition, $\frac{dv}{dt} = 0$. This is, at $t = \infty$, $i = 0$. This means that $t = \infty$ or in steady state, capacitor C acts as an open circuit. The final condition equivalent circuits of a capacitor is shown in Fig. 4.6.

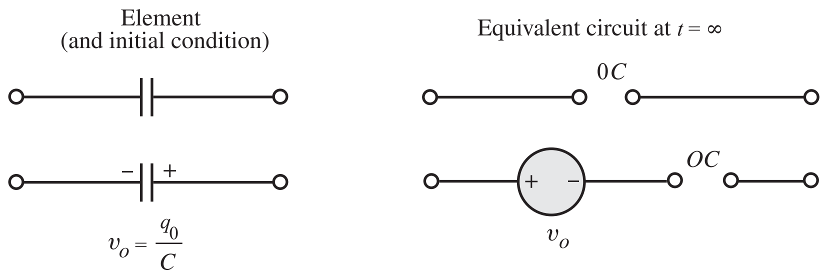


Figure 4.6 Final-condition equivalent circuits of a capacitor

4.2.3 The resistor

The cause–effect relation for an ideal resistor is given by $v = Ri$. From this equation, we find that the current through a resistor will change instantaneously if the voltage changes instantaneously. Similarly, voltage will change instantaneously if current changes instantaneously.

4.3 Procedure for evaluating initial conditions

There is no unique procedure that must be followed in solving for initial conditions. We usually solve for initial values of currents and voltages and then solve for the derivatives. For finding initial values of currents and voltages, an equivalent network of the original network at $t = 0^+$ is constructed according to the following rules:

(1) Replace all inductors with open circuit or with current sources having the value of current flowing at $t = 0^+$.

(2) Replace all capacitors with short circuits or with a voltage source of value $v_o = \frac{q_0}{C}$ if there is an initial charge.

(3) Resistors are left in the network without any changes.

EXAMPLE 4.1

Refer the circuit shown in Fig. 4.7(a). Find $i_1(0^+)$ and $i_L(0^+)$. The circuit is in steady state for $t < 0$.

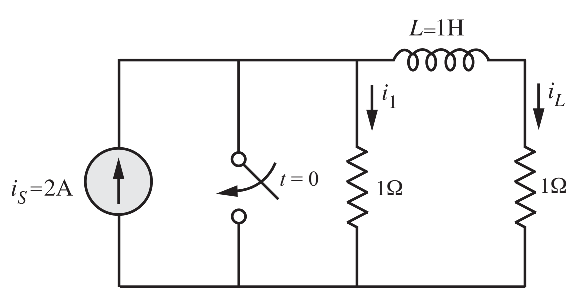


Figure 4.7(a)

SOLUTION

The symbol for the switch implies that it is open at $t = 0^-$ and then closed at $t = 0^+$. The circuit is in steady state with the switch open. This means that at $t = 0^-$, inductor L is short. Fig.4.7(b) shows the original circuit at $t = 0^-$.
Using the current division principle,

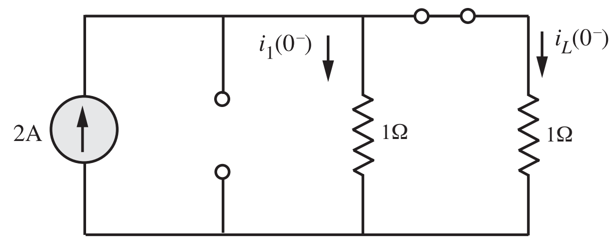


Figure 4.7(b)

$$i_L(0^-) = \frac{2 \times 1}{1 + 1} = 1\text{A}$$

Since the current in an inductor cannot change instantaneously, we have

$$i_L(0^+) = i_L(0^-) = 1\text{A}$$

At $t = 0^-$, $i_1(0^-) = 2 - 1 = 1\text{A}$. Please note that the current in a resistor can change instantaneously. Since at $t = 0^+$, the switch is just closed, the voltage across R_1 will be equal to zero because of the switch being short circuited and hence,

$$i_1(0^+) = 0\text{A}$$

Thus, the current in the resistor changes abruptly form 1A to 0A.

EXAMPLE 4.2

Refer the circuit shown in Fig. 4.8. Find $v_C(0^+)$. Assume that the switch was in closed state for a long time.

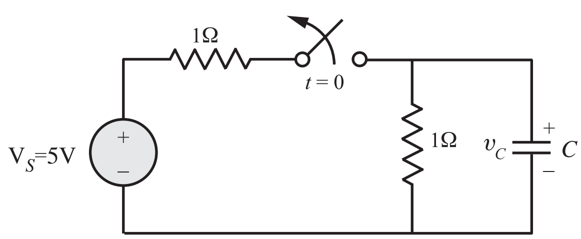


Figure 4.8

SOLUTION

The symbol for the switch implies that it is closed at $t = 0^-$ and then opens at $t = 0^+$. Since the circuit is in steady state with the switch closed, the capacitor is represented as an open circuit at $t = 0^-$. The equivalent circuit at $t = 0^-$ is as shown in Fig. 4.9.

$$v_C(0^-) = i(0^-)R_2$$

Using the principle of voltage divider,

$$v_C(0^-) = \frac{V_S}{R_1 + R_2} R_2 = \frac{5 \times 1}{1 + 1} = 2.5 \text{ V}$$

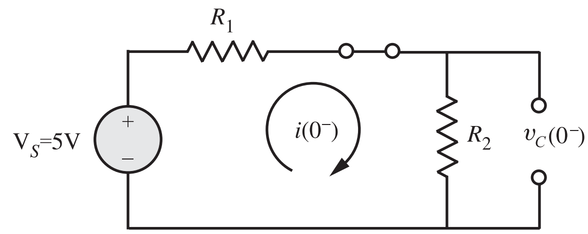


Figure 4.9

Since the voltage across a capacitor cannot change instaneously, we have

$$v_C(0^+) = v_C(0^-) = 2.5\text{V}$$

That is, when the switch is opened at $t = 0$, and if the source is removed from the circuit, still $v_C(0^+)$ remains at 2.5 V.

EXAMPLE 4.3

Refer the circuit shown in Fig 4.10. Find $i_L(0^+)$ and $v_C(0^+)$. The circuit is in steady state with the switch in closed condition.

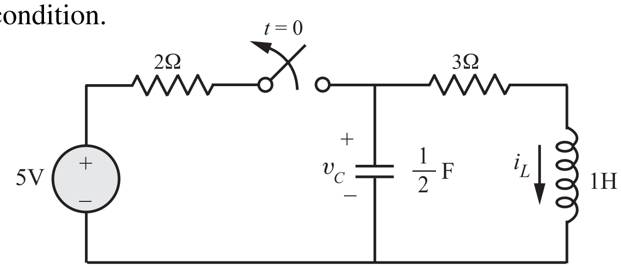


Figure 4.10

SOLUTION

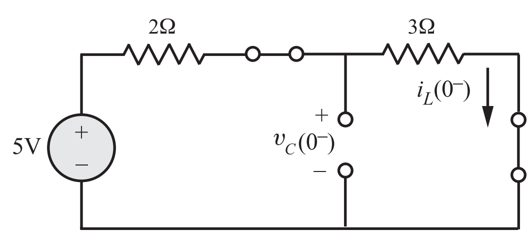


Figure 4.11

The symbol for the switch implies, it is closed at $t = 0^-$ and then opens at $t = 0^+$. In order to find $v_C(0^-)$ and $i_L(0^-)$ we replace the capacitor by an open circuit and the inductor by a short circuit, as shown in Fig.4.11, because in the steady state L acts as a short circuit and C as an open circuit.

$$i_L(0^-) = \frac{5}{2 + 3} = 1 \text{ A}$$

Using the voltage divider principle, we note that

$$v_C(0^-) = \frac{5 \times 3}{3 + 2} = 3 \text{ V}$$

Then we note that:

$$v_C(0^+) = v_C(0^-) = 3 \text{ V}$$
$$i_L(0^+) = i_L(0^-) = 2 \text{ A}$$

EXAMPLE 4.4

In the given network, K is closed at $t = 0$ with zero current in the inductor. Find the values of $i, \dfrac{di}{dt}, \dfrac{d^2i}{dt^2}$ at $t = 0^+$ if $R = 8\Omega$ and $L = 0.2\text{H}$. Refer the Fig. 4.12(a).

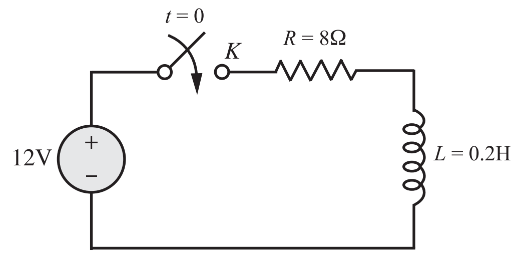


Figure 4.12(a)

SOLUTION

The symbol for the switch implies that it is open at $t = 0^-$ and then closes at $t = 0^+$. Since the current i through the inductor at $t = 0^-$ is zero, it implies that $i(0^+) = i(0^-) = 0$.

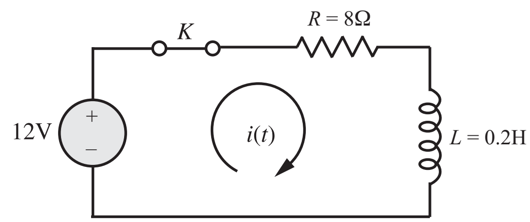


Figure 4.12(b)

To find $\dfrac{di(0^+)}{dt}$ and $\dfrac{d^2i(0^+)}{dt^2}$:

Applying KVL clockwise to the circuit shown in Fig. 4.12(b), we get

$$Ri + L\frac{di}{dt} = 12$$

$$\Rightarrow \quad 8i + 0.2\frac{di}{dt} = 12 \tag{4.1}$$

At $t = 0^+$, the equation (4.1) becomes

$$8i(0^+) + 0.2\frac{di(0^+)}{dt} = 12$$

$$\Rightarrow \quad 8 \times 0 + 0.2\frac{di(0^+)}{dt} = 12$$

$$\Rightarrow \quad \frac{di(0^+)}{dt} = \frac{12}{0.2} = \mathbf{60\ A/sec}$$

Differentiating equation (4.1) with respect to t, we get

$$8\frac{di}{dt} + 0.2\frac{d^2i}{dt^2} = 0$$

At $t = 0^+$, the above equation becomes

$$8\frac{di(0^+)}{dt} + 0.2\frac{d^2i(0^+)}{dt^2} = 0$$

$$\Rightarrow \quad 8 \times 60 + 0.2\frac{d^2i(0^+)}{dt^2} = 0$$

Hence

$$\frac{d^2i(0^+)}{dt^2} = \mathbf{-\,2400\ A/sec^2}$$

EXAMPLE 4.5

In the network shown in Fig. 4.13, the switch is closed at $t = 0$. Determine i, $\frac{di}{dt}$, $\frac{d^2i}{dt^2}$ at $t = 0^+$.

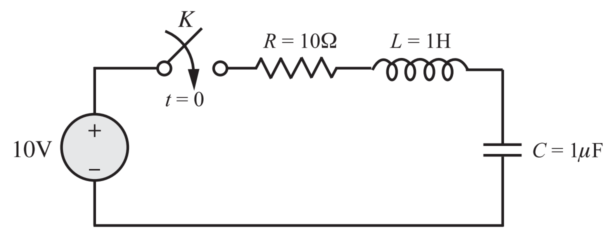


Figure 4.13

SOLUTION

The symbol for the switch implies that it is open at $t = 0^-$ and then closes at $t = 0^+$. Since there is no current through the inductor at $t = 0^-$, it implies that $i(0^+) = i(0^-) = 0$.

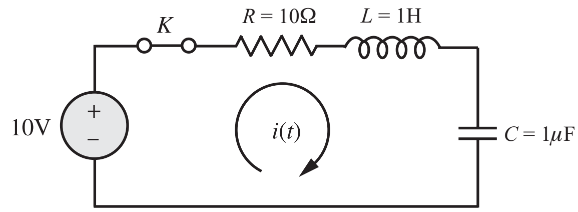


Figure 4.14

Writing *KVL clockwise* for the circuit shown in Fig. 4.14, we get

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_0^t i(\tau)d\tau = 10 \tag{4.2}$$

$$\Rightarrow \qquad Ri + L\frac{di}{dt} + v_C(t) = 10 \tag{4.2a}$$

Putting $t = 0^+$ in equation (4.2a), we get

$$Ri\left(0^+\right) + L\frac{di\left(0^+\right)}{dt} + v_C\left(0^+\right) = 10$$

$$\Rightarrow \qquad R \times 0 + L\frac{di\left(0^+\right)}{dt} + 0 = 10$$

$$\Rightarrow \qquad \frac{di\left(0^+\right)}{dt} = \frac{10}{L} = \mathbf{10\ A/sec}$$

Differentiating equation (4.2) with respect to t, we get

$$R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{i(t)}{C} = 0$$

At $t = 0^+$, the above equation becomes

$$R\frac{di\left(0^+\right)}{dt} + L\frac{d^2i\left(0^+\right)}{dt^2} + \frac{i\left(0^+\right)}{C} = 0$$

$$\Rightarrow \qquad R \times 10 + L\frac{d^2i\left(0^+\right)}{dt^2} + \frac{0}{C} = 0$$

$$\Rightarrow \qquad 100 + \frac{d^2i\left(0^+\right)}{dt^2} = 0$$

Hence at $t = 0^+$, $\qquad \dfrac{d^2i\left(0^+\right)}{dt^2} = \mathbf{-100\ A/sec^2}$

EXAMPLE 4.6

Refer the circuit shown in Fig. 4.15. The switch K is changed from position 1 to position 2 at $t = 0$. Steady-state condition having been reached at position 1. Find the values of i, $\dfrac{di}{dt}$, and $\dfrac{d^2i}{dt^2}$ at $t = 0^+$.

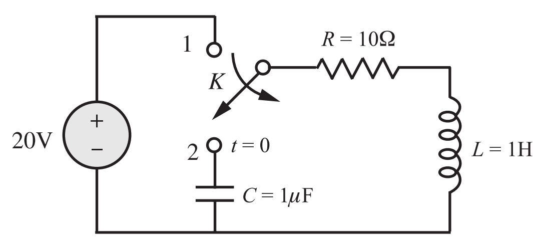


Figure 4.15

SOLUTION

The symbol for switch K implies that it is in position 1 at $t = 0^-$ and in position 2 at $t = 0^+$. Under steady-state condition, inductor acts as a short circuit. Hence at $t = 0^-$, the circuit diagram is as shown in Fig. 4.16.

$$i\left(0^-\right) = \frac{20}{10} = \mathbf{2A}$$

Since the current through an inductor cannot change instantaneously, $i\left(0^+\right) = i\left(0^-\right) = 2\text{A}$. Since there is no initial charge on the capacitor, $v_C\left(0^-\right) = 0$. Since the voltage across a capacitor cannot change instantaneously, $v_C\left(0^+\right) = v_C\left(0^-\right) = 0$. Hence at $t = 0^+$ the circuit diagram is as shown in Fig. 4.17(a).

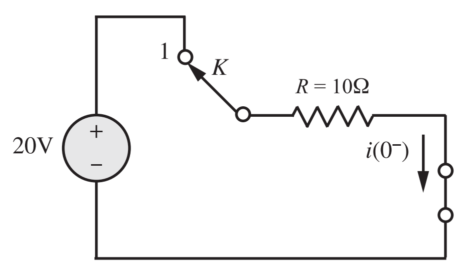


Figure 4.16

For $t \geq 0^+$, the circuit diagram is as shown in Fig. 4.17(b).

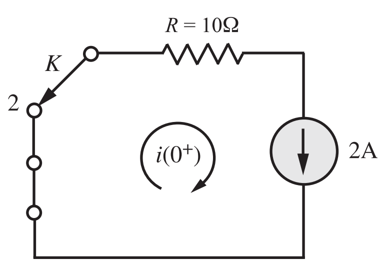


Figure 4.17(a)

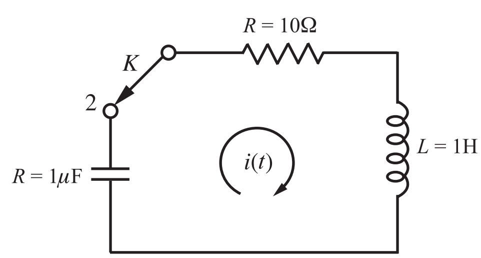


Figure 4.17(b)

Applying KVL clockwise to the circuit shown in Fig. 4.17(b), we get

$$Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int_{0^+}^{t} i(\tau)d\tau = 0 \tag{4.3}$$

$$\Rightarrow \qquad Ri(t) + L\frac{di(t)}{dt} + v_C(t) = 0 \tag{4.3a}$$

At $t = 0^+$ equation (4.3a) becomes

$$Ri\left(0^+\right) + L\frac{di\left(0^+\right)}{dt} + v_C\left(0^+\right) = 0$$

$$\Rightarrow \qquad R \times 2 + L\frac{di\left(0^+\right)}{dt} + 0 = 0$$

$$\Rightarrow \qquad 20 + \frac{di\left(0^+\right)}{dt} = 0$$

$$\Rightarrow \qquad \frac{di\left(0^+\right)}{dt} = \mathbf{-20\ A/sec}$$

Differentiating equation (4.3) with respect to t, we get

$$R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{i}{C} = 0$$

At $t = 0^+$, we get

$$R\frac{di\left(0^+\right)}{dt} + L\frac{d^2i\left(0^+\right)}{dt^2} + \frac{i\left(0^+\right)}{C} = 0$$

$$\Rightarrow \qquad R \times (-20) + L\frac{d^2i\left(0^+\right)}{dt^2} + \frac{2}{C} = 0$$

Hence, $$\frac{d^2i\left(0^+\right)}{dt^2} \approx \mathbf{-2 \times 10^6\ A/sec^2}$$

EXAMPLE 4.7

In the network shown in Fig. 4.18, the switch is moved from position 1 to position 2 at $t = 0$. The steady-state has been reached before switching. Calculate i, $\frac{di}{dt}$, and $\frac{d^2i}{dt^2}$ at $t = 0^+$.

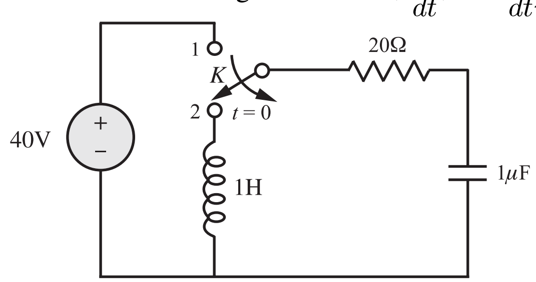


Figure 4.18

SOLUTION

The symbol for switch K implies that it is in position 1 at $t = 0^-$ and in position 2 at $t = 0^+$. Under steady-state condition, a capacitor acts as an open circuit. Hence at $t = 0^-$, the circuit diagram is as shown in Fig. 4.18(a).

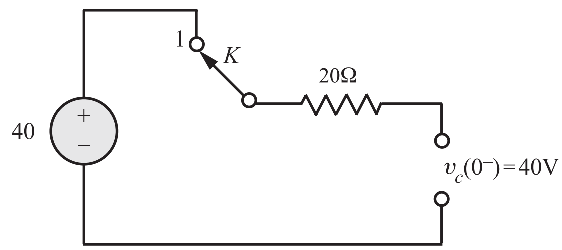


Figure 4.18(a)

We know that the voltage across a capacitor cannot change instantaneously. This means that $v_C\left(0^+\right) = v_C\left(0^-\right) = 40$ V.

At $t = 0^-$, inductor is not energized. This means that $i\left(0^-\right) = 0$. Since current in an inductor cannot change instantaneously, $i\left(0^+\right) = i\left(0^-\right) = \mathbf{0}$. Hence, the circuit diagram at $t = 0^+$ is as shown in Fig. 4.18(b).

The circuit diagram for $t \geq 0^+$ is as shown in Fig.4.18(c).

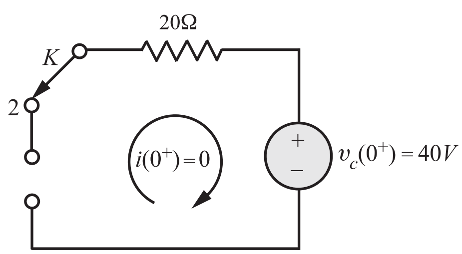


Figure 4.18(b)

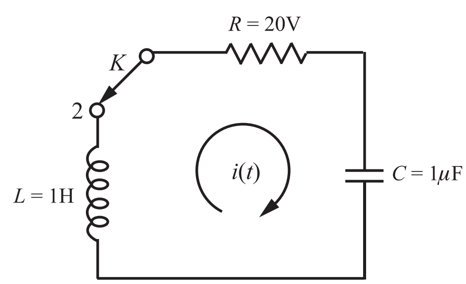


Figure 4.18(c)

Applying KVL clockwise, we get

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_{0^+}^{t} i(\tau)d\tau = 0 \tag{4.4}$$

$$\Rightarrow \qquad Ri + L\frac{di}{dt} + v_C(t) = 0$$

At $t = 0^+$, we get

$$Ri(0^+) + L\frac{di(0^+)}{dt} + v_C(0^+) = 0$$

$$\Rightarrow \qquad 20 \times 0 + 1\frac{di(0^+)}{dt} + 40 = 0$$

$$\Rightarrow \qquad \frac{di(0^+)}{dt} = \mathbf{-40A/\,sec}$$

Diferentiating equation (4.4) with respect to t, we get

$$R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{i}{C} = 0$$

Putting $t = 0^+$ in the above equation, we get

$$R\frac{di(0^+)}{dt} + L\frac{d^2i(0^+)}{dt^2} + \frac{i(0^+)}{C} = 0$$

$$\Rightarrow \qquad R \times (-40) + L\frac{d^2i(0^+)}{dt^2} + \frac{0}{C} = 0$$

Hence $$\frac{d^2i(0^+)}{dt^2} = \mathbf{800A/\,sec^2}$$

EXAMPLE 4.8

In the network shown in Fig. 4.19, $v_1(t) = e^{-t}$ for $t \geq 0$ and is zero for all $t < 0$. If the capacitor is initially uncharged, determine the value of $\frac{d^2v_2}{dt^2}$ and $\frac{d^3v_2}{dt^3}$ at $t = 0^+$.

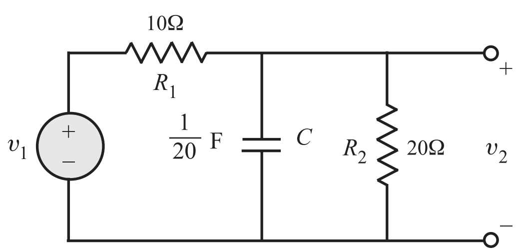


Figure 4.19

SOLUTION

Since the capacitor is initially uncharged, $v_2(0^+) = 0$

Referring to Fig. 4.19(a) and *applying KCL at node* $v_2(t)$:

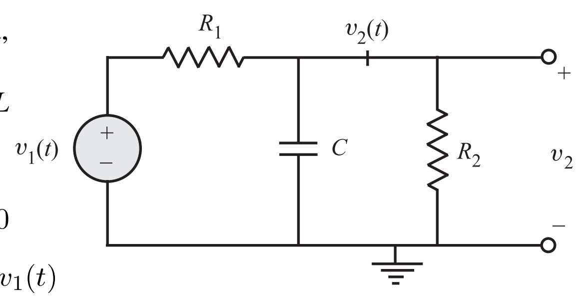


Figure 4.19(a)

$$\frac{v_2(t) - v_1(t)}{R_1} + C\frac{dv_2(t)}{dt} + \frac{v_2(t)}{R_2} = 0$$

$$\Rightarrow \qquad \left(\frac{1}{R_1} + \frac{1}{R_2}\right)v_2(t) + C\frac{dv_2(t)}{dt} = \frac{v_1(t)}{R_1}$$

$$\Rightarrow \qquad 0.15v_2 + 0.05\frac{dv_2}{dt} = 0.1e^{-t} \tag{4.5}$$

Putting $t = 0^+$, we get

$$0.15v_2(0^+) + 0.05\frac{dv_2(0^+)}{dt} = 0.1$$

$$\Rightarrow \qquad 0.15 \times 0 + 0.05\frac{dv_2(0^+)}{dt} = 0.1$$

$$\Rightarrow \qquad \frac{dv_2(0^+)}{dt} = \frac{0.1}{0.05} = 2 \text{ Volts/ sec}$$

Differentiating equation (4.5) with respect to t, we get

$$0.15\frac{dv_2}{dt} + 0.05\frac{d^2v_2}{dt^2} = -0.1e^{-t} \tag{4.6}$$

Putting $t = 0^+$ in equation (4.6), we find that

$$\frac{d^2v_2(0^+)}{dt^2} = \frac{-0.1 - 0.3}{0.05} = \mathbf{-8\ Volts/\,sec^2}$$

Again differentiating equation (4.6) with respect to t, we get

$$0.15\frac{d^2v_2}{dt^2} + 0.05\frac{d^3v_2}{dt^3} = 0.1e^{-t} \tag{4.7}$$

Putting $t = 0^+$ in equation (4.7) and solving for $\frac{d^3v_2}{dt^3}(0^+)$, we find that

$$\frac{d^3v_2(0^+)}{dt^3} = \frac{0.1 + 1.2}{0.05} = \mathbf{26\ Volts/\,sec^3}$$

EXAMPLE 4.9

Refer the circuit shown in Fig. 4.20. The circuit is in steady state with switch K closed. At $t = 0$, the switch is opened. Determine the voltage across the switch, v_K and $\frac{dv_K}{dt}$ at $t = 0^+$.

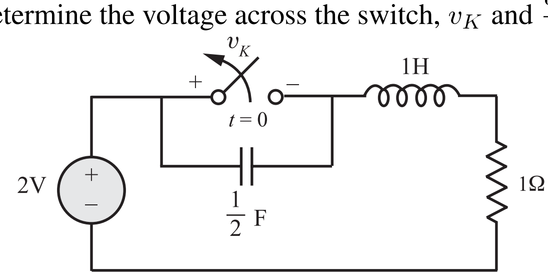


Figure 4.20

SOLUTION

The switch remains closed at $t = 0^-$ and open at $t = 0^+$. Under steady condition, inductor acts as a short circuit and hence the circuit diagram at $t = 0^-$ is as shown in Fig. 4.21(a).

$$\text{Therefore,}\quad v_K(0^+) = v_K(0^-)$$
$$= 0\ \text{V}$$

For $t \geq 0^+$ the circuit diagram is as shown in Fig. 4.21(b).

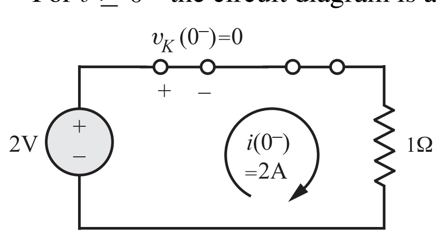


Figure 4.21(a)

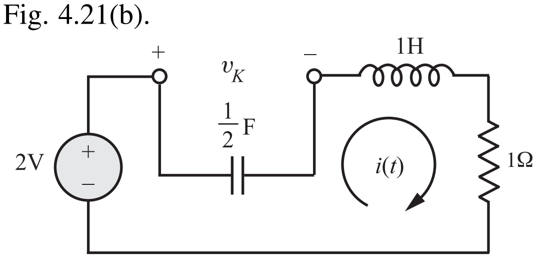


Figure 4.21(b)

$$i(t) = C\frac{dv_K}{dt}$$

At $(t) = 0^+$, we get

$$i(0^+) = C\frac{dv_K(0^+)}{dt}$$

Since the current through an inductor cannot change instantaneously, we get

$$i(0^+) = i(0^-) = 2\text{A}$$

Hence, $$2 = C\frac{dv_K(0^+)}{dt}$$

$$\frac{dv_K(0^+)}{dt} = \frac{2}{C} = \frac{2}{\frac{1}{2}} = \mathbf{4V/sec}$$

EXAMPLE 4.10

In the given network, the switch K is opened at $t = 0$. At $t = 0^+$, solve for the values of v, $\frac{dv}{dt}$ and $\frac{d^2v}{dt^2}$ if $I = 2$ A, $R = 200\ \Omega$ and $L = 1$ H

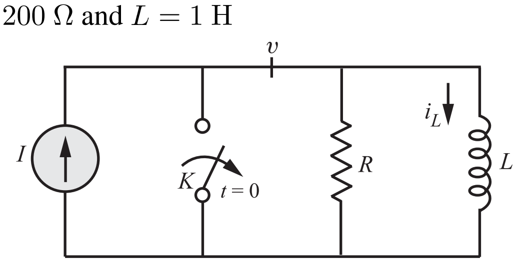


Figure 4.22

SOLUTION

The switch is opened at $t = 0$. This means that at $t = 0^-$, it is closed and at $t = 0^+$, it is open. Since $i_L(0^-) = 0$, we get $i_L(0^+) = 0$. The circuit at $t = 0^+$ is as shown in Fig. 4.23(a).

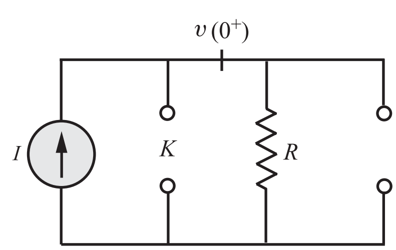


Figure 4.23(a)

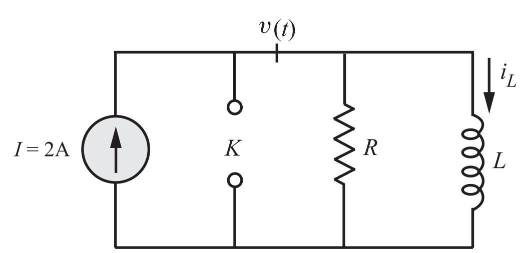


Figure 4.23(b)

$$\begin{aligned} v(0^+) &= IR \\ &= 2 \times 200 \\ &= \mathbf{400\ Volts} \end{aligned}$$

Refer to the circuit shown in Fig. 4.23(b).

For $t \geq 0^+$, the *KCL* at node $v(t)$ gives

$$I = \frac{v(t)}{R} + \frac{1}{L}\int_{0^+}^{t} v(\tau)d\tau \tag{4.8}$$

Differentiating both sides of equation (4.8) with respect to t, we get

$$0 = \frac{1}{R}\frac{dv(t)}{dt} + \frac{1}{L}v(t) \tag{4.8a}$$

At $t = 0^+$, we get

$$\frac{1}{R}\frac{dv(0^+)}{dt} + \frac{1}{L}v(0^+) = 0$$

$$\Rightarrow \quad \frac{1}{200}\frac{dv(0^+)}{dt} + \frac{1}{1} \times 400 = 0$$

$$\Rightarrow \quad \frac{dv(0^+)}{dt} = \mathbf{-8 \times 10^4\ V/sec}$$

Again differentiating equation (4.8a), we get

$$\frac{1}{R}\frac{d^2v(t)}{dt^2} + \frac{1}{L}\frac{dv(t)}{dt} = 0$$

At $t = 0^+$, we get

$$\frac{1}{200}\frac{d^2v(0^+)}{dt^2} + \frac{1}{1}\frac{dv(0^+)}{dt} = 0$$

$$\Rightarrow \quad \frac{d^2v(0^+)}{dt^2} = 200 \times 8 \times 10^4$$

$$= \mathbf{16 \times 10^6\ V/sec^2}$$

EXAMPLE 4.11

In the circuit shown in Fig. 4.24, a steady state is reached with switch K open. At $t = 0$, the switch is closed. For element values given, determine the values of $v_a(0^-)$ and $v_a(0^+)$.

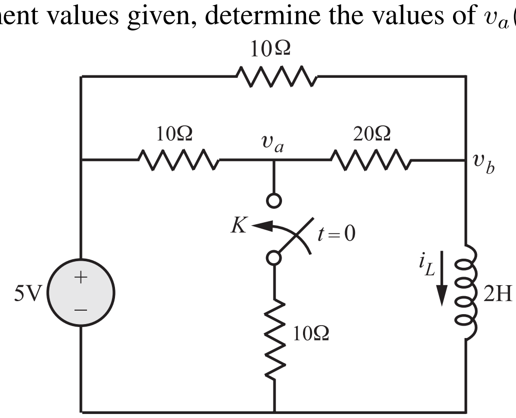


Figure 4.24

SOLUTION

At $t = 0^-$, the switch is open and at $t = 0^+$, the switch is closed. Under steady conditions, inductor L acts as a short circuit. Also the steady state is reached with switch K open. Hence, the circuit diagram at $t = 0^-$ is as shown in Fig.4.25(a).

$$i_L(0^-) = \frac{5}{30} + \frac{5}{10} = \frac{2}{3}\text{A}$$

Using the voltage divider principle:

$$v_a(0^-) = \frac{5 \times 20}{10 + 20} = \frac{\mathbf{10}}{\mathbf{3}}\ \mathbf{V}$$

Since the current in an inductor cannot change instantaneously,

$$i_L(0^+) = i_L(0^-) = \frac{2}{3}\ \text{A}.$$

At $t = 0^+$, the circuit diagram is as shown in Fig. 4.25(b).

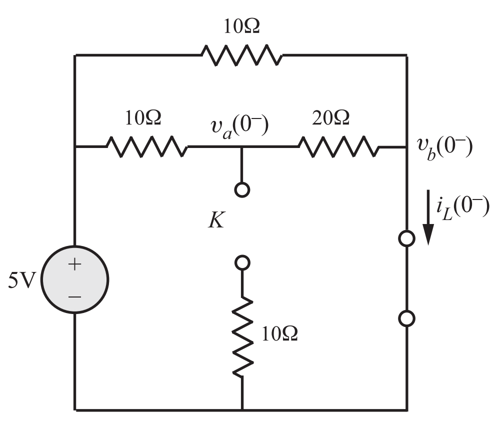


Figure 4.25(a)

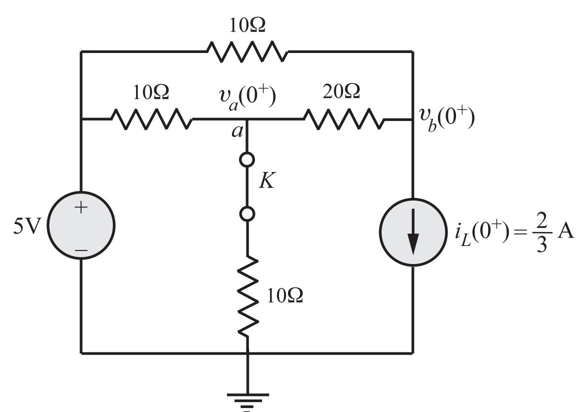


Figure 4.25(b)

Refer the circuit in Fig. 4.25(b).

KCL at node a:

$$\frac{v_a(0^+) - 5}{10} + \frac{v_a(0^+)}{10} + \frac{v_a(0^+) - v_b(0^+)}{20} = 0$$

$$\Rightarrow \qquad v_a(0^+)\left[\frac{1}{10} + \frac{1}{10} + \frac{1}{20}\right] - v_b(0^+)\left[\frac{1}{20}\right] = \frac{5}{10}$$

KCL at node b:

$$\frac{v_b(0^+) - v_a(0^+)}{20} + \frac{v_b(0^+) - 5}{10} + \frac{2}{3} = 0$$

$$\Rightarrow \qquad -v_a(0^+)\left[\frac{1}{20}\right] + v_b(0^+)\left[\frac{1}{20} + \frac{1}{10}\right] = \frac{5}{10} - \frac{2}{3}$$

Solving the above two nodal equations, we get,

$$v_a(0^+) = \frac{\mathbf{40}}{\mathbf{21}}\mathbf{V}$$

EXAMPLE 4.12

Find $i_L(0^+), v_C(0^+), \dfrac{dv_C(0^+)}{dt}$ and $\dfrac{di_L(0^+)}{dt}$ for the circuit shown in Fig. 4.26.

Assume that switch 1 has been opened and switch 2 has been closed for a long time and steady–state conditions prevail at $t = 0^-$.

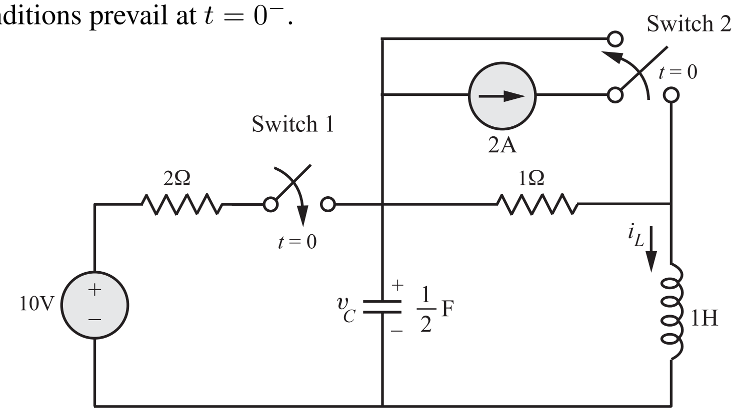


Figure 4.26

SOLUTION

At $t = 0^-$, switch 1 is open and switch 2 is closed, whereas at $t = 0^+$, switch 1 is closed and switch 2 is open.

First, let us redraw the circuit at $t = 0^-$ by replacing the inductor with a short circuit and the capacitor with an open circuit as shown in Fig. 4.27(a).

From Fig. 4.27(b), we find that $i_L(0^-) = 0$

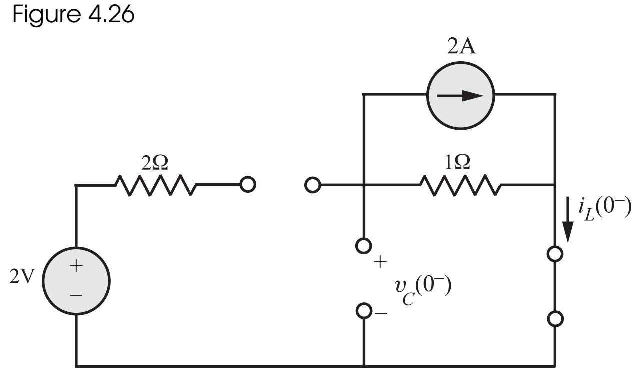


Figure 4.27(a)

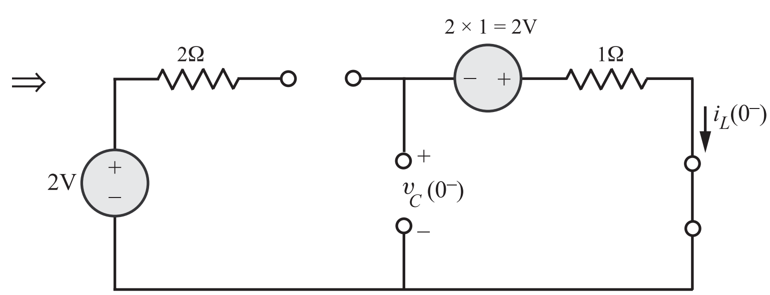


Figure 4.27(b)

Applying KVL clockwise to the loop on the right, we get

$$-v_C(0^-) - 2 + 1 \times 0 = 0$$

$$\Rightarrow \qquad v_C(0^-) = -2\text{ V}$$

Hence, at $t = 0^+ : \; i_L(0^+) = i_L(0^-) = \mathbf{0A}$

$$v_C(0^+) = v_C(0^-) = \mathbf{-2V}$$

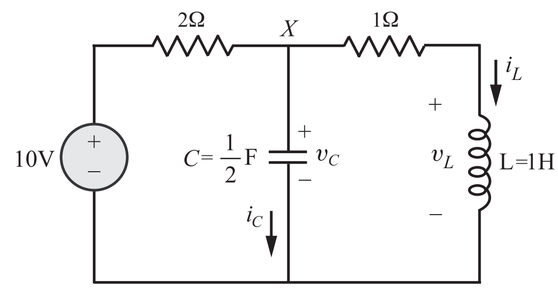


Figure 4.27(c)

The circuit diagram for $t \geq 0^+$ is shown in Fig. 4.27(c).

Applying KVL for right–hand mesh, we get

$$v_L - v_C + i_L = 0$$

At $t = 0^+$, we get

$$v_L(0^+) = v_C(0^+) - i_L(0^+)$$

$$= -2 - 0 = -2\text{ V}$$

We know that $\qquad v_L = L\dfrac{di_L}{dt}$

At $t = 0^+$, we get

$$\frac{di_L(0^+)}{dt} = \frac{v_L(0^+)}{L} = \frac{-2}{1} = \mathbf{-2A/\,sec}$$

Applying KCL at node X,

$$\frac{v_C - 10}{2} + i_C + i_L = 0$$

Consequently, at $t = 0^+$

$$i_C(0^+) = \frac{10 - v_C(0^+)}{2} - i_L(0^+) = 6 - 0 = 6\text{ A}$$

Since $\qquad i_C = C\dfrac{dv_C}{dt}$

We get, $\qquad \dfrac{dv_C(0^+)}{dt} = \dfrac{i_C(0^+)}{C} = \dfrac{6}{\frac{1}{2}} = \mathbf{12V/\,sec}$

EXAMPLE 4.13

For the circuit shown in Fig. 4.28, find:

(a) $i(0^+)$ and $v(0^+)$

(b) $\dfrac{di(0^+)}{dt}$ and $\dfrac{dv(0^+)}{dt}$

(c) $i(\infty)$ and $v(\infty)$

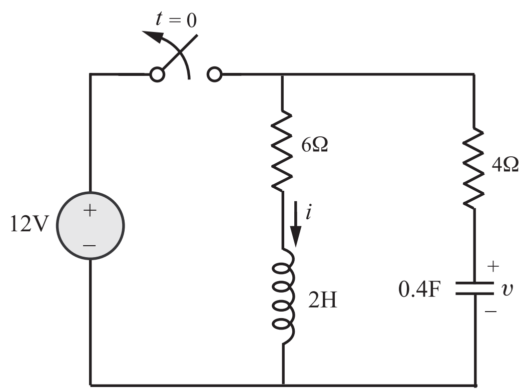


Figure 4.28

SOLUTION

(a) From the symbol of switch, we find that at $t = 0^-$, the switch is closed and $t = 0^+$, it is open. At $t = 0^-$, the circuit has reached steady state so that the equivalent circuit is as shown in Fig.4.29(a).

$$i(0^-) = \frac{12}{6} = 2\text{A}$$

$$v(0^-) = 12 \text{ V}$$

Therefore, we have

$$i(0^+) = i(0^-)$$

$$= \mathbf{2}\text{A}$$

$$v(0^+) = v(0^-) = \mathbf{12V}$$

(b) For $t \geq 0^+$, we have the equivalent circuit as shown in Fig.4.29(b).

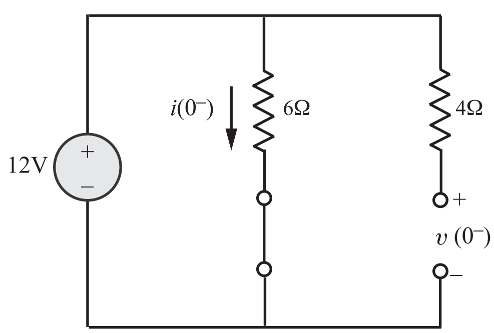


Figure 4.29(a)

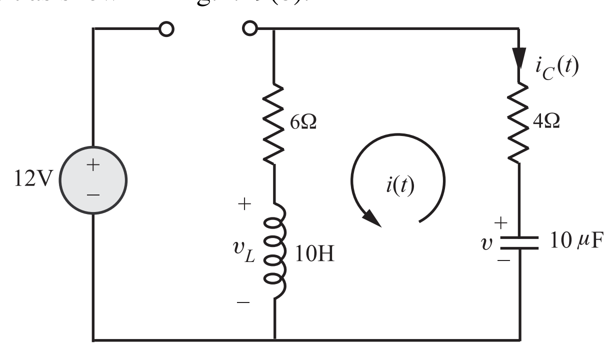


Figure 4.29(b)

Applying KVL anticlockwise to the mesh on the right, we get

$$v_L(t) - v(t) + 10i(t) = 0$$

Putting $t = 0^+$, we get

$$v_L(0^+) - v(0^+) + 10i(0^+) = 0$$

$$\Rightarrow \quad v_L(0^+) - 12 + 10 \times 2 = 0$$

$$\Rightarrow \quad v_L(0^+) = -8\text{V}$$

The voltage across the inductor is given by

$$v_L = L\frac{di}{dt}$$

$$\Rightarrow \quad v_L(0^+) = L\frac{di(0^+)}{dt}$$

$$\Rightarrow \quad \frac{di(0^+)}{dt} = \frac{1}{L}v_L(0^+)$$

$$= \frac{1}{10}(-8) = \mathbf{-0.8A/\,sec}$$

Similarly, the current through the capacitor is

$$i_C = C\frac{dv}{dt}$$

or $$\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = \frac{-i(0^+)}{C}$$

$$= \frac{-2}{10\times 10^{-6}} = \mathbf{-0.2\times 10^6 V/\,sec}$$

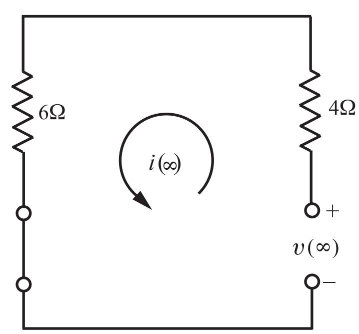


Figure 4.29(c)

(c) As t approaches infinity, the switch is open and the circuit has attained steady state. The equivalent circuit at $t = \infty$ is shown in Fig.4.29(c).

$$i(\infty) = \mathbf{0}$$

$$v(\infty) = \mathbf{0}$$

EXAMPLE 4.14

Refer the circuit shown in Fig.4.30. Find the following:

(a) $v(0^+)$ and $i(0^+)$

(b) $\dfrac{dv(0^+)}{dt}$ and $\dfrac{di(0^+)}{dt}$

(c) $v(\infty)$ and $i(\infty)$

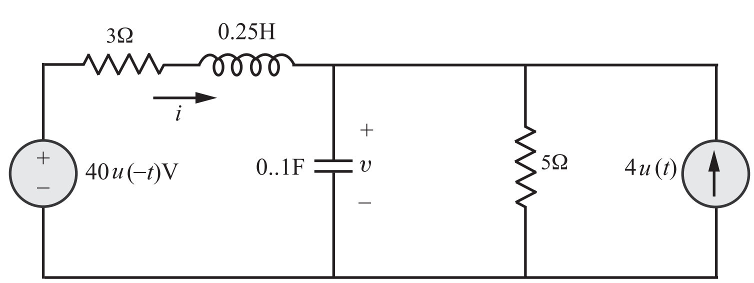


Figure 4.30

SOLUTION

From the definition of step function,

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

From Fig.4.31(a), $u(t) = 0$ at $t = 0^-$.

$$\text{Similarly,} \quad u(-t) = \begin{cases} 1, & -t > 0 \\ 0, & -t < 0 \end{cases}$$

$$\text{or} \qquad u(-t) = \begin{cases} 1, & t < 0 \\ 0, & t > 0 \end{cases}$$

From Fig.4.31(b), we find that $u(-t) = 1$, at $t = 0^-$.

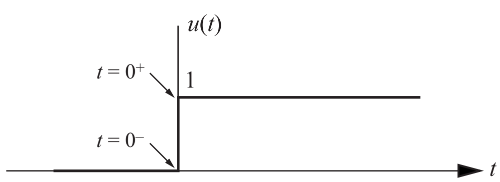


Figure 4.31(a)

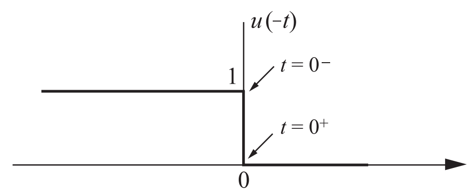


Figure 4.31(b)

Due to the presence of $u(-t)$ and $u(t)$ in the circuit of Fig.4.30, the circuit is an implicit switching circuit. We use the word implicit since there are no conventional switches in the circuit of Fig.4.30.

The equivalent circuit at $t = 0^-$ is shown in Fig.4.31(c). Please note that at $t = 0^-$, the independent current source is open because $u(t) = 0$ at $t = 0^-$ and the circuit is in steady state.

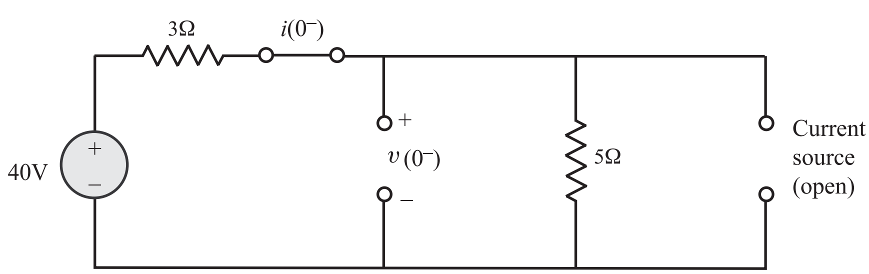


Figure 4.31(c)

$$i(0^-) = \frac{40}{3+5} = 5\text{A}$$

$$v(0^-) = 5i(0^-) = 25\text{V}$$

$$\text{Therefore} \quad i(0^+) = i(0^-) = \mathbf{5A}$$

$$v(0^+) = v(0^-) = \mathbf{25V}$$

(b) For $t \geq 0^+$, $u(-t) = 0$. This implies that the independent voltage source is zero and hence is represented by a short circuit in the circuit shown in Fig.4.31(d).

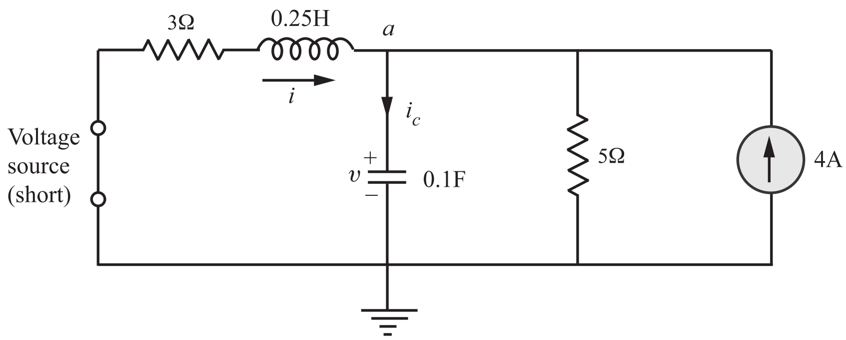


Figure 4.31(d)

Applying KVL at node a, we get

$$4 + i = C\frac{dv}{dt} + \frac{v}{5}$$

At $t = 0^+$, We get

$$4 + i(0^+) = C\frac{dv(0^+)}{dt} + \frac{v(0^+)}{5}$$

$$\Rightarrow \qquad 4 + 5 = 0.1\frac{dv(0^+)}{dt} + \frac{25}{5}$$

$$\Rightarrow \qquad \frac{dv(0^+)}{dt} = \mathbf{40V/\,sec}$$

Applying KVL to the left–mesh, we get

$$3i + 0.25\frac{di}{dt} + v = 0$$

Evaluating at $t = 0^+$, we get

$$3i(0^+) + 0.25\frac{di(0^+)}{dt} + v(0^+) = 0$$

$$\Rightarrow \qquad 3 \times 5 + 0.25\frac{di(0^+)}{dt} + 25 = 0$$

$$\Rightarrow \qquad \frac{di(0^+)}{dt} = \frac{-40}{\frac{1}{4}} = \mathbf{-160A/\,sec}$$

(c) As t approaches infinity, again the circuit is in steady state. The equivalent circuit at $t = \infty$ is shown in Fig.4.31(e).

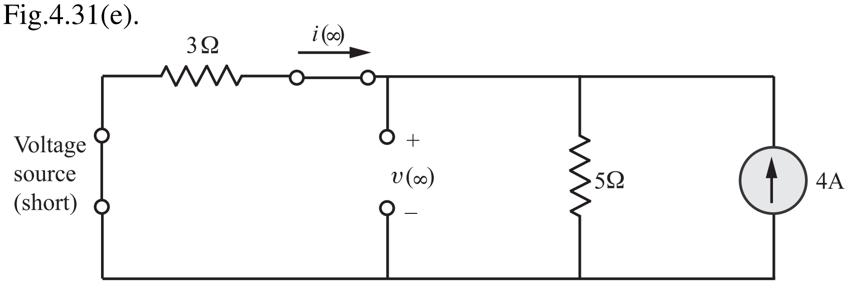


Figure 4.31(e)

Using the principle of current divider, we get

$$i(\infty) = -\left(\frac{4 \times 5}{3+5}\right) = \mathbf{-2.5A}$$
$$v(\infty) = (i(\infty) + 4)\,5$$
$$= (-2.5 + 4)5$$
$$= \mathbf{7.5V}$$

EXAMPLE 4.15

Refer the circuit shown in Fig.4.32. Find the following:

(a) $i(0^+)$ and $v(0^+)$

(b) $\dfrac{di(0^+)}{dt}$ and $\dfrac{dv(0^+)}{dt}$

(c) $i(\infty)$ and $v(\infty)$

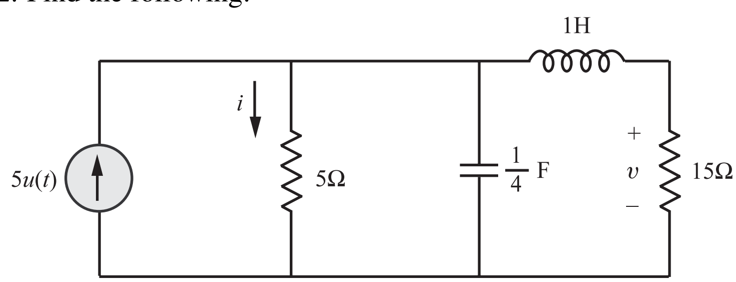


Figure 4.32

SOLUTION

Here the function $u(t)$ behaves like a switch. Mathematically,

$$u(t) = \begin{cases} 1, \; t > 0 \\ 0, \; t < 0 \end{cases}$$

The above expression means that the switch represented by $u(t)$ is open for $t < 0$ and remains closed for $t > 0$. Hence, the circuit diagram of Fig.4.32 may be redrawn as shown in Fig.4.33(a).

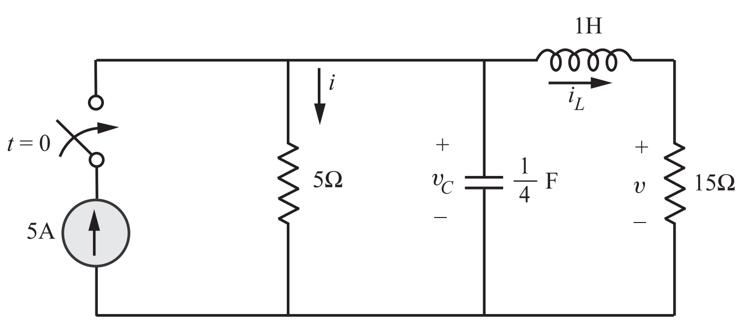


Figure 4.33(a)

For $t < 0$, the circuit is not active because switch is in open state, This implies that all the initial conditions are zero.

That is, $\quad i_L(0^-) = 0$ and $v_C(0^-) = 0$

for $t \geq 0^+$, the equivalent circuit is as shown in Fig.4.33(b).

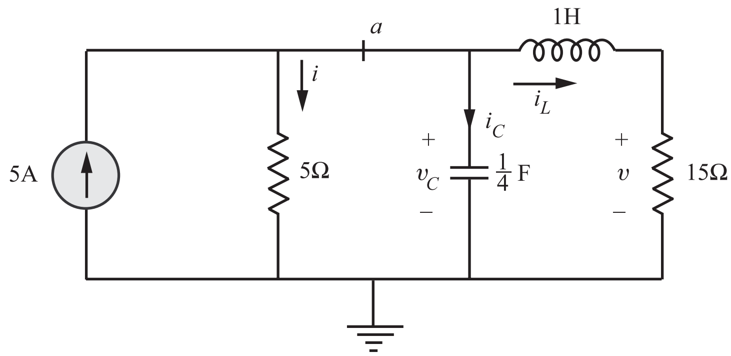


Figure 4.33(b)

From the circuit diagram of Fig.4.33(b), we find that

$$i = \frac{v_C}{5}$$

At $t = 0^+$, we get

$$i(0^+) = \frac{v_C(0^+)}{5} = \frac{v_C(0^-)}{5} = \frac{0}{5} = \mathbf{0A}$$

Also $$v = 15 i_L$$

Evaluating at $t = 0^+$, we get

$$\begin{aligned} v(0^+) &= 15 i_L(0^+) \\ &= 15 i_L(0^-) = 15 \times 0 = \mathbf{0V} \end{aligned}$$

(b) The equivalent circuit at $t = 0^+$ is shown in Fig.4.33(c).

We find from Fig.4.33(c) that

$$i_C(0^+) = 5\text{A}$$

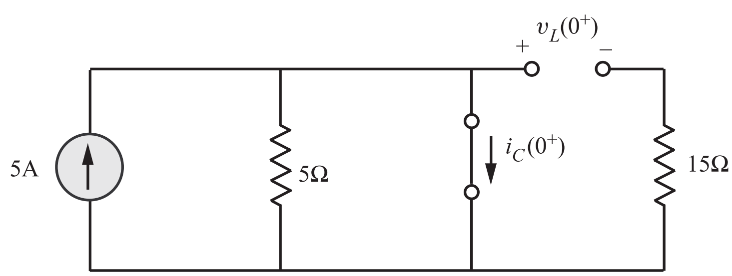


Figure 4.33(c)

From Fig.4.33(b), we can write

$$v_C = 5i$$

$$\Rightarrow \qquad \frac{dv_C}{dt} = 5\frac{di}{dt}$$

Multiplying both sides by C, we get

$$C\frac{dv_C}{dt} = 5C\frac{di}{dt}$$

$$\Rightarrow \qquad i_C = 5C\frac{di}{dt}$$

Putting $t = 0^+$, we get

$$\frac{di(0^+)}{dt} = \frac{1}{5C}i_C(0^+)$$
$$= \frac{1}{5\left(\frac{1}{4}\right)} \times 5$$
$$= \mathbf{4A/\,sec}$$

Also $\qquad v = 15i_L$

$$\Rightarrow \qquad \frac{dv}{dt} = 15\frac{di_L}{dt}$$

$$\Rightarrow \qquad \frac{dv}{dt} = 15\left[1 \times \frac{di_L}{dt}\right]$$

$$\Rightarrow \qquad \frac{dv}{dt} = 15v_L$$

At $t = 0^+$, we find that

$$\Rightarrow \qquad \frac{dv(0^+)}{dt} = 15v_L(0^+)$$

From Fig.4.33(b), we find that $v_L(0^+) = 0$

Hence, $\qquad \dfrac{dv(0^+)}{dt} = 15 \times 0$

$$= \mathbf{0V/\,sec}$$

EXAMPLE 4.16

In the circuit shown in Fig. 4.34, steady state is reached with switch K open. The switch is closed at $t = 0$.

Determine: $i_1,\ i_2,\ \dfrac{di_1}{dt}$ and $\dfrac{di_2}{dt}$ at $t = 0^+$

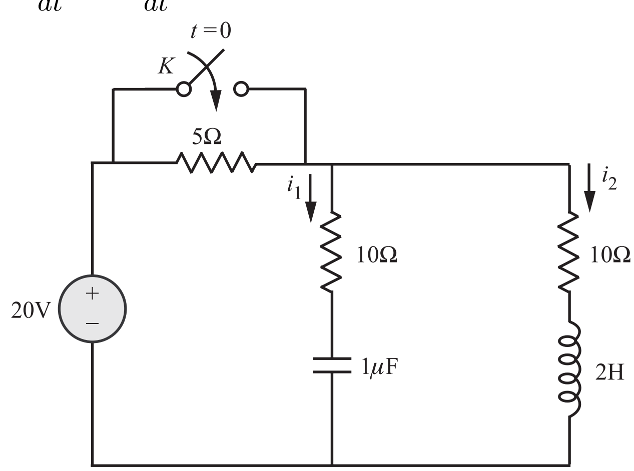


Figure 4.34

SOLUTION

At $t = 0^-$, switch K is open and at $t = 0^+$, it is closed. At $t = 0^-$, the circuit is in steady state and appears as shown in Fig.4.35(a).

$$i_2(0^-) = \frac{20}{10+5} = 1.33\text{A}$$

Hence, $$v_C(0^-) = 10 i_2(0^-) = 10 \times 1.33 = 13.3\text{V}$$

Since current through an inductor cannot change instantaneously, $i_2(0^+) = i_2(0^-) = \mathbf{1.33\ A}$.
Also, $v_C(0^+) = v_C(0^-) = 13.3\text{V}$.
The equivalent circuit at $t = 0^+$ is as shown in Fig.4.35(b).

$$i_1(0^+) = \frac{20 - 13.3}{10} = \frac{6.7}{10} = \mathbf{0.67A}$$

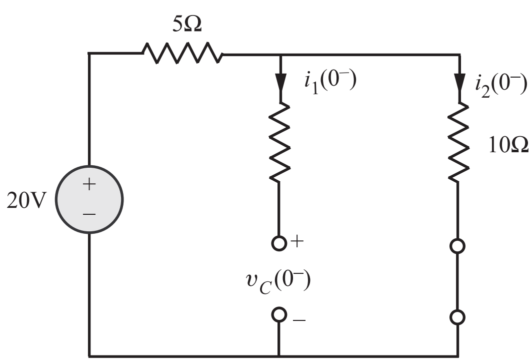


Figure 4.35(a)

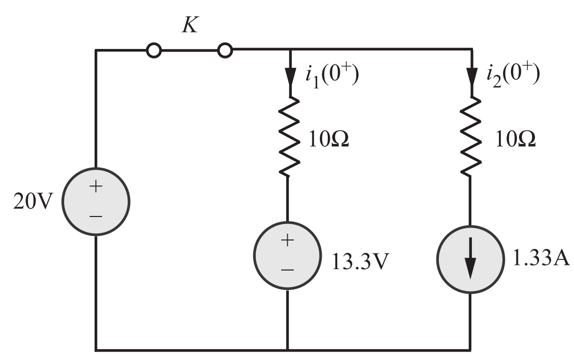


Figure 4.35(b)

For $t \geq 0^+$, the circuit is as shown in Fig.4.35(c).

Writing KVL clockwise for the left–mesh, we get

$$10 i_1 + \frac{1}{C}\int_{0^+}^{t} i_1(\tau)\,d\tau = 20$$

Differentiating with respect to t, we get

$$10\frac{di_1}{dt} + \frac{1}{C} i_1 = 0$$

Putting $t = 0^+$, we get

$$10\frac{di_1(0^+)}{dt} + \frac{1}{C} i_1(0^+) = 0$$

$$\Rightarrow \quad \frac{di_1(0^+)}{dt} = \frac{-1}{10 \times 1 \times 10^{-6}} i_1(0^+) = \mathbf{-0.67 \times 10^5 A/sec}$$

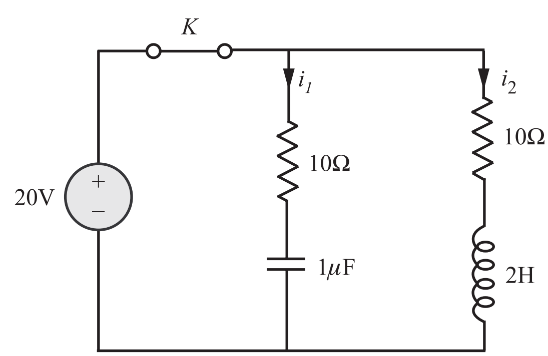


Figure 4.35(c)

Writing KVL equation to the path made of 20V $\rightarrow K \rightarrow 10\Omega \rightarrow$ 2H, we get

$$10i_2 + \frac{2di_2}{dt} = 20$$

At $t = 0^+$, the above equation becomes

$$10i_2(0^+) + \frac{2di_2(0^+)}{dt} = 20$$

$$\Rightarrow \qquad 10 \times 1.33 + \frac{2di_2(0^+)}{dt} = 20$$

$$\Rightarrow \qquad \frac{di_2(0^+)}{dt} = \mathbf{3.35A/\sec}$$

EXAMPLE 4.17

Refer the citcuit shown in Fig.4.36. The switch K is closed at $t = 0$. Find:

(a) v_1 and v_2 at $t = 0^+$

(b) v_1 and v_2 at $t = \infty$

(c) $\dfrac{dv_1}{dt}$ and $\dfrac{dv_2}{dt}$ at $t = 0^+$

(d) $\dfrac{d^2v_1}{dt^2}$ at $t = 0^+$

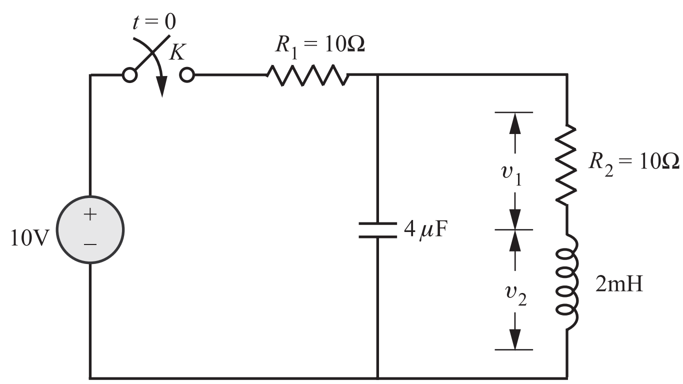


Figure 4.36

SOLUTION

(a) The circuit symbol for switch conveys that at $t = 0^-$, the switch is open and $t = 0^+$, it is closed. At $t = 0^-$, since the switch is open, the circuit is not activated. This implies that all initial conditions are zero. Hence, at $t = 0^+$, inductor is open and capacitor is short. Fig 4.37(a) shows the equivalent circuit at $t = 0^+$.

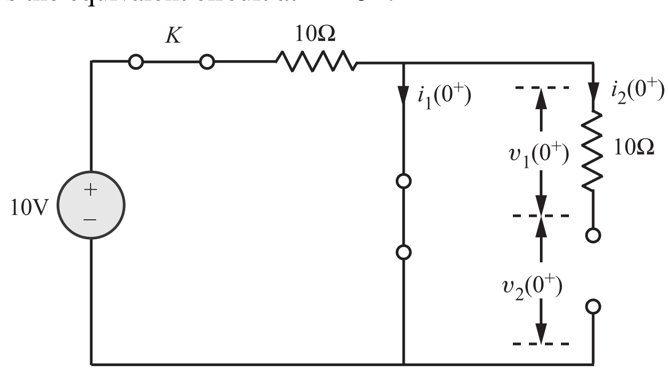


Figure 4.37(a)

$$i_1(0^+) = \frac{10}{10} = 1\text{A}$$
$$v_1(0^+) = 0, \quad i_2(0^+) = 0$$

Applying KVL to the path, 10V source $\rightarrow K \rightarrow 10\Omega \rightarrow 10\Omega \rightarrow$ 2mH, we get

$$-10 + 10i_1(0^+) + v_1(0^+) + v_2(0^+) = 0$$
$$\Rightarrow \quad -10 + 10 + 0 + v_2(0^+) = 0$$
$$\Rightarrow \quad v_2(0^+) = \mathbf{0}$$

(b) At $t = \infty$, switch K remains closed and circuit is in steady state. Under steady state conditions, capacitor C is open and inductor L is short. Fig. 4.37(b) shows the equivalent circuit at $t = \infty$.

$$i_2(\infty) = \frac{10}{10+10} = \mathbf{0.5A}$$
$$i_1(\infty) = \mathbf{0}$$
$$v_1(\infty) = 0.5 \times 10 = \mathbf{5V}$$
$$v_2(\infty) = \mathbf{0}$$

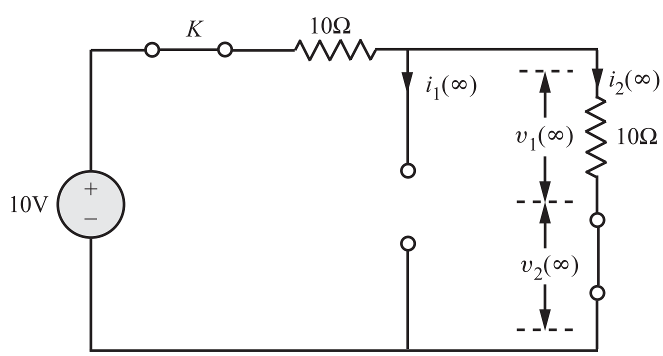


Figure 4.37(b)

(c) For $t \geq 0^+$, the circuit is as shown in Fig. 4.37(c).

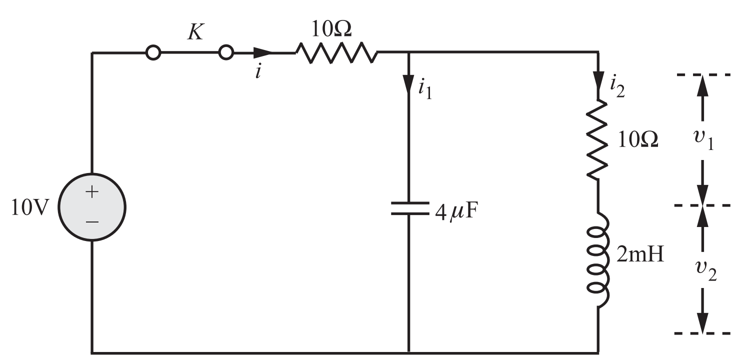


Figure 4.37(c)

$$i_2 = \frac{1}{L}\int\limits_{0^+}^{t} v_2(\tau)d\tau = \frac{v_1(t)}{R_2}$$

Differentiating with respect to t, we get

$$\frac{v_2}{L} = \frac{1}{R_2}\frac{dv_1}{dt}$$

Evaluating at $t = 0^+$ we get

$$\frac{dv_1(0^+)}{dt} = \frac{R_2}{L_2}v_2(0^+)$$

$$\Rightarrow \qquad \frac{dv_1(0^+)}{dt} = \mathbf{0V/\,sec}$$

Applying KVL clockwise to the path 10 V source $\rightarrow K \rightarrow 10\Omega \rightarrow 4\mu$F, we get

$$-10 + 10i + \frac{1}{C}\int\limits_{0^+}^{t} [i(\tau) - i_2(\tau)]d\tau = 0$$

Differentiating with respect to t, we get

$$10\frac{di}{dt} + \frac{1}{C}[i - i_2] = 0$$

Evaluating at $t = 0^+$, we get

$$\begin{aligned}\frac{di(0^+)}{dt} &= \frac{i_2(0^+) - i(0^+)}{C \times 10}\\ &= \frac{0-1}{10 \times 4 \times 10^{-6}} \qquad \left[\begin{aligned}\because\ i(0^+) &= i_1(0^+) + i_2(0^+)\\ &= 1 + 0\\ &= 1\text{A}\end{aligned}\right]\\ &= \mathbf{-25000A/\,sec}\end{aligned}$$

Applying KVL clockwise to the path 10 V source $\rightarrow K \rightarrow 10\Omega \rightarrow 10\Omega \rightarrow 2$ mH, we get

$$-10 + 10i + 10i_2 + v_2 = 0$$

$$\Rightarrow \qquad 10i + v_1 + v_2 = 10$$

Differentiating with respect to t, we get

$$10\frac{di}{dt} + \frac{dv_1}{dt} + \frac{dv_2}{dt} = 0$$

At $t = 0^+$, we get

$$10\frac{di(0^+)}{dt} + \frac{dv_1(0^+)}{dt} + \frac{dv_2(0^+)}{dt} = 0$$

$$\Rightarrow \quad 10(-25000) + 0 + \frac{dv_2(0^+)}{dt} = 0$$

$$\Rightarrow \quad \frac{dv_2(0^+)}{dt} = \mathbf{25 \times 10^4 V/\sec}$$

(d) From part (c), we have

$$\frac{1}{L}\int_{0^+}^{t} v_2(\tau)d\tau = \frac{v_1}{10}$$

Differentiating with respect to t twice, we get

$$\frac{1}{L}\frac{dv_2}{dt} = \frac{1}{10}\frac{d^2v_1}{dt^2}$$

At $t = 0^+$, we get

$$\frac{1}{L}\frac{dv_2(0^+)}{dt} = \frac{1}{10}\frac{d^2v_1(0^+)}{dt^2}$$

Hence, $$\frac{d^2v_1(0^+)}{dt^2} = \mathbf{125 \times 10^7 V/\sec^2}$$

EXAMPLE 4.18

Refer the network shown in Fig. 4.38. Switch K is changed from a to b at $t = 0$ (a steady state having been established at position a).

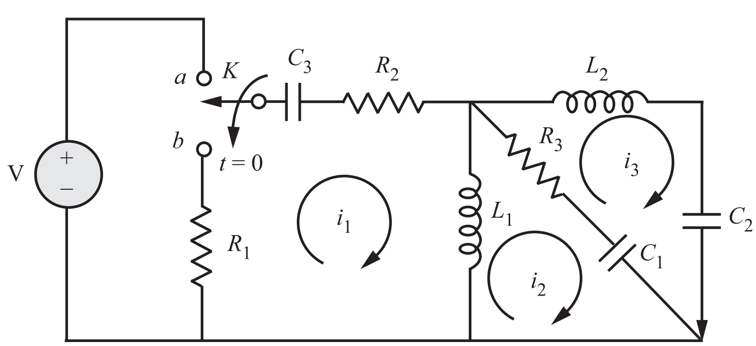


Figure 4.38

Show that at $t = 0^+$.

$$i_1 = i_2 = \frac{-V}{R_1 + R_2 + R_3}, \quad i_3 = 0$$

SOLUTION

The symbol for switch indicates that at $t = 0^-$, it is in position a and at $t = 0^+$, it is in position b. The circuit is in steady state at $t = 0^-$. Fig 4.39(a) refers to the equivalent circuit at $t = 0^-$. Please remember that at steady state C is open and L is short.

$$i_{L_1}(0^-) = 0, \;\; i_{L_2}(0^-) = 0, \;\; v_{C_2}(0^-) = 0, \;\; v_{C_1}(0^-) = 0$$

Applying KVL clockwise to the left-mesh, we get

$$-V + v_{C_3}(0^-) + 0 \times R_2 + 0 = 0$$

$$\Rightarrow \qquad v_{C_3}(0^-) = \text{V volts.}$$

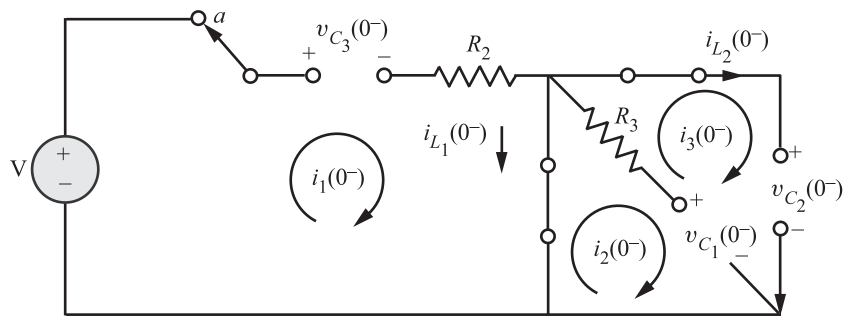


Figure 4.39(a)

Since current in an inductor and voltage across a capacitor cannot change instantaneously, the equivalent circuit at $t = 0^+$ is as shown in Fig. 4.39(b).

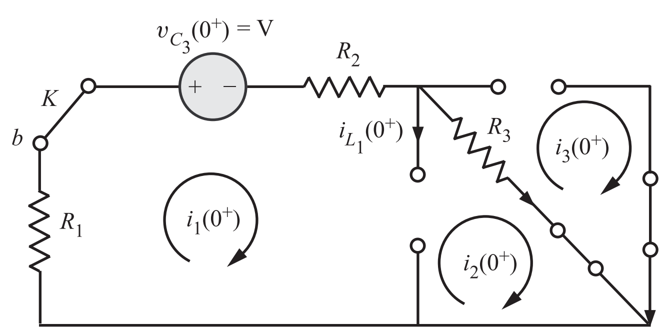


Figure 4.39(b)

$$i_1(0^+) = i_2(0^+) \text{ since } \; i_{L_1}(0^+) = 0$$
$$i_3(0^+) = 0 \text{ since } \; i_{L_2}(0^+) = 0$$

Applying KVL to the path $v_{C_3}(0^+) \rightarrow R_2 \rightarrow R_3 \rightarrow R_1 \rightarrow K$ we get,

$$V + R_2 i_1(0^+) + R_3 i_2(0^+) + R_1 i_1(0^+) = 0$$

Since $i_1(0^+) = i_2(0^+)$, the above equation becomes

$$-V = [R_1 + R_2 + R_3]\, i_1(0^+)$$

Hence,
$$i_1(0^+) = i_2(0^+) = \frac{\boldsymbol{-V}}{\boldsymbol{R_1 + R_2 + R_3}}\,\mathbf{A}$$

EXAMPLE 4.19

Refer the circuit shown in Fig. 4.40. The switch K is closed at $t = 0$.

Find (a) $\dfrac{di_1(0^+)}{dt}$ and (b) $\dfrac{di_2(0^+)}{dt}$

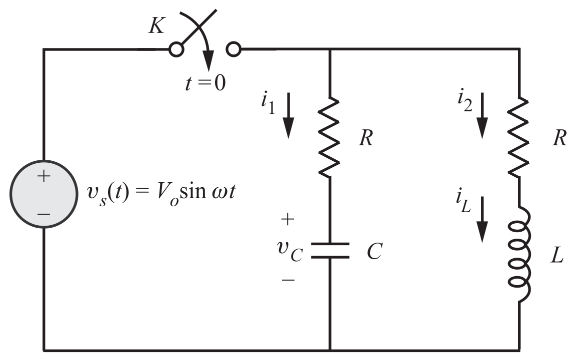


Figure 4.40

SOLUTION

The circuit symbol for the switch shows that at $t = 0^-$, it is open and at $t = 0^+$, it is closed. Hence, at $t = 0^-$, the circuit is not activated. This implies that all initial conditions are zero. That is, $v_C(0^-) = 0$ and $i_L(0^-) = i_2(0^-) = 0$. The equivalent circuit at $t = 0^+$ keeping in mind that $v_C(0^+) = v_C(0^-)$ and $i_L(0^+) = i_L(0^-)$ is as shown in Fig. 4.41 (a).

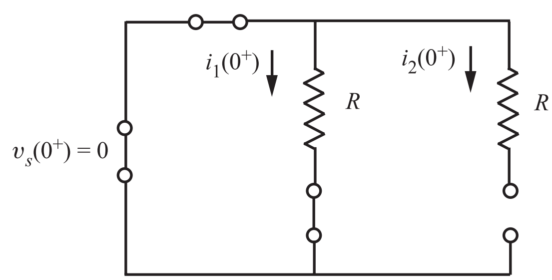


Figure 4.41(a)

$$i_1(0^+) = 0 \text{ and } i_2(0^+) = 0.$$

Figure. 4.41(b) shows the circuit diagram for $t \geq 0^+$.

$$V_o \sin \omega t = i_1 R + \frac{1}{C}\int_{0^+}^{t} i_1(\tau)\,d\tau$$

Differentiating with respect to t, we get

$$V_o \omega \cos \omega t = R\frac{di_1}{dt} + \frac{i_1}{C}$$

At $t = 0^+$, we get

$$V_o\omega = R\frac{di_1(0^+)}{dt} + \frac{i_1(0^+)}{C}$$

$$\Rightarrow \quad \frac{di_1(0^+)}{dt} = \frac{\boldsymbol{V_0\omega}}{\boldsymbol{R}}\,\mathbf{A/\,sec}$$

Also, $\quad V_o \sin\omega t = i_2 R + L\dfrac{di_2}{dt}$

Evaluating at $t = 0^+$, we get

$$0 = i_2(0^+)R + L\frac{di_2(0^+)}{dt}$$

$$\Rightarrow \quad \frac{di_2(0^+)}{dt} = \mathbf{0A/\,sec}$$

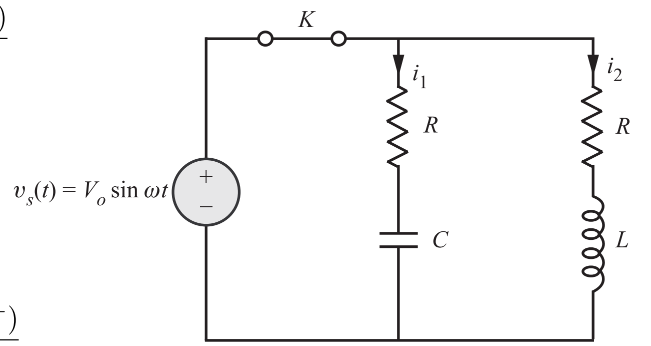


Figure 4.41(b)

EXAMPLE 4.20

In the network of the Fig. 4.42, the switch K is opened at $t = 0$ after the network has attained steady state with the switch closed.

(a) Find the expression for v_K at $t = 0^+$.

(b) If the parameters are adjusted such that $i(0^+) = 1$, and $\dfrac{di(0^+)}{dt} = -1$, what is the value of the derivative of the voltage across the switch at $t = 0^+$, $\left(\dfrac{dv_K}{dt}(0^+)\right)$?

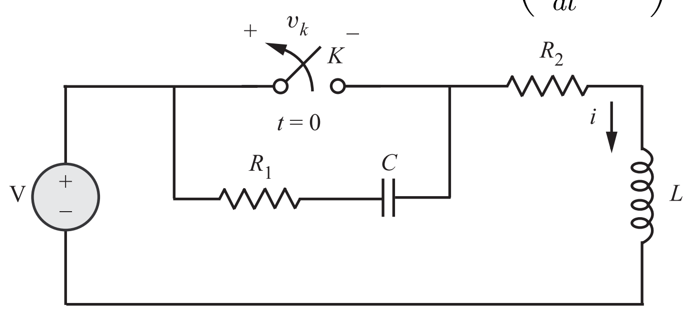


Figure 4.42

SOLUTION

At $t = 0^-$, switch is in the closed state and at $t = 0^+$, it is open. Also at $t = 0^-$, the circuit is in steady state. The equivalent circuit at $t = 0^-$ is as shown in Fig. 4.43(a).

$$i(0^-) = \frac{V}{R_2} \text{ and } v_C(0^-) = 0$$

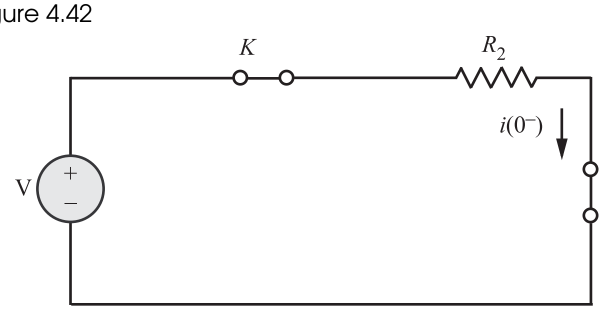


Figure 4.43(a)

For $t \geq 0^+$, the equivalent circuit is as shown in Fig. 4.43(b).
From Fig. 4.43 (b),

$$v_K = R_1 i + \frac{1}{C}\int_{0^+}^{t} i(\tau)d\tau$$

$$\Rightarrow \qquad v_K = R_1 i + v_C(t)$$

At $t = 0^+$, $v_K(0^+) = R_1 i(0^+) + v_C(0^+)$

$$\Rightarrow \qquad v_K(0^+) = R_1\frac{V}{R_2} + v_C(0^-)$$

$$= \boldsymbol{R_1\frac{V}{R_2}}\textbf{volts}$$

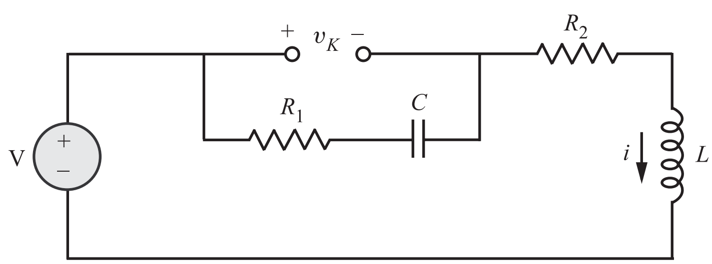


Figure 4.43(b)

(b)

$$v_K = R_1 i + \frac{1}{C}\int_{0^+}^{t} i(\tau)d\tau$$

$$\Rightarrow \qquad \frac{dv_K}{dt} = R_1\frac{di}{dt} + \frac{i}{C}$$

Evaluating at $t = 0^+$, we get

$$\frac{dv_K(0^+)}{dt} = R_1\frac{di(0^+)}{dt} + \frac{i(0^+)}{C}$$

$$= R_1 \times (-1) + \frac{1}{C}$$

$$= \boldsymbol{\frac{1}{C} - R_1}\textbf{volts/ sec}$$

Reinforcement Problems

R.P 4.1

Refer the circuit shown in Fig RP.4.1(a). If the switch is closed at $t = 0$, find the value of $\dfrac{d^2 i_L(0^+)}{dt^2}$ at $t = 0^+$.

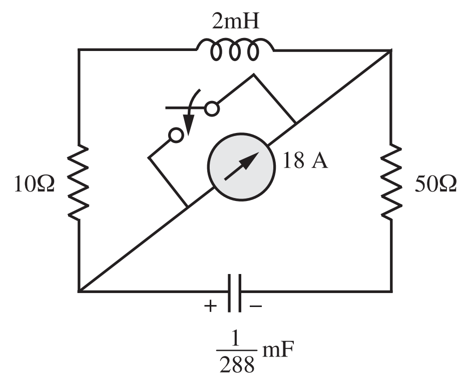


Figure R.P.4.1(a)

SOLUTION

The circuit at $t = 0^-$ is as shown in Fig RP 4.1(b).

Since current through an inductor and voltage across a capacitor cannot change instantaneously, it implies that $i_L(0^+) = 18$A and $v_C(0^+) = -180$ V.

The circuit for $t \geq 0^+$ is as shown in Fig. RP 4.1 (c).

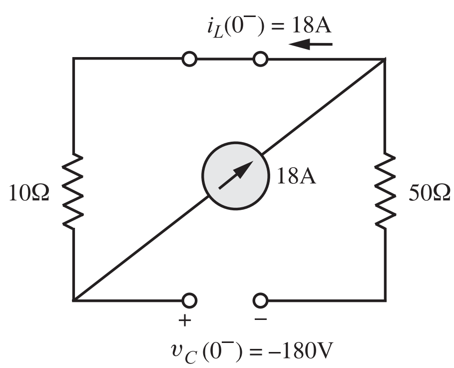


Figure R.P.4.1(b)

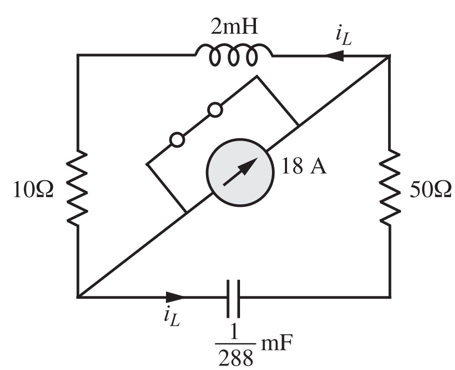


Figure R.P.4.1(c)

Referring Fig RP 4.1 (c), we can write

$$2 \times 10^{-3} \frac{di_L}{dt} + 60 i_L + 288 \times 10^3 \int_{0^+}^{t} i_L(t) dt = 0 \tag{4.9}$$

At $t = 0^+$, we get

$$\frac{di_L(0^+)}{dt} = \frac{-60 \times 18 + 180}{2 \times 10^{-3}}$$
$$= -450 \times 10^3 \text{ A/ sec}$$

Differentiating equation (4.9) with respect to t, we get

$$2 \times 10^{-3}\frac{d^2 i_L}{dt^2} + 60\frac{di_L}{dt} + 288 \times 10^3 i_L = 0$$

At $t = 0^+$, we get

$$\frac{d^2 i_L(0^+)}{dt^2} = \frac{60(450)10^3 - 288 \times 10^3(18)}{2 \times 10^{-3}}$$
$$= 1.0908 \times 10^{10} \text{ A/ sec}^2$$

R.P 4.2

For the circuit shown in Fig. RP 4.2, determine $\dfrac{d^2 v_C(0^+)}{dt^2}$ and $\dfrac{d^3 v_C(0^+)}{dt^3}$.

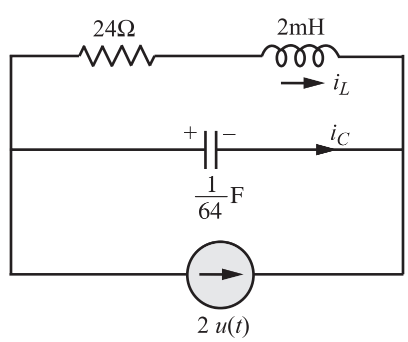


Figure R.P.4.2

SOLUTION

Given

$$i(t) = 2u(t) = \begin{cases} 2, & t \geq 0^+ \\ 0, & t \leq 0^- \end{cases}$$

Hence, at $t = 0^-$, $v_C(0^-) = 0$ and $i_L(0^-) = 0$.
For $t \geq 0^+$, the circuit equations are

$$\frac{1}{64}\frac{dv_C(t)}{dt} + \frac{1}{2}\int_{0^+}^{t} v_L(t)dt = -2 \tag{4.10}$$

$$\Rightarrow \quad \frac{1}{64}\frac{dv_C(t)}{dt} + i_L(t) = -2 \tag{4.11}$$

[**Note** : $i_C + i_L = -2$ because of the capacitor polarity]

At $t = 0^+$, equation (4.10) gives

$$\frac{1}{64}\frac{dv_C(0^+)}{dt} + i_L(0^+) = -2$$

Since, $i_L(0^+) = i_L(0^-) = 0$, we get

$$\frac{1}{64}\frac{dv_C(0^+)}{dt} + 0 = -2$$

$$\Rightarrow \qquad \frac{dv_C(0^+)}{dt} = -128 \text{ volts/ sec}$$

Differentiating equation (4.10) with respect to t we get

$$\frac{1}{64}\frac{d^2v_C(t)}{dt} + \frac{1}{2}v_L(t) = 0 \tag{4.12}$$

Also,
$$\frac{v_C - v_L}{24} = \frac{1}{2}\int_{0^+}^{t} v_L dt = i_L \tag{4.13}$$

At $t = 0^+$, we get

$$\frac{v_C(0^+) - v_L(0^+)}{24} = i_L(0^+)$$

Since $v_C(0^+) = 0$ and $i_L(0^+) = 0$, we get $v_L(0^+) = 0$.

At $t = 0^+$, equation (4.12) becomes

$$\frac{1}{64}\frac{d^2v_C(0^+)}{dt^2} + \frac{1}{2}v_L(0^+) = 0$$

$$\Rightarrow \qquad \frac{1}{64}\frac{d^2v_C(0^+)}{dt^2} + \frac{1}{2} \times 0 = 0$$

$$\Rightarrow \qquad \frac{d^2v_C(0^+)}{dt^2} = 0$$

Differentiating equation (4.12) with respect to t we get

$$\Rightarrow \qquad \frac{1}{64}\frac{d^3v_C}{dt^3} + \frac{1}{2}\frac{dv_L}{dt} = 0 \tag{4.14}$$

Differentiating equation (4.13) with respect to t, we get

$$\frac{\dfrac{dv_C}{dt} - \dfrac{dv_L}{dt}}{24} = \frac{1}{2}v_L$$

At $t = 0^+$, we get

$$\Rightarrow \qquad \frac{\dfrac{dv_C(0^+)}{dt} - \dfrac{dv_L(0^+)}{dt}}{24} = \frac{1}{2}v_L(0^+)$$

$$\Rightarrow \qquad \frac{-128 - \dfrac{dv_L(0^+)}{dt}}{24} = 0$$

$$\Rightarrow \qquad \frac{dv_L(0^+)}{dt} = -128 \text{ volts/ sec}$$

At $t = 0^+$, equation (4.14) becomes

$$\frac{1}{64}\frac{d^3v_C(0^+)}{dt^3} + \frac{1}{2}\frac{dv_L(0^+)}{dt} = 0$$

$$\Rightarrow \qquad \frac{d^3v_C(0^+)}{dt^3} = 4096 \text{ volts/ sec}^3$$

R.P 4.3

In the network of Fig RP 4.3 (a), switch K is closed at $t = 0$. At $t = 0^-$ all the capacitor voltages and all the inductor currents are zero. Three node-to-datum voltages are identified as v_1, v_2 and v_3. Find at $t = 0^+$:

(i) v_1, v_2 and v_3

(ii) $\dfrac{dv_1}{dt}$, $\dfrac{dv_2}{dt}$ and $\dfrac{dv_3}{dt}$

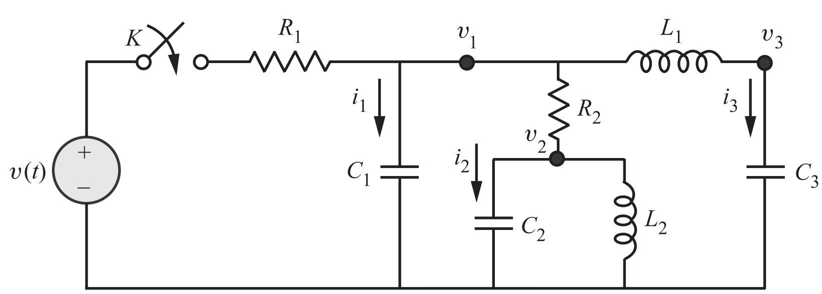


Figure R.P.4.3(a)

SOLUTION

The network at $t = 0^+$ is as shown in Fig RP-4.3 (b).

Since v_C and i_L cannot change instantaneously, we have from the network shown in Fig. RP-4.3 (b),

$$v_1(0^+) = 0$$
$$v_2(0^+) = 0$$
$$v_3(0^+) = 0$$

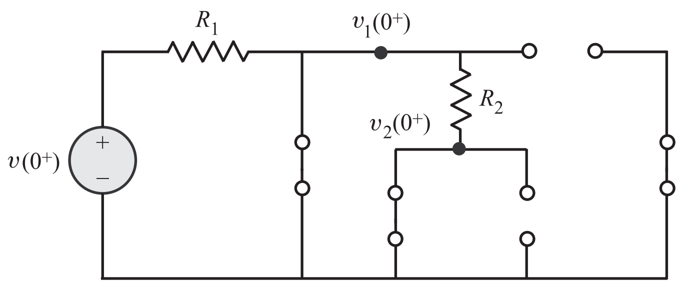


Figure R.P.4.3(b)

For $t \geq 0^+$, the circuit equations are

$$\left.\begin{aligned} v_{C_1} &= \frac{1}{C_1}\int_{0^+}^{t} i_1\,dt \\ v_{C_2} &= \frac{1}{C_2}\int_{0^+}^{t} i_2\,dt \\ v_{C_3} &= \frac{1}{C_3}\int_{0^+}^{t} i_3\,dt \end{aligned}\right\} \tag{4.15}$$

From Fig. RP-4.3 (b), we can write

$$i_1(0^+) = \frac{v(0^+)}{R_1},$$

$$i_2(0^+) = \frac{v_1(0^+) - v_2(0^+)}{R_2}$$

and

$$i_3(0^+) = 0$$

Differentiating equation (4.15) with respect to t, we get

$$\frac{dv_{C_1}}{dt} = \frac{i_1}{C_1}, \frac{dv_{C_2}}{dt} = \frac{i_2}{C_2} \text{ and } \frac{dv_{C_3}}{dt} = \frac{i_3}{C_3}$$

At $t = 0^+$, the above equations give

$$\frac{dv_1(0^+)}{dt} = \frac{i_1(0^+)}{C_1} = \frac{v(0^+)}{R_1C_1}$$

$$\frac{dv_2(0^+)}{dt} = \frac{i_2(0^+)}{C_2} = \frac{v_1(0^+) - v_2(0^+)}{R_2C_2} = 0$$

and

$$\frac{dv_3(0^+)}{dt} = \frac{i_3(0^+)}{C_3} = 0$$

R.P 4.4

For the network shown in Fig RP 4.4 (a) with switch K open, a steady-state is reached. The circuit papmeters are $R_1 = 10\Omega$, $R_2 = 20\Omega$, $R_3 = 20\Omega$, $L = 1$ H and $C = 1\mu$F. Take $V = 100$ volts. The switch is closed at $t = 0$.

(a) Write the integro-differential equation after the switch is closed.

(b) Find the voltage V_o across C before the switch is closed and give its polarity.

(c) Find i_1 and i_2 at $t = 0^+$.

(d) Find $\frac{di_1}{dt}$ and $\frac{di_2}{dt}$ at $t = 0^+$.

(e) What is the value of $\frac{di_1}{dt}$ at $t = \infty$?

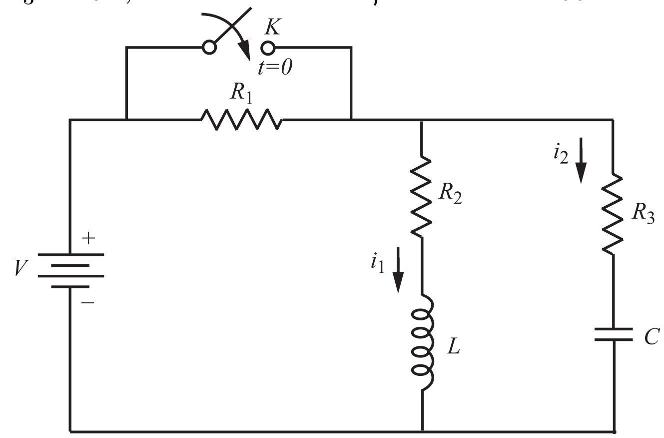


Figure R.P.4.4(a)

SOLUTION

The switch is in open state at $t = 0^-$. The network at $t = 0^-$ is as shown in Fig RP 4.4 (b).

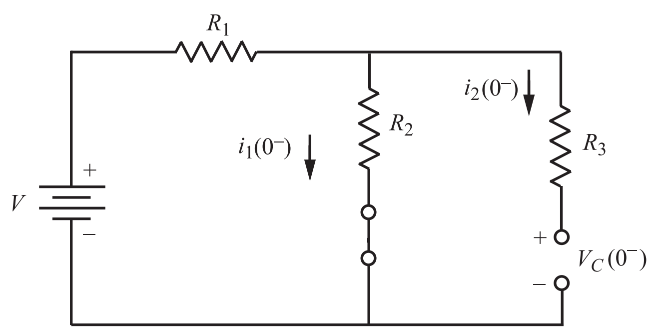


Figure R.P.4.4(b)

$$i_1(0^-) = \frac{V}{R_1 + R_2} = \frac{100}{30} = \frac{10}{3}\text{ A}$$

$$V_C(0^-) = i_1(0^-)R_2 = \frac{10}{3} \times 20 = \frac{200}{3}\text{ volts}$$

Note that L is short and C is open under steady-state condition.

For $t \geq 0^+$ (switch in closed state),

we have
$$20i_1 + \frac{di_1}{dt} = 100 \tag{4.16}$$

and
$$20i_2 + 10^6 \int_{0^+}^{t} i_2 dt = 100 \tag{4.17}$$

Also $$i_1(0^+) = i_1(0^-) = \frac{10}{3}\ \text{A}$$

and $$V_C(0^+) = V_C(0^-) = \frac{200}{3}\ \text{Volts}$$

From equation (4.16) at $t = 0^+$,

we have $$\frac{di_1(0^+)}{dt} = 100 - 20 \times \frac{10}{3}$$
$$= \frac{100}{3}\ \text{A/sec}$$

From equation (4.17), at $t = 0^+$, we have

$$i_2(0^+) = \frac{1}{20}\left[100 - \frac{200}{3}\right] = \frac{5}{3}\ \text{A}$$

Differentiating equation (4.17), we get

$$20\frac{di_2}{dt} + 10^6 i_2 = 0 \tag{4.18}$$

From equation (4.18) at $t = 0^+$, we get

$$\frac{20di_2(0^+)}{dt} + 10^6 i_2(0^+) = 0$$

$$\Rightarrow \quad \frac{di_2(0^+)}{dt} = \frac{-10^6 \times \frac{5}{3}}{20}$$
$$= \frac{-10^6}{12}\ \text{A/ sec}$$

At $t = \infty$,

$$i_1(\infty) = \frac{100}{20} = 5\ \text{A}$$
$$\frac{di_1}{dt}(\infty) = 0$$

R.P 4.5

For the network shown in Fig RP 4.5 (a), find $\frac{d^2 i_1(0^+)}{dt^2}$.

The switch K is closed at $t = 0$.

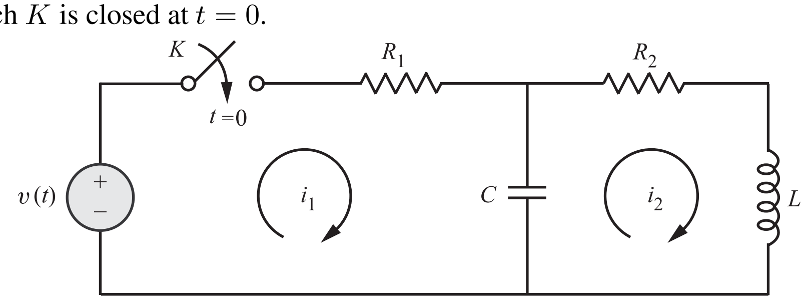


Figure R.P.4.5(a)

SOLUTION

At $t = 0^-$, we have $v_C(0^-) = 0$ and $i_2(0^-) = i_L(0^-) = 0$. Because of the switching property of L and C, we have $v_C(0^+) = 0$ and $i_2(0^+) = 0$. The network at $t = 0^+$ is as shown in Fig RP 4.5 (b).

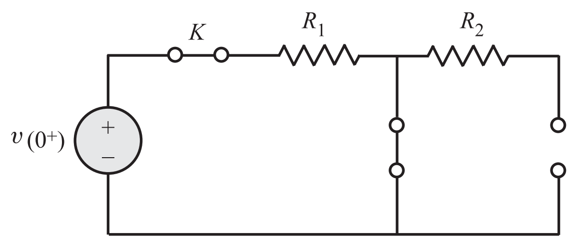


Figure R.P.4.5(b)

Referring Fig RP 4.5 (b), we find that

$$i_1(0^+) = \frac{v(0^+)}{R_1}$$

The circuit equations for $t \geq 0^+$ are

$$R_1 i_1 + \frac{1}{C}\int_{0^+}^{t}(i_1 - i_2)dt = v(t) \tag{4.19}$$

and

$$R_2 i_2 + \underbrace{\frac{1}{C}\int_{0^+}^{t}(i_2 - i_1)dt}_{v_C(t)} + L\frac{di_2}{dt} = 0 \tag{4.20}$$

At $t = 0^+$, equation (4.20) becomes

$$R_2 i_2(0^+) + v_C(0^+) + L\frac{di_2(0^+)}{dt} = 0$$

$$\Rightarrow \qquad \frac{di_2(0^+)}{dt} = 0 \tag{4.21}$$

Differentiating equation (4.19), we get

$$R_1\frac{di_1}{dt} + \frac{1}{C}(i_1 - i_2) = \frac{dv(t)}{dt} \tag{4.22}$$

Letting $t = 0^+$ in equation (4.22), we get

$$R_1\frac{di_1(0^+)}{dt} + \frac{1}{C}\left\{i_1(0^+) - i_2(0^+)\right\} = \frac{dv(0^+)}{dt}$$

$$\Rightarrow \qquad \frac{di_1(0^+)}{dt} = \frac{1}{R_1}\left\{\frac{dv(0^+)}{dt} - \frac{v(0^+)}{R_1 C}\right\} \tag{4.23}$$

Differentiating equation (4.22) gives

$$R_1\frac{d^2i_1}{dt^2}+\frac{1}{C}\left[\frac{di_1}{dt}-\frac{di_2}{dt}\right]=\frac{d^2v(t)}{dt^2}$$

Letting $t=0^+$, we get

$$R_1\frac{d^2i_1(0^+)}{dt^2}+\frac{1}{C}\left[\frac{di_1(0^+)}{dt}-\frac{di_2(0^+)}{dt}\right]=\frac{d^2v(0^+)}{dt^2}$$

$$\Rightarrow \qquad R_1\frac{d^2i_1(0^+)}{dt^2}=-\frac{1}{C}\frac{di_1(0^+)}{dt}+\frac{d^2v(0^+)}{dt^2}$$

$$\Rightarrow \qquad \frac{d^2i_1(0^+)}{dt^2}=-\frac{1}{R_1C}\left\{\frac{1}{R_1}\frac{dv(0^+)}{dt}-\frac{1}{R_1^2C}v(0^+)\right\}+\frac{d^2v(0^+)}{dt^2}$$

R.P 4.6

Determine $v_a(0^-)$ and $v_a(0^+)$ for the network shown in Fig RP 4.6 (a). Assume that the switch is closed at $t=0$.

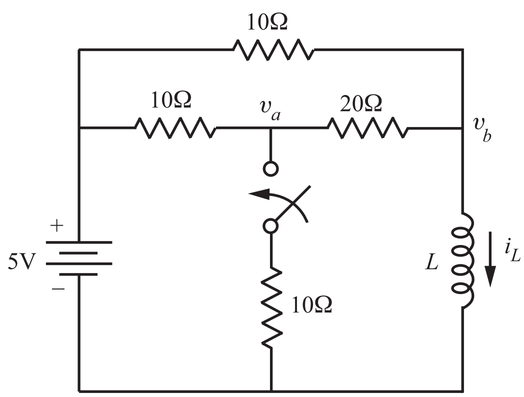


Figure R.P.4.6(a)

SOLUTION

Since L is short for DC at steady state, the network at $t=0^-$ is as shown in Fig. RP 4.6 (b).

Applying KCL at junction a, we get

$$\frac{v_a(0^-)-5}{10}+\frac{v_a(0^-)-v_b(0^-)}{20}=0$$

Since $v_b(0^-)=0$, we get

$$\frac{v_a(0^-)-5}{10}+\frac{v_a(0^-)-0}{20}=0$$

$$\Rightarrow \qquad v_a(0^-)=\frac{0.5}{0.1+0.05}=\frac{10}{3}\text{ volts}$$

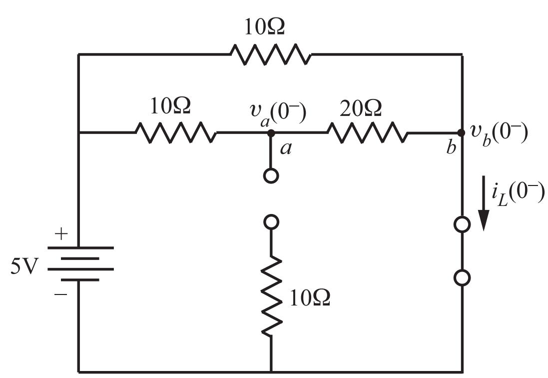


Figure R.P.4.6(b)

Also, $$i_L(0^-) = i_L(0^+) = \frac{v_a(0^-)}{20} + \frac{5}{10}$$
$$= \frac{2}{3}\ \text{A}$$

For $t \geq 0^+$, we can write

$$\frac{v_a - 5}{10} + \frac{v_a}{10} + \frac{v_a - v_b}{20} = 0$$

and $$\frac{v_b - v_a}{20} + \frac{v_b - 5}{10} + i_L = 0$$

Simplifying at $t = 0^+$, we get

$$\frac{1}{4}v_a(0^+) - \frac{1}{20}v_b(0^+) = \frac{1}{2}$$

and $$-\frac{1}{20}v_a(0^+) + \frac{3}{20}v_b(0^+) = \frac{-1}{6}$$

Solving we get, $v_a(0^+) = \dfrac{40}{21} = 1.905$ volts

Exercise problems

E.P 4.1

Refer the circuit shown in Fig. E.P. 4.1 Switch K is closed at $t = 0$.

Find $i(0^+)$, $\dfrac{di(0^+)}{dt}$ and $\dfrac{d^2i(0^+)}{dt^2}$.

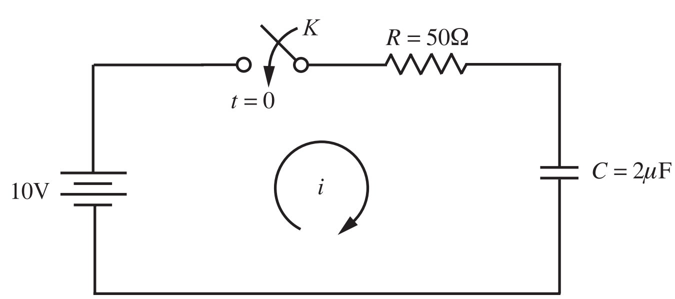


Figure E.P.4.1

Ans: $\boldsymbol{i(0^+) = 0.2\text{A}, \quad \dfrac{di(0^+)}{dt} = -2 \times 10^3 \text{A/sec}, \quad \dfrac{d^2i(0^+)}{dt^2} = 20 \times 10^6 \text{A/sec}^2}$

E.P 4.2

Refer the circuit shown in Fig. E.P. 4.2. Switch K is closed at $t = 0$. Find the values of i, $\dfrac{di}{dt}$ and $\dfrac{d^2i}{dt^2}$ at $t = 0^+$.

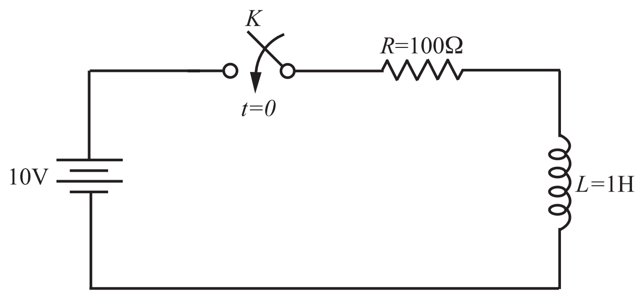


Figure E.P.4.2

$$\textbf{Ans: } i(0^+) = 0, \quad \frac{di(0^+)}{dt} = 10 \text{ A/sec}, \quad \frac{d^2i(0^+)}{dt^2} = -1000 \text{ A/sec}^2$$

E.P 4.3

Refering to the circuit shown in Fig. E.P. 4.3, switch is changed from position 1 to position 2 at $t = 0$. The circuit has attained steady state before switching. Determine i, $\dfrac{di}{dt}$ and $\dfrac{d^2i}{dt^2}$ at $t = 0^+$.

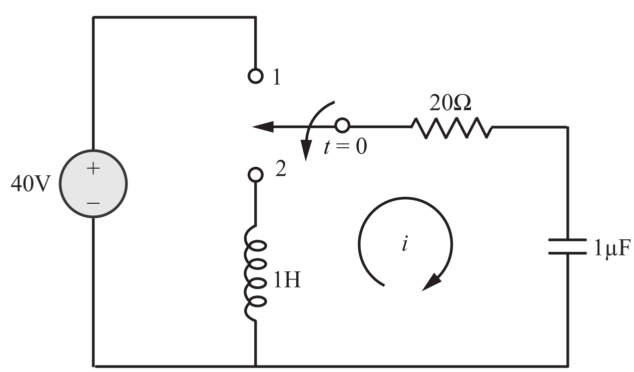


Figure E.P.4.3

$$\textbf{Ans: } i(0^+) = 0, \quad \frac{di(0^+)}{dt} = -40 \text{ A/sec}, \quad \frac{d^2i(0^+)}{dt^2} = 800 \text{ A/sec}^2$$

E.P 4.4

In the network shown in Fig. E.P.4.4, the initial voltage on C_1 is V_a and on C_2 is V_b such that $v_1(0^-) = V_a$ and $v_2(0^-) = V_b$. Find the values of $\frac{dv_1}{dt}$ and $\frac{dv_2}{dt}$ at $t = 0^+$.

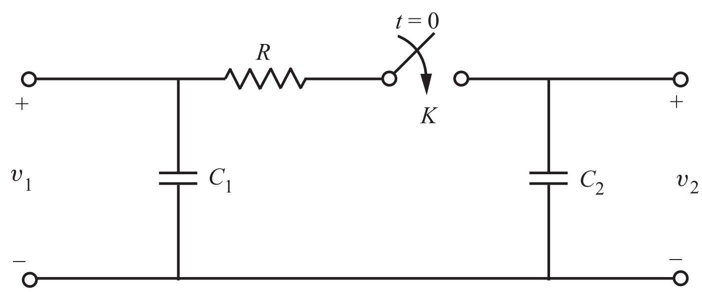


Figure E.P.4.4

Ans: $$\frac{dv_1(0^+)}{dt} = \frac{V_b - V_a}{C_1 R}\,\mathrm{V/sec}, \quad \frac{dv_2(0^+)}{dt} = \frac{V_a - V_b}{C_2 R}\,\mathrm{V/sec}$$

E.P 4.5

In the network shown in Fig E.P. 4.5, switch K is closed at $t = 0$ with zero capacitor voltage and zero inductor current. Find $\frac{d^2 v_2}{dt^2}$ at $t = 0^+$.

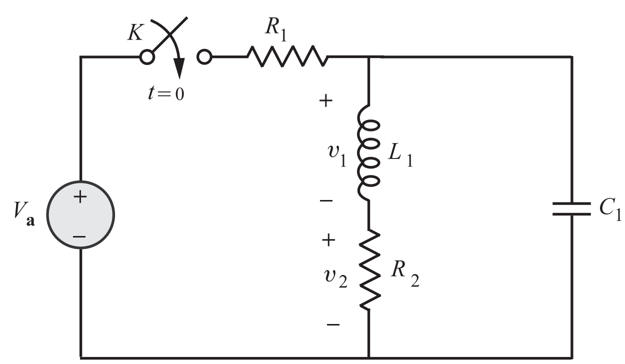


Figure E.P.4.5

Ans: $$\frac{d^2 v_2(0^+)}{dt^2} = \frac{R_2 V_a}{R_1 L_1 C_1}\,\mathrm{V/sec^2}$$

E.P 4.6

In the network shown in Fig. E.P. 4.6, switch K is closed at $t = 0$. Find $\dfrac{d^2v_1}{dt^2}$ at $t = 0^+$.

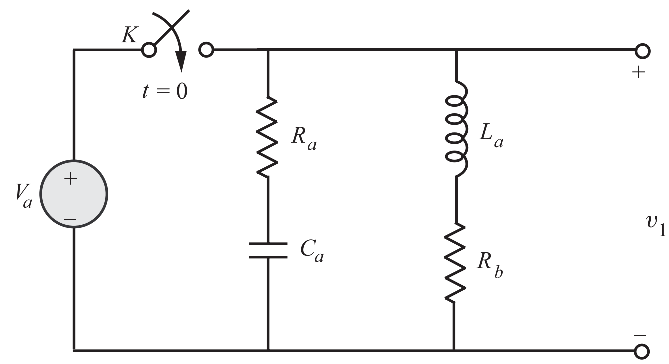


Figure E.P.4.6

Ans: $\dfrac{\boldsymbol{d^2v_1(0^+)}}{\boldsymbol{dt^2}} = \mathbf{0\ V/\,sec^2}$

E.P 4.7

The switch in Fig. E.P. 4.7 has been closed for a long time. It is open at $t = 0$. Find $\dfrac{di(0^+)}{dt}$, $\dfrac{dv(0^+)}{dt}$, $i(\infty)$ and $v(\infty)$.

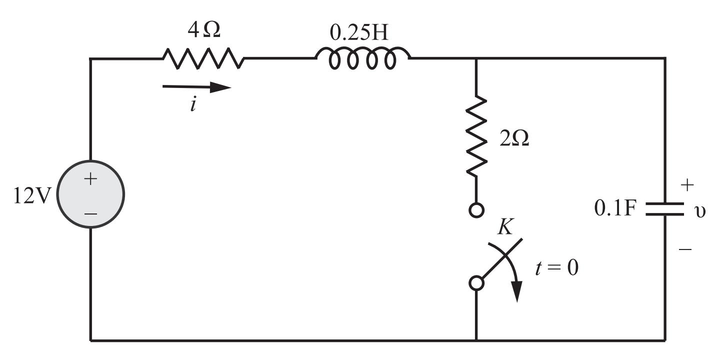


Figure E.P.4.7

Ans: $\dfrac{\boldsymbol{di(0^+)}}{\boldsymbol{dt}} = \mathbf{0A/\,sec}, \quad \dfrac{\boldsymbol{dv(0^+)}}{\boldsymbol{dt}} = \mathbf{20A/\,sec}, \quad \boldsymbol{i(\infty)} = \mathbf{0A}, \quad \boldsymbol{v(\infty)} = \mathbf{12V}$

E.P 4.8

In the circuit of Fig E.P. 4.8, calculate $i_L(0^+)$, $\dfrac{di_L(0^+)}{dt}$, $\dfrac{dv_C(0^+)}{dt}$, $v_R(\infty)$, $v_C(\infty)$ and $i_L(\infty)$.

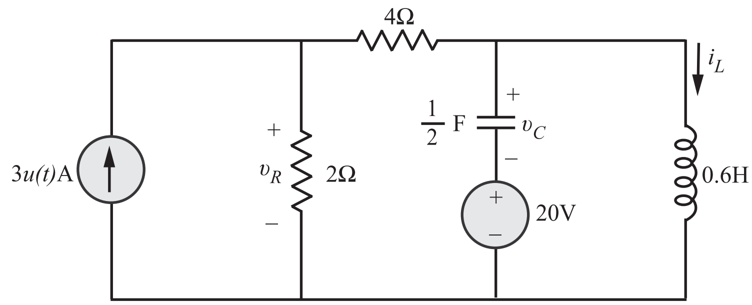


Figure E.P.4.8

Ans: $\boldsymbol{i_L(0^+) = 0\text{ A}, \dfrac{di_L(0^+)}{dt} = 0\text{ A/sec}}$

$$\boldsymbol{\frac{dv_C(0^+)}{dt} = 2\text{ V/sec}, \quad v_R(\infty) = 4\text{V}, \quad v_C(\infty) = -20\text{V}, \quad i_L(\infty) = 1\text{A}}$$

E.P 4.9

Refer the circuit shown in Fig. E.P. 4.9. Assume that the switch was closed for a long time for $t < 0$. Find $\dfrac{di_L(0^+)}{dt}$ and $i_L(0^+)$. Take $v(0^+) = 8$ V.

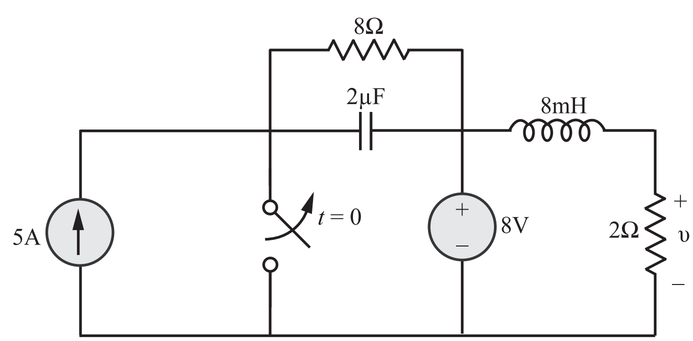


Figure E.P.4.9

Ans: $\boldsymbol{i_L(0^+) = 4\text{ A}, \dfrac{di_L(0^+)}{dt} = 0\text{ A/sec}}$

E.P 4.10

Refer the network shown in Fig. E.P. 4.10. A steady state is reached with the switch K closed and with $i = 10$A. At $t = 0$, switch K is opened. Find $v_2(0^+)$ and $\dfrac{dv_2(0^+)}{dt}$.

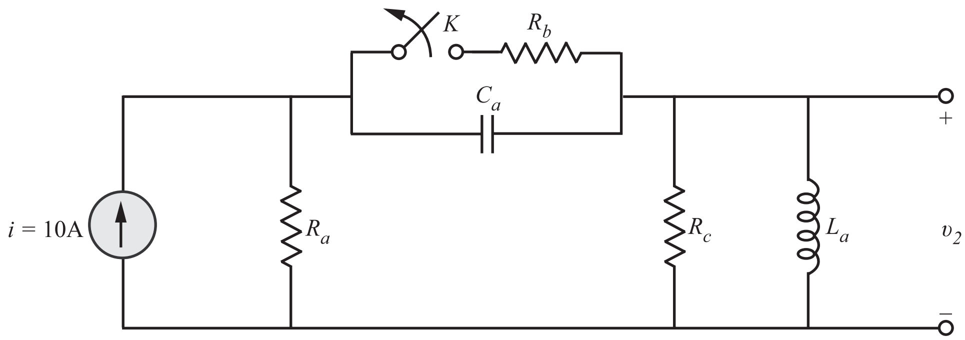


Figure E.P.4.10

Ans: $\boldsymbol{v_2(0^+) = 0,} \qquad \dfrac{\boldsymbol{dv_2(0^+)}}{\boldsymbol{dt}} = \dfrac{\boldsymbol{10R_aR_c}}{\boldsymbol{C_a(R_a+R_b)(R_a+R_c)}}$ V/ sec.

E.P 4.11

Refer the network shown in Fig. E.P. 4.11. The network is in steady state with switch K closed. The switch is opened at $t = 0$. Find $v_k(0^+)$ and $\dfrac{dv_k(0^+)}{dt}$.

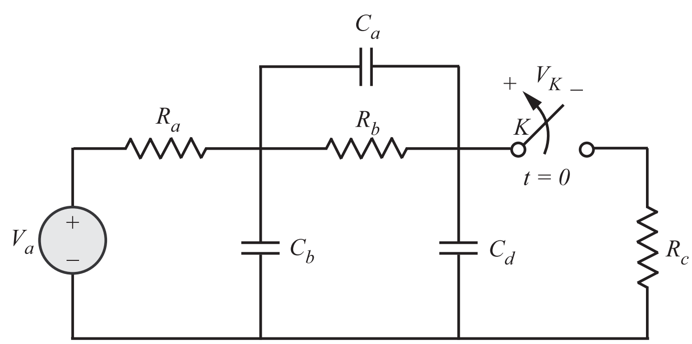


Figure E.P.4.11

Ans: $\boldsymbol{v_k(0^+) = \dfrac{V_aR_c}{R_a+R_b+R_c}}$ **Volts,**

$$\boldsymbol{\frac{dv_k(0^+)}{dt} = \frac{V_a(C_a+C_b)}{(R_a+R_b+R_c)(C_aC_d+C_bC_a+C_bC_d)}}\textbf{V/ sec}$$

E.P 4.12

Refer the network shown in Fig. E.P. 4.12. Find $\dfrac{d^2i_1(0^+)}{dt^2}$.

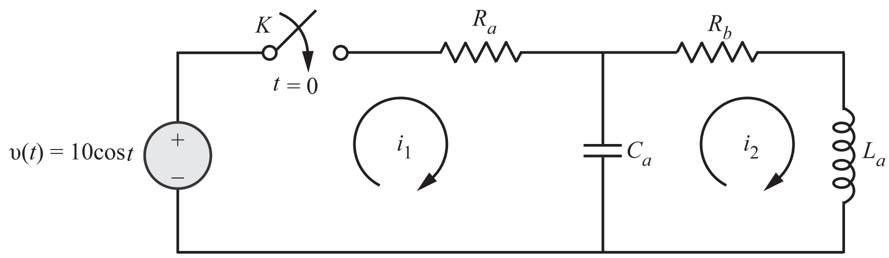


Figure E.P.4.12

Ans: $$\frac{d^2 i_1(0^+)}{dt^2} = \frac{1}{R_a}\left[-10 + \frac{10}{R_a^2 C_a^2}\right] \text{A/sec}^2$$

E.P 4.13

Refer the circuit shown in Fig. E.P. 4.13. Find $\dfrac{di_1(0^+)}{dt}$. Assume that the circuit has attained steady state at $t = 0^-$.

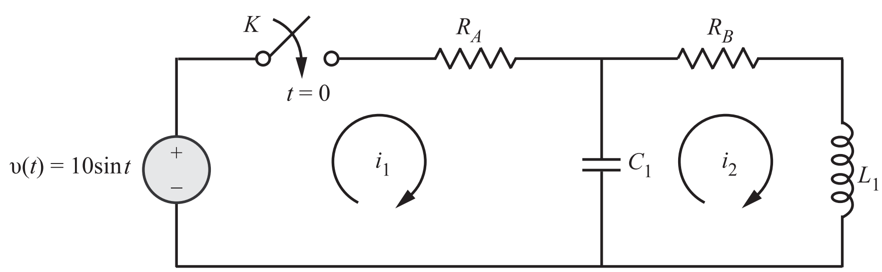


Figure E.P.4.13

Ans: $$\frac{di_1(0^+)}{dt} = \frac{10}{R_A} \text{A/sec}$$

E.P 4.14

Refer the network shown in Fig. E.P.4.14. The circuit reaches steady state with switch K closed. At a new reference time, $t = 0$, the switch K is opened. Find $\dfrac{dv_1(0^+)}{dt}$ and $\dfrac{d^2 v_2(0^+)}{dt^2}$.

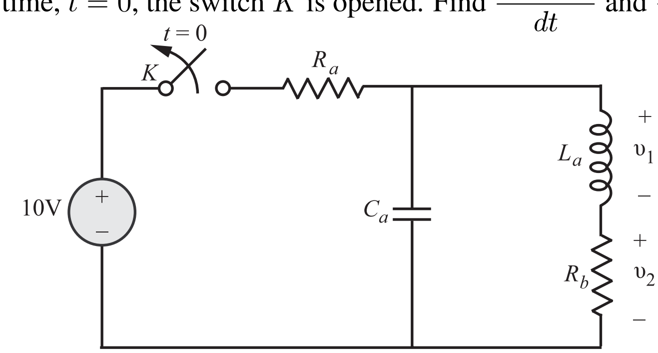


Figure E.P.4.14

Ans: $\boldsymbol{\dfrac{dv_1(0^+)}{dt} = \dfrac{-10}{C_a(R_a+R_b)}\ \text{V/sec}, \quad \dfrac{d^2v_2(0^+)}{dt^2} = \dfrac{-10R_b}{L_aC_a(R_a+R_b)}\ \text{V/sec}^2}$

E.P 4.15

The switch shown in Fig. E.P. 4.15 has been open for a long time before closing at $t = 0$. Find: $i_0(0^-), i_L(0^-)\ i_0(0^+), i_L(0^+), i_0(\infty), i_L(\infty)$ and $v_L(\infty)$.

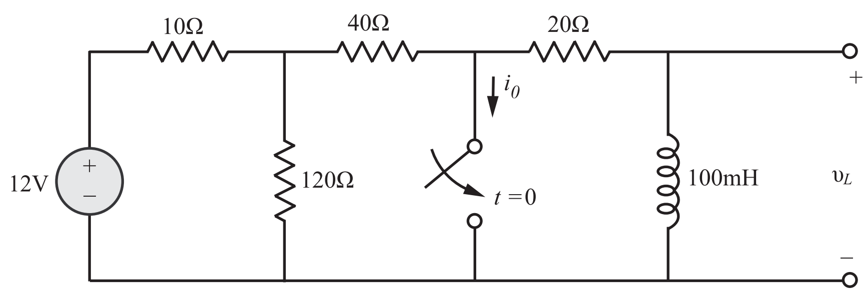


Figure E.P.4.15

Ans: $\boldsymbol{i(0^-) = 0,\ \ i_L(0^-) = 160\text{mA},\ \ i_0(0^+) = 65\text{mA},\ \ i_L(0^+) = 160\text{mA},}$
$\boldsymbol{i_0(\infty) = 225\text{mA}, i_L(\infty) = 0, v_L(\infty) = 0}$

E.P 4.16

The switch shown in Fig. E.P. 4.16 has been closed for a long time before opeing at $t = 0$.
Find: $i_1(0^-), i_2(0^-), i_1(0^+), i_2(0^+)$. Explain why $i_2(0^-) \neq i_2(0^+)$.

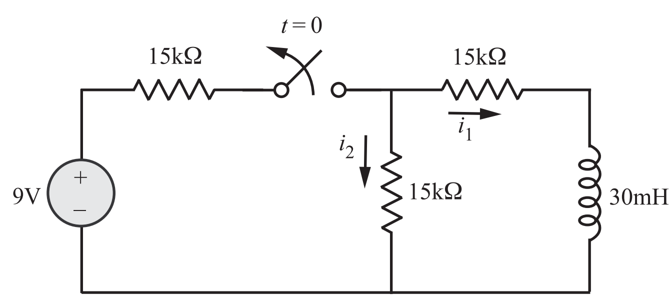


Figure E.P.4.16

Ans: $\boldsymbol{i_1(0^-) = i_2(0^-) = 0.2\text{mA}, i_2(0^+) = -i_1(0^+) = -0.2\text{mA}}$

E.P 4.17

The switch in the circuit of Fig E.P.4.17 is closed at $t = 0$ after being open for a long time. Find:

(a) $i_1(0^-)$ and $i_2(0^-)$

(b) $i_1(0^+)$ and $i_2(0^+)$

(c) Explain why $i_1(0^-) = i_1(0^+)$

(d) Explain why $i_2(0^-) \neq i_2(0^+)$

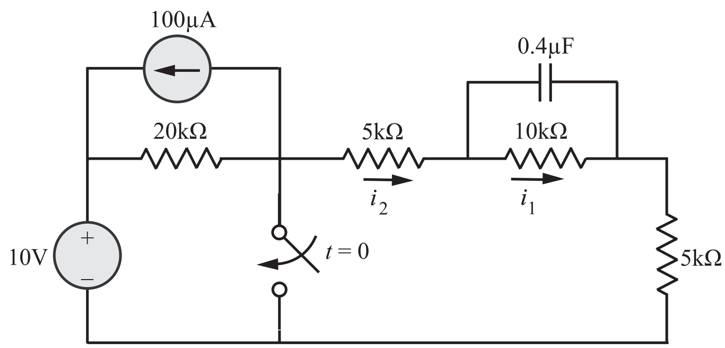


Figure E.P.4.17

Ans: $\boldsymbol{i_1(0^-) = i_2(0^-) = 0.2 \text{ mA},\ i_1(0^+) = 0.2 \text{ mA},\ i_2(0^+) = -0.2\text{mA}}$

Chapter 5 Laplace Transform

5.1 Introduction

In this chapter, we will introduce Laplace transform. This is an extremely important technique. For a given set of initial conditions, it will give the total response of the circuit comprising of both natural and forced responses in one operation. The idea of Laplace transform is analogous to any familiar transform. For example, Logarithms are used to change a multiplication or division problem into a simpler addition or subtraction problem and Antilogs are used to carry out the inverse process. This example points out the essential feature of a transform: They are designed to create a new domain to make mathematical manipulations easier. After evaluating the unknown in the new domain, we use inverse transform to get the evaluated unknown in the original domain. The Laplace transform enables the circuit analyst to convert the set of integrodifferential equations describing a circuit to the complex frequency domain, where thay become a set of linear algebraic equations. Then using algebraic manipulations, one may solve for the variables of interest. Finally, one uses the inverse transform to get the variable of interest in time domain. Also, in this chapter, we express the impedance in s domain or complex frequency domain. Hence, we may analyze a circuit using one of the reduction techniques such as Thevenin theorem or source transformation discussed in earlier chapters.

5.2 Definition of Laplace transorm

A transform is a change in the mathematical description of a physical variable to facilitate computation. Keeping this definition in mind, Laplace transform of a function $f(t)$ is defined as

$$\mathscr{L}\{f(t)\} = F(s) = \int_{0}^{\infty} f(t)e^{-st}dt \tag{5.1}$$

Here the complex frequency is $s = \sigma + j\omega$. Since the argument of the exponent e in equation (5.1) must be dimensionless, it follows that s has the dimensions of frequency and units of inverse seconds (sec^{-1}).

The notation implies that once the integral has been evaluated, $f(t)$, a time domain function is transformed to $F(s)$, a frequency domain function.

If the lower limit of integration in equation (5.1) is $-\infty$, then it is called the bilateral Laplace transform. However for circuit applications, the lower limit is taken as zero and accordingly the transform is unilateral in nature.

The lower limit of integration is sometimes chosen to be 0^- to permit $f(t)$ to include $\delta(t)$ or its derivatives. Thus we should note immediately that the integration from 0^- to 0^+ is zero except when an impulse function or its derivatives are present at the origin.

Region of convergence

The Laplace transform of a signal $f(t)$ as seen from equation (5.1) is an integral operation. It exists if $f(t)e^{-\sigma t}$ is absolutely integrable. That is $\int\limits_0^\infty f(t)e^{-\sigma t}dt < \infty$. Cleary, only typical choices of σ will make the integral converge. The range of σ that ensures the existence of $X(s)$ defines the region of convergence (ROC) of the Laplace transform. As an example, let us take $x(t) = e^{3t}$, $t \geq 0$. Then

$$X(s) = \int\limits_0^\infty x(t)e^{-(\sigma+j\omega)t}dt$$

$$= \int\limits_0^\infty e^{(-\sigma+3)t}e^{-j\omega t}dt$$

The above integral converges if and only if $-\sigma + 3 < 0$ or $\sigma > 3$. Thus, $\sigma > 3$ defines the ROC of $X(s)$. Since, we shall deal only with causal signals($t \geq 0$) we avoid explicit mention of ROC.

Due to the convergence factor, $e^{-\sigma t}$, a number of important functions have Laplace transforms, even though Fourier transforms for these functions do not exist. But this does not mean that every mathematical function has Laplace transform. The reader should be aware that, for example, a function of the form e^{t^2} does not have Laplace transform.

The inverse Laplace transform is defined by the relationship:

$$\mathscr{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi j}\int\limits_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds \qquad (5.2)$$

where σ is real. The evaluation of integral in equation (5.2) is based on complex variable theory, and hence we will avoid its use by developing a set of Laplace transform pairs.

5.3 Three important singularity functions

The three important singularity functions employed in circuit analysis are:

(i) unit step function, $u(t)$
(ii) delta function, $\delta(t)$
(iii) ramp function, $r(t)$.

They are called singularity functions because they are either not finite or they do not possess finite derivatives everywhere.

The mathematical definition of unit step function is

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases} \tag{5.3}$$

The step function is not defined at $t = 0$. Thus, the unit step function $u(t)$ is 0 for negative values of t, and 1 for positive values of t. Often it is advantageous to define the unit step function as follows:

$$u(t) = \begin{cases} 1, & t \geq 0^+ \\ 0, & t \leq 0^- \end{cases}$$

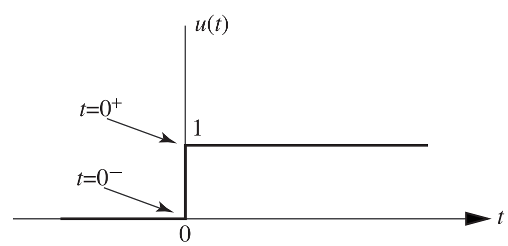


Figure 5.1 The unit step function

A discontinuity may occur at time other than $t = 0$; for example, in sequential switching, the unit step function that occurs at $t = a$ is expressed as $u(t - a)$.

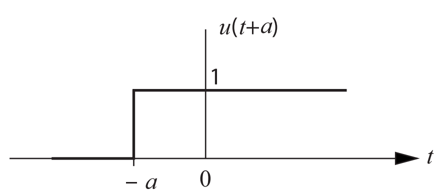


Figure 5.2 The step function occuring at $\boldsymbol{t = a}$

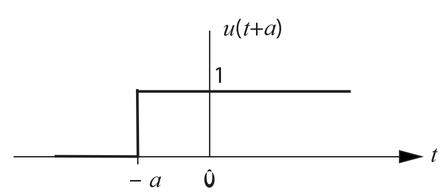


Figure 5.3 The step function occuring at $\boldsymbol{t = -a}$

Thus, $$u(t - a) = \begin{cases} 0, & t - a < 0 \text{ or } t < a \\ 1, & t - a > 0 \text{ or } t > a \end{cases}$$

Similarly, the unit step function that occurs at $t = -a$ is expressed as $u(t + a)$.

Thus, $$u(t + a) = \begin{cases} 0, & t + a < 0 \text{ or } t < -a \\ 1, & t + a > 0 \text{ or } t > -a \end{cases}$$

We use step function to represent an abrupt change in voltage or current, like the changes that occur in the circuits of control engineering and digital systems. For example, the voltage

$$v(t) = \begin{cases} 0, & t < a \\ K, & t > a \end{cases}$$

may be expressed in terms of the unit step function as

$$v(t) = Ku(t-a) \tag{5.4}$$

The derivative of the unit step function $u(t)$ is the unit impulse function $\delta(t)$.

That is,
$$\delta(t) = \frac{d}{dt}u(t) = \begin{cases} 0, & t < 0 \\ \text{undefined}, & t = 0 \\ 0, & t > 0 \end{cases} \tag{5.5}$$

The unit impulse function also known as dirac delta fucntion is shown in Fig. 5.4.

The unit impulse may be visualized as very short duration pulse of unit area. This may be expressed mathematically as:

$$\int_{0^-}^{0^+} \delta(t)dt = 1 \tag{5.6}$$

where $t = 0^-$ denotes the time just before $t = 0$ and $t = 0^+$ denotes the time just after $t = 0$. Since the area under the unit impulse is unity, it is a practice to write '1' beside the arrow that is used to symbolize the unit impulse function as shown in Fig. 5.4. When the impulse has a strength other than unity, the area of the impulse function is equal to its strength. For example, an impulse function $5\delta(t)$ has an area of 5 units. Figure 5.5 shows impulse functions, $2\delta(t+2)$, $5\delta(t)$ and $-2\delta(t-3)$.

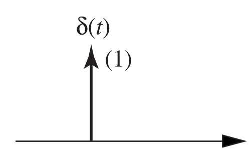


Figure 5.4 The circuit impulse function

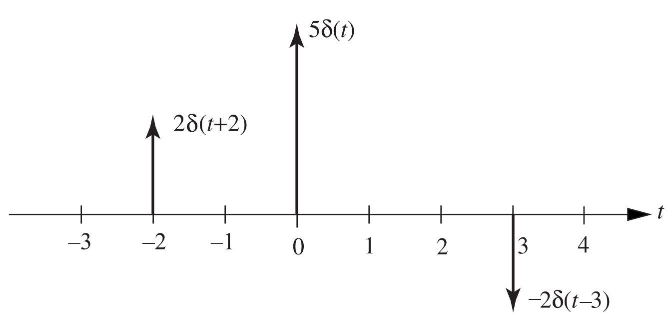


Figure 5.5 Three impulse functions

An important property of the unit impulse function is what is often called the sifting property; which is exhibited by the following integral:

$$\int_{t_1}^{t_2} f(t)\delta(t-t_0)dt = \begin{cases} f(t_0), & t_1 < t_0 < t_2 \\ 0, & t_1 > t_0 > t_2 \end{cases}$$

for a fintie t_0 and any $f(t)$ continuous at t_0.

Integrating the unit step function results in the unit ramp function $r(t)$.

$$r(t) = \int_{-\infty}^{t} u(\tau)d\tau = tu(t) \tag{5.7}$$

$$\text{or} \qquad r(t) = \begin{cases} 0, & t \leq 0 \\ t, & t \geq 0 \end{cases}$$

Figure 5.6 shows the ramp function.

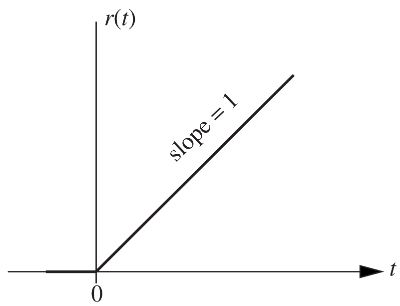


Figure 5.6 The unit ramp function

In general, a ramp is a function that changes at a constant rate.

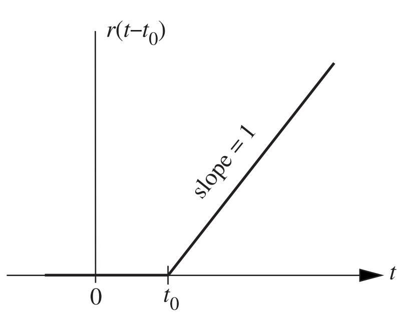


Figure 5.7 The unit ramp function delayed by t_0

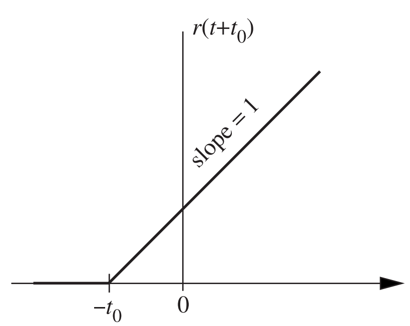


Figure 5.8 The unit ramp function advanced by t_0

A delayed ramp function is shown in Fig. 5.7. Mathematically, it is described as follows:

$$r(t-t_0) = \begin{cases} 0, & t \leq t_0 \\ t-t_0, & t \geq t_0 \end{cases}$$

An advanced ramp function is shown in Fig. 5.8. Mathematically, it is described as follows:

$$r(t+t_0) = \begin{cases} 0, & t \le -t_0 \\ t+t_0, & t \ge -t_0 \end{cases}$$

It is very important to note that the three sigularity functions are related by differentiation as

$$\delta(t) = \frac{du(t)}{dt}, \qquad u(t) = \frac{dr(t)}{dt}$$

or by integration as

$$u(t) = \int_{-\infty}^{t} \delta(t)dt, \quad r(t) = \int_{-\infty}^{t} u(\tau)d\tau$$

5.4 Functional transforms

A functional transform is simply the Laplace transform of a specified function of t. Here we make an assumption that $f(t)$ is zero for $t < 0$.

5.4.1 Decaying exponential function

$f(t) = e^{-at}u(t)$, where $a > 0$ and $u(t)$ is the unit step function.

$$\begin{aligned} \mathcal{L}\left\{e^{-at}u(t)\right\} = F(s) &= \int_0^\infty f(t)dt \\ &= \int_0^\infty e^{-at}e^{-st}dt \\ &= \left.\frac{-e^{-(s+a)t}}{(s+a)}\right|_{t=0}^{\infty} \\ &= \frac{1}{s+a} \end{aligned}$$

5.4.2 Unit step function

$$f(t) = u(t)$$

$$\mathcal{L}\{u(t)\} = F(s) = \int_0^\infty e^{-st}dt = \frac{1}{s}$$

5.4.3 Impulse function

$$f(t) = \delta(t)$$
$$\mathscr{L}\{\delta(t)\} = F(s) = \int_{0^-}^{\infty} \delta(t)e^{-st}dt = \left. e^{-st}\right|_{t=0} = 1$$

Please note that we have used the sifting property of an impulse function.

5.4.4 Sinusoidal function

$$f(t) = \sin \omega t, \qquad t \geq 0$$

Since
$$\sin \omega t = \frac{1}{2j}\left[e^{j\omega t} - e^{-j\omega t}\right]$$
and
$$\mathscr{L}\{e^{-at}\} = \frac{1}{s+a}$$
we have
$$\begin{aligned}\mathscr{L}\{\sin \omega t\} = F(s) &= \frac{1}{2j}\int_{0}^{\infty} \left(e^{j\omega t} - e^{-j\omega t}\right) e^{-st}dt \\ &= \frac{1}{2j}\left[\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right] \\ &= \frac{\omega}{s^2+\omega^2}\end{aligned}$$

Table 5.1 gives a list of important Laplace transform pairs. It includes the functions of most interest in an introductory course on circuit applications.

Table 5.1 Important transform pairs

$f(t)(t \geq 0)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
t	$\frac{1}{s^2}$
e^{-at}	$\frac{1}{s+a}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$

$f(t) (t \geq 0)$	$F(s)$
t^n	$\dfrac{n!}{s^{n+1}}$
te^{-at}	$\dfrac{1}{(s+a)^2}$
$e^{-at}\sin\omega t$	$\dfrac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos\omega t$	$\dfrac{s+a}{(s+a)^2+\omega^2}$

All functions in the above table are represented without multiplied by $u(t)$, since we have explicity declared that $t \geq 0$.

5.5 Operational transforms (properties of Laplace transform)

Operational transforms indicate how mathematical operations performed on either $f(t)$ or $F(s)$ are converted into the opposite domain. Following operations are of primary interest.

***Note*:** The symbol $\triangleq$ means **"by the definition"**.

5.5.1 Linearity

If $$\mathscr{L}\{f_1(t)\} = F_1(s) \text{ and } \mathscr{L}\{f_2(t)\} = F_2(s)$$

then $$\mathscr{L}\{a_1 f_1(t) + a_2 f_2(t)\} = a_1 F_1(s) + a_2 F_2(s)$$

***Proof*:**

$$\begin{aligned}\mathscr{L}\{a_1 f_1(t) + a_2 f_2(t)\} &\triangleq \int_0^\infty [a_1 f_1(t) + a_2 f_2(t)] e^{-st} dt \\ &= a_1 \int_0^\infty f_1(t) e^{-st} dt + a_2 \int_0^\infty f_2(t) e^{-st} dt \\ &= a_1 F_1(s) + a_2 F_2(s)\end{aligned}$$

EXAMPLE 5.1

Find the Laplace transform of $f(t) = (A + Be^{-bt}u(t))$.

SOLUTION

We have the transform pair

$$\mathscr{L}\{u(t)\} = \frac{1}{s}$$

and $$\mathscr{L}\{e^{-bt}u(t)\} = \frac{1}{s+b}$$

Thus, using linearity property,

$$\begin{aligned}\mathscr{L}\{f(t)\} = F(s) &= \mathscr{L}\{Au(t)\} + \mathscr{L}\{Be^{-bt}u(t)\}\\ &= \frac{A}{s} + \frac{B}{s+b}\\ &= \boldsymbol{\frac{(A+B)s + Ab}{s(s+b)}}\end{aligned}$$

5.5.2 Time shifting

If $\mathscr{L}\{x(t)\} = X(s)$, then for any real number t_0,

$$\mathscr{L}\{x(t-t_0)u(t-t_0)\} = e^{-t_0 s}X(s)$$

***Proof*:**

$$\mathscr{L}\{x(t-t_0)u(t-t_0)\} \triangleq \int_0^\infty x(t-t_0)u(t-t_0)e^{-st}dt$$

Since,
$$u(t-t_0) = \begin{cases} 1, & t - t_0 > 0 \text{ or } t > t_0 \\ 0, & t - t_0 < 0 \text{ or } t < t_0 \end{cases}$$

we get,
$$\mathscr{L}\{x(t-t_0)u(t-t_0)\} = \int_{t_0}^\infty x(t-t_0)e^{-st}dt$$

Using the transformation of variable,

$$t = \tau + t_0$$

we get,
$$\begin{aligned}\mathscr{L}\{x(t-t_0)u(t-t_0)\} &= \int_0^\infty x(\tau)e^{-s(\tau+t_0)}d\tau\\ &= e^{-t_0 s}\int_0^\infty x(\tau)e^{-s\tau}d\tau\\ &= e^{-t_0 s}X(s)\end{aligned}$$

EXAMPLE 5.2

Find the Laplace transform of $x(t)$, shown in Fig. 5.9.

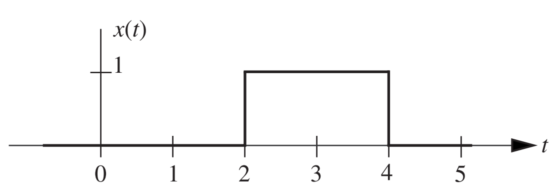


Figure 5.9

SOLUTION

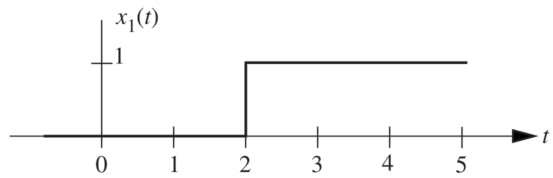


Figure 5.10(a)

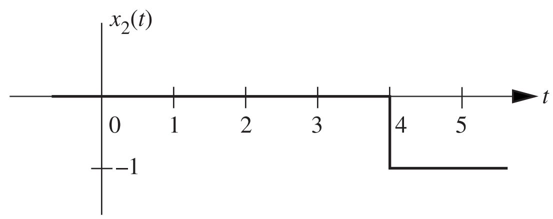


Figure 5.10(b)

Using Figs. 5.10(a) and 5.10(b), we can write

$$x(t) = x_1(t) + x_2(t) = u(t-2) - u(t-4)$$

We know that, $\mathscr{L}\{u(t)\} = \frac{1}{s}$ and using time shifting property, we have

$$\mathscr{L}\{x(t)\} = X(s) = \frac{1}{s}e^{-2s} - \frac{1}{s}e^{-4s}$$

$$\Rightarrow \qquad \boldsymbol{X(s) = \frac{1}{s}(e^{-2s} - e^{-4s})}$$

5.5.3 Shifting in s domain (Frequency-domain shifting)

If $\mathscr{L}\{x(t)\} = X(s)$, then

$$\mathscr{L}\{e^{s_0 t}x(t)\} = X(s - s_0)$$

***Proof*:**

$$\begin{aligned}\mathscr{L}\{e^{s_0 t}x(t)\} &\triangleq \int_0^\infty e^{s_0 t}x(t)e^{-st}dt \\ &= \int_0^\infty x(t)e^{-(s-s_0)t}dt \\ &= X(s - s_0)\end{aligned}$$

EXAMPLE 5.3

Find the Laplace transform of $x(t) = Ae^{-at}\cos(\omega_0 t + \theta)u(t)$.

SOLUTION

Given

$$\begin{aligned} x(t) &= Ae^{-at}\cos(\omega_0 t + \theta)u(t) \\ &= Ae^{-at}[\cos\omega_0 t\cos\theta - \sin\omega_0 t\sin\theta]u(t) \\ &= A\cos\theta e^{-at}\cos\omega_0 tu(t) - A\sin\theta e^{-at}\sin\omega_0 tu(t) \end{aligned}$$

We know the transform pairs,

$$\mathscr{L}\{\cos\omega_0 tu(t)\} = \frac{s}{s^2+\omega_0^2}$$

and

$$\mathscr{L}\{\sin\omega_0 tu(t)\} = \frac{\omega_0}{s^2+\omega_0^2}$$

Applying frequency shifting property, we get

$$\begin{aligned} \mathscr{L}\left\{e^{-at}\cos\omega_0 tu(t)\right\} &= \left.\frac{s}{s^2+\omega_0^2}\right|_{s\to s+a} \\ &= \frac{s+a}{(s+a)^2+\omega_0^2} \end{aligned}$$

and

$$\begin{aligned} \mathscr{L}\left\{e^{-at}\sin\omega_0 tu(t)\right\} &= \left.\frac{\omega_0}{s^2+\omega_0^2}\right|_{s\to s+a} \\ &= \frac{\omega_0}{(s+a)^2+\omega_0^2} \end{aligned}$$

Finally, applying linearity property, we get

$$\begin{aligned} \mathscr{L}\{Ae^{-at}\cos(\omega_0 t+\theta)u(t)\} &= A\cos\theta\,\mathscr{L}\{e^{-at}\cos\omega_0 tu(t)\} - A\sin\theta\,\mathscr{L}\{e^{-at}\sin\omega_0 tu(t)\} \\ &= \frac{A\cos\theta(s+a)}{(s+a)^2+\omega_0^2} - A\sin\theta\frac{\omega_0}{(s+a)^2+\omega_0^2} \\ &= \boldsymbol{\frac{A[(s+a)\cos\theta - \omega_0\sin\theta]}{(s+a)^2+\omega_0^2}} \end{aligned}$$

5.5.4 Time scaling

If $\mathscr{L}\{x(t)\} = X(s)$, then

$$\mathscr{L}\{x(at)\} = \frac{1}{a}X\left(\frac{s}{a}\right)$$

***Proof*:**

$$\mathscr{L}\{x(at)\} \triangleq \int_0^\infty x(at)e^{-st}dt$$

put $$at = \tau$$

$$\Rightarrow \quad adt = d\tau$$

Hence $$\mathscr{L}\{x(at)\} = \int_0^\infty x(\tau)e^{-s\frac{\tau}{a}}\frac{1}{a}d\tau$$

$$= \frac{1}{a}\int_0^\infty x(\tau)e^{-\frac{s}{a}\tau}d\tau = \frac{1}{a}X\left(\frac{s}{a}\right)$$

EXAMPLE 5.4

Find the Laplace transform of $x(t) = \sin(2\omega_0 t)u(t)$.

SOLUTION

We know the transform pair,

$$\mathscr{L}\{\sin\omega_0 tu(t)\} = \frac{\omega_0}{s^2+\omega_0^2}$$

Applying scaling property,

$$\mathscr{L}\{\sin 2\omega_0 tu(t)\} = \frac{1}{2}\left[\frac{\omega_0}{\left(\frac{s}{2}\right)^2+\omega_0^2}\right]$$

$$= \boldsymbol{\frac{2\omega_0}{s^2+4\omega_0^2}}$$

5.5.5 Time differentiation

If $\mathscr{L}\{x(t)\} = X(s)$, then

$$\mathscr{L}\left\{\frac{dx(t)}{dt}\right\} = sX(s) - x(0)$$

***Proof*:**

Let $$y(t) = \frac{dx(t)}{dt}$$

Then $$\mathscr{L}\{y(t)\} = Y(s) \triangleq \int_0^\infty y(t)e^{-st}dt$$

$$= \int_0^\infty \frac{dx(t)}{dt}e^{-st}dt$$

Integrating by parts yields

$$Y(s) = e^{-st}x(t)\Big|_0^\infty - \int_0^\infty x(t)(-se^{-st})dt$$

$$= \lim_{t\to\infty}[e^{-st}x(t)] - x(0) + s\int_0^\infty x(t)e^{-st}dt$$

$$= 0 - x(0) + sX(s)$$

Hence, $$\mathscr{L}\left\{\frac{dx(t)}{dt}\right\} = Y(s) = sX(s) - x(0)$$

Therefore, differentiation in time domain is equivalent to multiplication by s in the s domain.

Whenever $x(t)$ is discontinuous at $t = 0$ (like a step function), then $x(0)$ should be read as $x(0^-)$.

The differentiation property can be extended to yield

$$\mathscr{L}\left\{\frac{d^n x(t)}{dt^n}\right\} = s^n X(s) - s^{n-1}x(0)\cdots - x^{n-1}(0)$$

When $x(t)$ is discontinuous at the origin, the argument 0 on the right side of the above equation should be read as 0^-. Accordingly for a discontinuous function $x(t)$ at the origin, we get

$$\mathscr{L}\left\{\frac{d^n x(t)}{dt^n}\right\} = s^n X(s) - s^{n-1}x(0^-)\cdots - x^{n-1}(0^-)$$

EXAMPLE 5.5

Find the Laplace transform of $x(t) = \sin^2\omega_0 t u(t)$.

SOLUTION

We find that, $x(0) = 0$

$$\frac{dx(t)}{dt} = 2\omega_0 \sin\omega_0 t\cos\omega_0 t u(t)$$

$$= \omega_0 \sin 2\omega_0 t u(t) \tag{5.8}$$

We know that, $$\mathscr{L}\{\sin\omega_0 t u(t)\} = \frac{\omega_0}{s^2+\omega_0^2}$$

Applying time scaling property,

$$\mathscr{L}\{\sin 2\omega_0 t u(t)\} = \frac{1}{2}\left[\frac{\omega_0}{\left(\frac{s}{2}\right)^2 + \omega_0^2}\right]$$
$$= \frac{2\omega_0}{s^2 + 2(\omega_0)^2}$$

Taking Laplace transform on both the sides of equation (5.8), we get

$$\mathscr{L}\left\{\frac{dx(t)}{dt}\right\} = \omega_0\, \mathscr{L}\{\sin \omega_0 t u(t)\}$$
$$\Rightarrow \qquad sX(s) - x(0) = \frac{2\omega_0^2}{s^2 + (2\omega_0)^2}$$
$$\Rightarrow \qquad \boldsymbol{X(s) = \frac{2\omega_0^2}{s[s^2 + (2\omega_0)^2]}}$$

EXAMPLE 5.6

Solve the second order linear differential equation

$$y''(t) + 5y'(t) + 6y(t) = x(t)$$

with the initial conditions, $y(0) = 2$, $y'(0) = 1$ and $x(t) = e^{-t}u(t)$.

SOLUTION

Taking Laplace transform on both the sides of the given differential equation, we get

$$\left|s^2Y(s) - sy(0) - y'(0)\right| + 5\left|sY(s) - y(0)\right| + 6Y(s) = X(s)$$

where $$X(s) = \mathscr{L}\{e^{-t}u(t)\} = \frac{1}{s+1}$$

Substituting the initial conditions, we get

$$(s^2 + 5s + 6)Y(s) = \frac{1}{s+1} + 2s + 11$$
$$\Rightarrow \qquad Y(s) = \frac{2s^2 + 13s + 12}{(s+1)(s+2)(s+3)}$$

Using partial fraction expansion, we get

$$Y(s) = \frac{1}{2}\left[\frac{1}{s+1}\right] + 6\left[\frac{1}{s+2}\right] - \frac{9}{2}\left[\frac{1}{s+3}\right]$$

Taking inverse Laplace transform, we get

$$\boldsymbol{y(t) = \frac{1}{2}e^{-t} + 6e^{-2t} - \frac{9}{2}e^{-3t}, \qquad t \geq 0}$$

5.5.6 Integration in time domain

For a causal signal $x(t)$,

If
$$y(t) = \int_0^t x(\tau)d\tau,$$

then
$$\mathscr{L}\{y(t)\} = Y(s) = \frac{X(s)}{s}$$

***Proof*:**

$$\boldsymbol{\mathscr{L}\{x(t)\} = X(s) \triangleq \int_0^\infty x(t)e^{-st}dt}$$

Dividing both sides by s yields

$$\frac{X(s)}{s} = \int_0^\infty x(t)\frac{e^{-st}}{s}dt$$

Integrating the right-hand side by parts, we get

$$\frac{X(s)}{s} = \frac{e^{-ts}}{s}y(t)\bigg|_{t=0}^{\infty} - \int_0^\infty y(t)\frac{e^{-ts}}{s}(-s)dt$$

$$\frac{X(s)}{s} = y(t)\ \frac{e^{-st}}{s}\bigg|_{t=0}^{\infty} + \int_0^\infty y(t)e^{-st}dt$$

The first term on the right-hand side evaluates to zero at both limits, because

$$e^{-\infty} = 0 \text{ and } y(0) = \int_0^0 x(\tau)d\tau = 0$$

Hence,
$$Y(s) = \frac{X(s)}{s}$$

Thus, integration in time domain is equivalent to division by s in the s domain.

EXAMPLE 5.7

Consider the RC circuit shown in Fig. 5.11. The input is the rectangular pulse shown in Fig. 5.12. Find $i(t)$ by assuming circuit is initially relaxed.

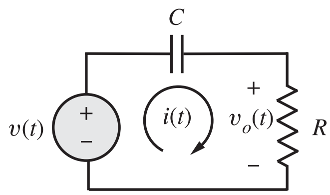


Figure 5.11

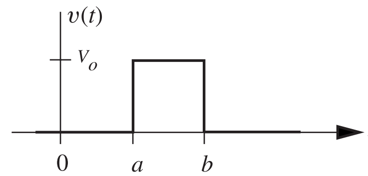


Figure 5.12

SOLUTION

Applying *KVL* to the circuit represented by Fig. 5.11, we get

$$Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = v(t)$$

$$\Rightarrow \qquad Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = V_o[u(t-a) - u(t-b)]$$

Taking Laplace transforms on both the sides, we get

$$R\mathbf{I}(s) + \frac{1}{Cs}\mathbf{I}(s) = \frac{V_o}{s}(e^{-as} - e^{-bs})$$

$$\Rightarrow \qquad \mathbf{I}(s) = \frac{\frac{V_o}{R}}{s + \frac{1}{RC}}(e^{-as} - e^{-bs})$$

We know the transform pair,

$$\mathcal{L}\{e^{-at}u(t)\} = \frac{1}{s+a}$$

and then using the time-shift property, we can find inverse of $\mathbf{I}(s)$.

That is,
$$i(t) = \frac{V_o}{R}e^{-\frac{t}{RC}}u(t)\bigg|_{t\to t-a} - \frac{v_o}{R}e^{-\frac{t}{RC}}u(t)\bigg|_{t\to t-b}$$

$$\Rightarrow \quad \boldsymbol{i(t) = \frac{V_o}{R}\left[e^{-\frac{(t-a)}{RC}}u(t-a) - e^{-\frac{(t-b)}{RC}}u(t-b)\right]}$$

5.5.7 Differentiation in the s domain

For a signal $x(t)$, $t \geq 0$, we have

$$\mathcal{L}\{-tx(t)\} = \frac{dX(s)}{ds}$$

***Proof*:**
For a causal signal, $x(t)$, the Laplace transform is given by

$$\mathscr{L}\{x(t)\} = X(s) = \int_0^\infty x(t)e^{-st}dt$$

Differentiating both the sides with respect to s, we get

$$\frac{dX(s)}{ds} = \int_0^\infty x(t)(-te^{-st})dt$$

$$\Rightarrow \qquad \frac{dX(s)}{ds} = \int_0^\infty [-tx(t)]e^{-st}dt$$

Hence, $$\mathscr{L}\{-tx(t)\} = \frac{dX(s)}{ds} \quad \text{or} \quad \mathscr{L}\{tx(t)\} = \frac{-dX(s)}{ds}$$

In general, $$\mathscr{L}\{t^n x(t)\} = (-1)^n \frac{d^n X(s)}{ds^n}$$

EXAMPLE 5.8

Find the Laplace transform of $x_1(t) = te^{-3t}u(t)$.

SOLUTION

We know that,

$$\mathscr{L}\{e^{-at}u(t)\} = \frac{1}{s+a}$$

Hence $$\mathscr{L}\{e^{-3t}u(t)\} = \frac{1}{s+3}$$

Using the differentiation in s domain property,

$$\mathscr{L}\{x_1(t)\} = X_1(s) = \frac{-d}{ds}\left[\frac{1}{s+3}\right]$$

$$= \boldsymbol{\frac{1}{(s+3)^2}}$$

5.5.8 Convolution

If $$\mathscr{L}\{x(t)\} = X(s)$$

and $$\mathscr{L}\{h(t)\} = H(s)$$

then $$\mathscr{L}\{x(t) * h(t)\} = X(s)H(s)$$

where $*$ indicates the convolution operator.

***Proof*:**

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Since $x(t)$ and $h(t)$ are causal signals, the convolution in this case reduces to

$$x(t) * h(t) = \int_{0}^{\infty} x(\tau)h(t-\tau)d\tau$$

Hence, $$\mathscr{L}\{x(t) * h(t)\} = \int_{0}^{\infty} \left[\int_{0}^{\infty} x(\tau)h(t-\tau)d\tau\right] e^{-st}dt$$

Interchanging the order of integrals, we get

$$\mathscr{L}\{x(t) * h(t)\} = \int_{0}^{\infty} x(\tau) \left[\int_{0}^{\infty} h(t-\tau)e^{-st}dt\right] d\tau$$

Using the change of variable $\lambda = t - \tau$ in the inner integral, we get

$$\begin{aligned}\mathscr{L}\{x(t) * h(t)\} &= \int_{0}^{\infty} x(\tau)e^{-s\tau} \left[\int_{0}^{\infty} h(\lambda)e^{-s\lambda}d\lambda\right] d\tau \\ &= X(s)H(s)\end{aligned}$$

Please note that this theorem reduces the complexity of evaluating the convolution integral to a simple multiplication.

EXAMPLE 5.9

Find the convolution of $h(t) = e^{-t}$ and $f(t) = e^{-2t}$.

SOLUTION

$$\begin{aligned}h(t) * f(t) &= \mathscr{L}^{-1}\{H_1(s)F(s)\} \\ &= \mathscr{L}^{-1}\left\{\left(\frac{1}{s+1}\right)\left(\frac{1}{s+2}\right)\right\} \\ &= \mathscr{L}^{-1}\left\{\frac{1}{s+1} + \frac{-1}{s+2}\right\} \\ &= \boldsymbol{e^{-t} - e^{-2t}, \quad t \geq 0}\end{aligned}$$

EXAMPLE * 5.10

Find the convolution of two indentical rectangular pulses. Each rectangular pulse has unit amplitude and duration equal to $2T$ seconds. Also, the pulse is centered at $t = T$.

SOLUTION

Let the pulse be as shown in Fig. 5.13.
From the Fig. 5.13, we can write

$$x(t) = u(t) - u(t - 2T)$$

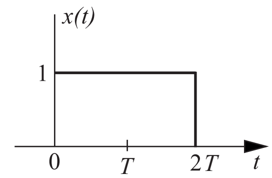


Figure 5.13

Taking Laplace transform, we get

$$X(s) = \frac{1}{s} - \frac{1}{s}e^{-2Ts}$$
$$= \frac{1}{s}(1 - e^{-2Ts})$$

Let $\quad y(t) = x(t) * x(t)$

Then, $\quad Y(s) = X^2(s)$

$$= \left[\frac{1 - e^{-2Ts}}{s}\right]^2$$

$$\Rightarrow \quad Y(s) = \frac{1}{s^2} - \frac{2}{s^2}e^{-2Ts} + \frac{1}{s^2}e^{-4Ts}$$

Taking inverse Laplace transform, we get

$$y(t) = tu(t) - 2(t - 2T)u(t - 2T) + (t - 4T)u(t - 4T)$$
$$= r(t) - 2r(t - 2T) + r(t - 4T)$$

$$\boldsymbol{y(t) = x(t) * x(t)}$$

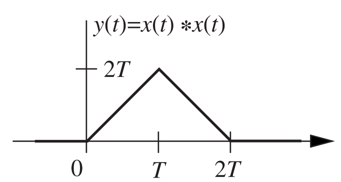


Figure 5.14

5.5.9 Initial-value theorem

The initial-value theorem allows us to find the initial value $x(0)$ directly from its Laplace transform $X(s)$.

If $x(t)$ is a causal signal,

then, $$x(0) = \lim_{s \to \infty} sX(s) \tag{5.9}$$

***Proof*:**
To prove this theorem, we use the time differentiation property.

$$\mathscr{L}\left\{\frac{dx(t)}{dt}\right\} = sX(s) - x(0) = \int_0^\infty \frac{dx}{dt}e^{-st}dt \tag{5.10}$$

---

* The problems with $*$ are better understood after the inverse Laplace transforms are studied.

If we let $s \to \infty$, then the integral on the right side of equation (5.10) vanishes due to damping factor, e^{-st}.

Thus,
$$\lim_{s\to\infty}[sX(s) - x(0)] = 0$$
$$\Rightarrow \qquad x(0) = \lim_{s\to\infty} sX(s)$$

EXAMPLE 5.11

Find the initial value of
$$F(s) = \frac{s+1}{(s+1)^2 + 3^2}$$

SOLUTION

$$f(0) = \lim_{s\to\infty} sX(s) = \lim_{s\to\infty} s\left[\frac{s+1}{(s+1)^2+3^2}\right]$$
$$= \lim_{s\to\infty}\frac{s^2+s}{(s+1)^2+3^2}$$
$$= \lim_{s\to\infty}\frac{s^2\left[1+\frac{1}{s}\right]}{s^2\left[1+\frac{2}{s}+\frac{10}{s^2}\right]} = 1$$

We know the transform pair:
$$\mathscr{L}\{e^{-bt}\cos at\} = \frac{s+b}{(s+b)^2+a^2}$$

Hence, inverse Laplace transform of $F(s)$ yields
$$\boldsymbol{f(t) = e^{-t}\cos 3t}$$

At $t = 0$, we get $f(0) = 1$.
This verifies the theorem.

5.5.10 Final-value theorem

The final-value theorem allows us to find the final value $x(\infty)$ directly from its Laplace transform $X(s)$.

If $x(t)$ is a causal signal,

then
$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

***Proof*:**

The Laplace transform of $\dfrac{dx(t)}{dt}$ is given by
$$sX(s) - x(0) = \int_0^\infty \frac{dx(t)}{dt}e^{-st}dt$$

Taking the limit $s \to 0$ on both the sides, we get

$$\begin{aligned}
\lim_{s\to 0}[sX(s) - x(0)] &= \lim_{s\to 0}\int_0^\infty \frac{dx(t)}{dt}e^{-st}dt \\
&= \int_0^\infty \frac{dx(t)}{dt}\left[\lim_{s\to 0} e^{-st}\right]dt \\
&= \int_0^\infty \frac{dx(t)}{dt}dt \\
&= x(t)|_0^\infty \\
&= x(\infty) - x(0)
\end{aligned}$$

Since, $$\lim_{s\to 0}[sX(s) - x(0)] = \lim_{s\to 0}[sX(s)] - x(0)$$

we get, $$x(\infty) - x(0) = \lim_{s\to 0}[sX(s) - x(0)]$$

Hence, $$x(\infty) = \lim_{s\to 0}[sX(s)]$$

This proves the final value theorem.

The final value theorem may be applied if, and only if, all the poles* of $X(s)$ have a real part that is negative.

The final value theorem is very useful since we can find $x(\infty)$ from $X(s)$. However, one must be careful in using final value theorem since the function $x(t)$ may not have a final value as $t \to \infty$. For example, consider $x(t) = \sin at$ having $X(s) = \frac{a}{s^2 + a^2}$. Now we know $\lim_{t\to\infty} \sin at$ does not exit. However, if we uncarefully use the final value theorem in this case, we would obtain:

$$\lim_{s\to 0} sX(s) = \lim_{s\to 0} s\frac{a}{s^2 + a^2} = 0$$

Note that the actual function $x(t)$ does not have a limiting value as $t \to \infty$. The final value theorem has failed because the poles of $X(s)$ lie on the $j\omega$ axis. Therefore, we conclude that for final value theorem to give a valid result, poles of $X(s)$ should not lie to right side of the s-plane or on the $j\omega$ axis.

EXAMPLE 5.12

Find the final value of

$$X(s) = \frac{10}{(s+1)^2 + 10^2}$$

---

* Consider a function, $X(s) = \frac{P(s)}{Q(s)}$. The roots of the denomoniator polymial, $Q(s)$ are called poles ($\times$) and the roots of the numerator polynomial, $P(s)$ are called zeros (O).

SOLUTION

$$\lim_{t\to\infty} x(t) = x(\infty)$$
$$= \lim_{s\to 0}[sX(s)] = \lim_{s\to 0} \frac{s10}{(s+1)^2+10^2} = \mathbf{0}$$

We know the Laplace transform pair

$$\mathscr{L}\left[e^{-at}\sin bt\right] = \frac{b}{(s+a)^2+b^2}$$

Hence, $$x(t) = \mathscr{L}^{-1}\{X(s)\}$$
$$= \mathscr{L}^{-1}\left\{\frac{10}{(s+1)^2+10^2}\right\} = e^{-t}\sin 10t$$

Thus, $$x(\infty) = 0$$

This verifies the result obtained from final-value theorem.

5.5.11 Time periodicity

Let us consider a function $x(t)$ that is periodic as shown in Fig. 5.15. The function $x(t)$ can be represented as the sum of time-shifted functions as shown in Fig. 5.16.

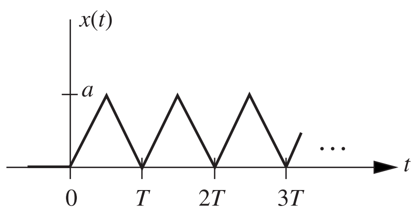


Figure 5.15 A periodic function

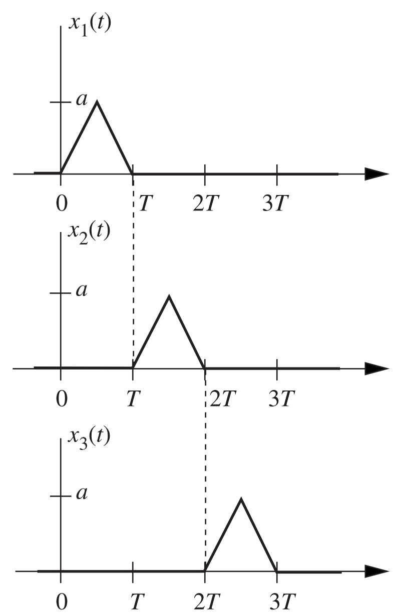


Figure 5.16 Decomposition of periodic function

Hence, $$x(t) = x_1(t) + x_2(t) + x_3(t) + \cdots$$
$$= x_1(t) + x_1(t-T)u(t-T) + x_1(t-2T)u(t-2T) + \cdots \tag{5.11}$$

where $x_1(t)$ is the waveform described over the first period of $x(t)$. That is, $x_1(t)$ is the same as the function $x(t)$ gated* over the interval $0 < t < T$.

*gating means the function $x(t)$ is multiplied by 1 over the interval $0 \le t \le T$ and elsewhere by 0.

Taking the Laplace transform on both sides of equation (5.11) with the time-shift property applied, we get

$$X(s) = X_1(s) + X_1(s)e^{-Ts} + X_1(s)e^{-2Ts} + \cdots$$

$$\Rightarrow \qquad X(s) = X_1(s)(1 + e^{-Ts} + e^{-2Ts} + \cdots)$$

But $$1 + a + a^2 + \cdots = \frac{1}{1-a}, \qquad |a| < 1$$

Hence, we get $$X(s) = X_1(s)\left[\frac{1}{1-e^{-Ts}}\right] \tag{5.12}$$

In equation (5.12), $X_1(s)$ is the Laplace transform of $x(t)$ defined over first period only. Hence, we have shown that the Laplace transform of a periodic function is the Laplace transform evaluated over its first period divided by $1 - e^{-Ts}$.

EXAMPLE 5.13

Find the Laplace transform of the periodic signal $x(t)$ shown in Fig. 5.17.

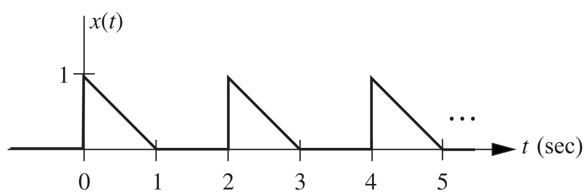


Figure 5.17

SOLUTION

From Fig. 5.17, we find that $T = 2$ Seconds.

The signal $x(t)$ considered over one period is donoted as $x_1(t)$ and shown in Fig. 5.18(a).

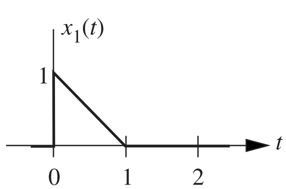


Figure 5.18(a)

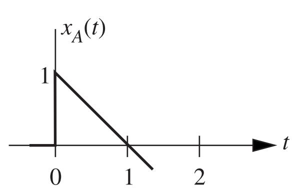


Figure 5.18(b)

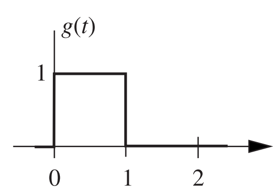


Figure 5.18(c)

The signal $x_1(t)$ may be viewed as the multiplication of $x_A(t)$ and $g(t)$.

$$\begin{aligned}
\text{That is,}\quad x_1(t) &= x_A(t)g(t) \\
&= [-t+1][u(t)-u(t-1)] \\
\Rightarrow \quad x_1(t) &= -tu(t)+tu(t-1)+u(t)-u(t-1) \\
&= -tu(t)+(t-1+1)u(t-1)+u(t)-u(t-1) \\
&= -tu(t)+(t-1)u(t-1)+u(t-1)+u(t)-u(t-1) \\
&= u(t)-tu(t)+(t-1)u(t-1) \\
&= u(t)-r(t)+r(t-1)
\end{aligned}$$

Taking Laplace Transform, we get

$$\begin{aligned}
X_1(s) &= \frac{1}{s}-\frac{1}{s^2}+\frac{1}{s^2}e^{-s} \\
&= \frac{s-1+e^{-s}}{s^2}
\end{aligned}$$

Hence,

$$\boldsymbol{X(s)} = \frac{\boldsymbol{X_1(s)}}{\boldsymbol{1-e^{-sT}}} = \frac{\boldsymbol{(s-1+e^{-s})}}{\boldsymbol{s^2(1-e^{-2s})}}$$

5.6 Inverse Laplace transform

The inverse Laplace transform of $X(s)$ is defined by an integral operation with respect to variable s as follows:

$$x(t) = \frac{1}{2\pi}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds \tag{5.13}$$

Since s is complex, the solution requries a knowledge of complex variables. In otherwords, the evaluation of integral in equation (5.13) requires the use of contour integration in the complex plane, which is very difficult. Hence, we will avoid using equation (5.13) to compute inverse Laplace transform.

In many situations, the Laplace transform can be expressed in the form

$$X(s) = \frac{P(s)}{Q(s)} \tag{5.14}$$

where

$$\begin{aligned}
P(s) &= b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0 \\
Q(s) &= a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0, \qquad a_n \neq 0
\end{aligned}$$

The function $X(s)$ as defined by equation (5.14) is said to be rational function of s, since it is a ratio of two polynomials. The denominator $Q(s)$ can be factored into linear factors.

A partial fraction expansion allows a strictly proper rational function $\dfrac{P(s)}{Q(s)}$ to be expressed as a factor of terms whose numerators are constants and whose denominator corresponds to linear or a combination of linear and repeated factors. This in turn allows us to relate such terms to their corresponding inverse transform.

For performing partial fraction technique on $X(s)$, the function $X(s)$ has to meet the following conditions:

(i) $X(s)$ must be a proper fraction. That is, $m < n$. When $X(s)$ is improper, we can use long division to reduce it to proper fraction.

(ii) $Q(s)$ should be in the factored form.

EXAMPLE 5.14

Find the inverse Laplace transform of

$$X(s) = \frac{2s+4}{s^2+4s+3}$$

SOLUTION

$$X(s) = \frac{2s+4}{s^2+4s+3}$$

$$= \frac{2(s+2)}{(s+1)(s+3)} = \frac{K_1}{s+1} + \frac{K_2}{s+3}$$

where,

$$K_1 = (s+1)X(s)\big|_{s=-1}$$

$$= \left.\frac{2(s+2)}{(s+3)}\right|_{s=-1} = 1$$

$$K_2 = (s+3)X(s)\big|_{s=-3}$$

$$= \left.\frac{2(s+2)}{(s+1)}\right|_{s=-3} = 1$$

Hence,

$$X(s) = \frac{1}{s+1} + \frac{1}{s+3}$$

We know that:

$$\mathscr{L}\{e^{-\alpha t}u(t)\} = \frac{1}{s+\alpha}$$

Therefore,

$$\boldsymbol{x(t) = [e^{-t} + e^{-3t}]u(t)}$$

EXAMPLE 5.15

Find the inverse Laplace transform of

$$X(s) = \frac{s^2+2s+5}{(s+3)(s+5)^2}$$

SOLUTION

Let
$$X(s) = \frac{K_1}{s+3} + \frac{K_2}{s+5} + \frac{K_3}{(s+5)^2}$$

where
$$K_1 = (s+3)X(s)|_{s=-3}$$
$$= \left.\frac{s^2+2s+5}{(s+5)^2}\right|_{s=-3} = 2$$
$$K_2 = \frac{1}{1!}\frac{d}{ds}[(s+5)^2X(s)]\bigg|_{s=-5}$$
$$= \frac{d}{ds}\left[\frac{s^2+2s+5}{s+3}\right]\bigg|_{s=-5}$$
$$= \left.\frac{s^2+6s+1}{(s+3)^2}\right|_{s=-5} = -1$$
$$K_3 = (s+5\,)^2X(s)|_{s=-5}$$
$$= \left.\frac{s^2+2s+5}{(s+3)}\right|_{s=-5} = -10$$

Then
$$X(s) = \frac{2}{s+3} - \frac{1}{s+5} - \frac{10}{(s+5)^2}$$

Taking inverse Laplace transform, we get

$$x(t) = 2e^{-3t} - e^{-5t} - 10te^{-5t}, \qquad t \geq 0$$

or
$$\boldsymbol{x(t) = (2e^{-3t} - e^{-5t} - 10te^{-5t})u(t)}$$

Reinforcement problems

R.P 5.1

Find the Laplace transform of: (a) $\cosh(at)$ (b) $\sinh(at)$

SOLUTION

(a) $\cosh(at) = \frac{1}{2}[e^{at} + e^{-at}]$

We know the Laplace transform pair:

$$\mathscr{L}\{e^{-at}\} = \frac{1}{s+a}$$

and
$$\mathscr{L}\{e^{at}\} = \frac{1}{s-a}$$

Applying linearity property, we get,

$$\begin{aligned}\mathscr{L}\{\cosh(at)\} &= \frac{1}{2}\mathscr{L}\{e^{at}\} + \frac{1}{2}\mathscr{L}\{e^{-at}\} \\ &= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] \\ &= \frac{s}{s^2-a^2}\end{aligned}$$

(b) $\sinh at = \frac{1}{2}[e^{at} - e^{-at}]$

Applying linearity property,

$$\begin{aligned}\mathscr{L}\{\sinh(at)\} &= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] \\ &= \boldsymbol{\frac{a}{s^2-a^2}}\end{aligned}$$

R.P 5.2

Find the Laplace transform of $f(t) = \cos(\omega t + \theta)$.

SOLUTION

Given
$$\begin{aligned}f(t) &= \cos(\omega t + \theta) \\ &= \cos\theta \cos\omega t - \sin\theta \sin\omega t\end{aligned}$$

Applying linearity property, we get,

$$\begin{aligned}\mathscr{L}\{f(t)\} &= F(s) \\ &= \cos\theta\, \mathscr{L}\{\cos\omega t\} - \sin\theta\, \mathscr{L}\{\sin\omega t\} \\ &= \cos\theta \frac{s}{s^2+\omega^2} - \sin\theta \frac{\omega}{s^2+\omega^2} \\ &= \boldsymbol{\frac{s\cos\theta - \omega\sin\theta}{s^2+\omega^2}}\end{aligned}$$

R.P 5.3

Find the Laplace transform of each of the following functions:

(a) $x(t) = t^2\cos(2t + 30°)u(t)$

(b) $x(t) = 2tu(t) - 4\frac{d}{dt}\delta(t)$

(c) $x(t) = 5u\left(\frac{t}{3}\right)$

(d) $x(t) = 5e^{-\frac{t}{2}}u(t)$

SOLUTION

(a) Let us first find the Laplace transform of $\cos(2t+30°)u(t)$

$$\begin{aligned}\mathscr{L}\{\cos(2t+30°)u(t)\} &= \mathscr{L}\{\cos 30° \cos 2tu(t) - \sin 30° \sin 2tu(t)\}\\ &= \cos 30° \mathscr{L}\{\cos 2tu(t)\} - \sin 30° \mathscr{L}\{\sin 2tu(t)\}\\ &= \cos 30° \left[\frac{s}{s^2+4}\right] - \sin 30° \left[\frac{2}{s^2+4}\right]\\ &= \frac{s\cos 30° - 2\sin 30°}{s^2+4}\end{aligned}$$

The Laplace transform of $x(t)$ is now found by using differentiation in s domain property.

$$\begin{aligned}\mathscr{L}\left\{t^2\cos(2t+30°)\right\} &= \frac{d^2}{ds^2}\left[\mathscr{L}\{\cos(2t+30°)u(t)\}\right]\\ &= \frac{d^2}{ds^2}\left[\frac{s\cos 30° - 2\sin 30°}{s^2+4}\right]\\ &= \frac{d^2}{ds^2}\left[\frac{\frac{\sqrt{3}}{2}s-1}{s^2+4}\right]\\ &= \frac{d}{ds}\frac{d}{ds}\left[\frac{\frac{\sqrt{3}}{2}s-1}{s^2+4}\right]\\ &= \frac{d}{ds}\frac{d}{ds}\left[\left(\frac{\sqrt{3}}{2}s-1\right)\left(s^2+4\right)^{-1}\right]\\ &= \frac{d}{ds}\left[\left(\frac{\sqrt{3}}{2}\left(s^2+4\right)^{-1}\right) - 2s\left(\frac{\sqrt{3}}{2}s-1\right)\left(s^2+4\right)^{-2}\right]\\ &= \frac{\frac{\sqrt{3}}{2}(-2s)}{\left(s^2+4\right)^2} - 2\frac{\frac{\sqrt{3}}{2}s-1}{\left(s^2+4\right)^2} - \frac{2s\frac{\sqrt{3}}{2}}{\left(s^2+4\right)^2} + \frac{8s^2\left(\frac{\sqrt{3}}{2}s-1\right)}{\left(s^2+4\right)^3}\\ &= \boldsymbol{\frac{8-12\sqrt{3}s-6s^2+\sqrt{3}s^2}{\left(s^2+4\right)^3}}\end{aligned}$$

(b) $x(t) = 2tu(t) - 4\dfrac{d}{dt}\delta(t)$

$$\mathscr{L}\{x(t)\} = X(s) = 2\mathscr{L}\{tu(t)\} - 4\mathscr{L}\left\{\frac{d}{dt}\delta(t)\right\}$$

We know that whenever a function $f(t)$ is discontinuous at the origin, we have $\mathscr{L}\left\{\frac{d}{dt}f(t)\right\}$ $= sF(s) - f(0^-)$. Applying this relation to the second term on the right side of the above equation, we get

$$\begin{aligned} X(s) &= 2\frac{1}{s^2} - 4[s \times 1 - \delta(0^-)] \\ &= \frac{2}{s^2} - 4[s - 0] \\ &= \boldsymbol{\frac{2}{s^2} - 4s} \end{aligned}$$

(c) $x(t) = 5u\left(\frac{t}{3}\right)$

Using scaling property,

$$\mathscr{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$$

we get,

$$\begin{aligned} \mathscr{L}\{x(t)\} = X(s) &= 5 \times \frac{1}{1/3}\,\mathscr{L}\{u(t)\}_{s\to\left(\frac{s}{\frac{1}{3}}\right)} \\ &= 5 \times \frac{1}{1/3} \times \left[\frac{1}{s}\right]_{s\to\left(\frac{s}{\frac{1}{3}}\right)} \\ &= \boldsymbol{\frac{5}{s}} \end{aligned}$$

(d) $x(t) = 5e^{-\frac{t}{2}}u(t)$

We know the Laplace transform pair:

$$\mathscr{L}\{e^{-at}u(t)\} = \frac{1}{s+a}$$

Hence,

$$\begin{aligned} \mathscr{L}\{x(t)\} = X(s) &= 5\mathscr{L}\left\{e^{-\frac{1}{2}t}u(t)\right\} \\ &= 5\frac{1}{s+\frac{1}{2}} = \boldsymbol{\frac{10}{2s+1}} \end{aligned}$$

R.P 5.4

Find the Laplace transform of the following functions:

(a) $x(t) = t\cos at$

(b) $x(t) = \frac{1}{2a^2}\sin at\sinh(at)$

(c) $x(t) = \frac{\sin^2\omega t}{t}$

SOLUTION

(a) $x(t) = t\cos at$

We know that $$\mathscr{L}\{tf(t)\} = -\frac{d}{ds}F(s)$$

Let $$f(t) = \cos at$$

$$\Rightarrow \qquad F(s) = \frac{s}{s^2+a^2}$$

Hence $$\mathscr{L}\{t\cos at\} = \mathscr{L}\{tf(t)\} = -\frac{d}{ds}\left[\frac{s}{s^2+a^2}\right]$$
$$= \boldsymbol{\frac{s^2-a^2}{(s^2+a^2)^2}}$$

(b) $$x(t) = \frac{1}{2a^2}\sin at \sinh at$$
$$= \frac{1}{2a^2}\left[\frac{1}{2}e^{at}\sin at - \frac{1}{2}e^{-at}\sin at\right]$$
$$= \frac{1}{4a^2}\left[e^{at}\sin at - e^{-at}\sin at\right]$$

We know the shifting in s domain property:

$$\mathscr{L}\{e^{s_0 t}f(t)\} = F(s)|_{s\to(s-s_0)}$$

Applying this property along with linearity property, we get

$$\mathscr{L}\{x(t)\} = X(s)$$
$$= \frac{1}{4a^2}\left[\mathscr{L}\{e^{at}\sin at\} - \mathscr{L}\{e^{-at}\sin at\}\right]$$
$$= \frac{1}{4a^2}\left[\left.\frac{a}{s^2+a^2}\right|_{s\to s-a} - \left.\frac{a}{s^2+a^2}\right|_{s\to s+a}\right]$$
$$= \frac{1}{4a^2}\left[\frac{a}{(s-a)^2+a^2} - \frac{a}{(s+a)^2+a^2}\right]$$
$$= \boldsymbol{\frac{s}{[(s-a)^2+a^2]\,[(s+a)^2+a^2]}}$$

(c) $x(t) = \dfrac{1}{t}\sin^2 \omega t$

We know that $$\mathscr{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt$$

Hence,
$$\int_s^\infty F(s)ds = \int_0^\infty f(t)\int_s^\infty e^{-st}dsdt$$
$$= \int_0^\infty f(t)\left[\frac{e^{-st}}{-t}\right]_s^\infty dt$$
$$= \int_0^\infty \frac{f(t)}{t}e^{-st}dt$$
$$= \mathscr{L}\left[\frac{f(t)}{t}\right]$$

In the present case,
$$f(t) = \sin^2 \omega t$$
$$= \left[\frac{1}{j2}e^{j\omega t} - \frac{1}{j2}e^{-j\omega t}\right]^2$$
$$= \frac{e^{j2\omega t} - 2 + e^{-j2\omega t}}{-4}$$

Hence,
$$F(s) = -\frac{1}{4}\left[\frac{1}{s-j2\omega}\right] + \frac{1}{2}\left(\frac{1}{s}\right) - \frac{1}{4}\left[\frac{1}{s+j2\omega}\right]$$

Hence,

$$X(s) = \mathscr{L}\left\{\frac{1}{t}\sin^2 \omega t\right\}$$
$$= \mathscr{L}\left\{\frac{1}{t}f(t)\right\}$$
$$= \int_s^\infty F(s)ds = \lim_{x\to\infty}\int_s^x f(x)dx$$
$$= \lim_{x\to\infty}\left[\frac{\ln(x-j2\omega) - \ln(s-j2\omega) - 2\ln x + 2\ln s + \ln(x+j2\omega) - \ln(s+j2\omega)}{-4}\right]$$
$$= -\frac{1}{4}\ln\left(\frac{x^2+4\omega^2}{x^2}\right)_{x\to\infty} + \frac{1}{4}\ln\left[\frac{s^2+4\omega^2}{s^2}\right]$$
$$= \boldsymbol{\frac{1}{4}\ln\left[\frac{s^2+4\omega^2}{s^2}\right]}$$

R.P 5.5

Consider the pulse shown in Fig. R.P. 5.5, where $f(t) = e^{2t}$ for $0 < t < T$. Find $F(s)$ for the pulse.

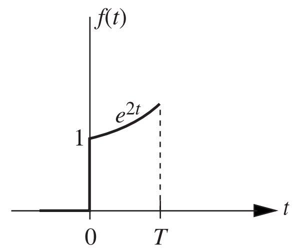


Figure R.P. 5.5

SOLUTION

The discrete pulse $f(t)$ could be imagined as the product of signal $x(t)$ and $g(t)$ as shown in Fig. R.P. 5.5(a) and (b) respectively.

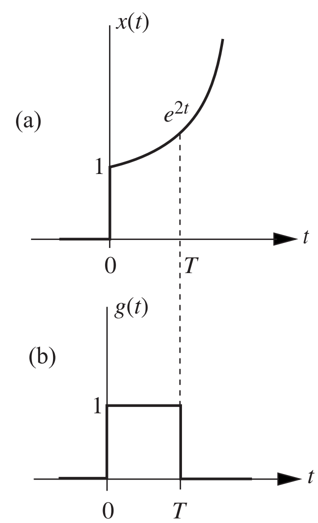


Figure R.P. 5.5(a)

$$\begin{aligned}
\text{That is,} \qquad f(t) &= x(t)g(t) \\
&= e^{2t}[u(t) - u(t-T)] \\
&= e^{2t}u(t) - e^{2t}u(t-T) \\
&= e^{2t}u(t) - e^{2(t-T+T)}u(t-T) \\
&= e^{2t}u(t) - e^{2T}e^{2(t-T)}u(t-T)
\end{aligned}$$

$$\begin{aligned}
\text{Hence,} \quad \mathscr{L}\{f(t)\} = F(s) &= \frac{1}{s-2} - \frac{e^{2T}e^{-Ts}}{s-2} \\
&= \frac{1}{s-2} - \frac{e^{-T(s-2)}}{s-2} \\
&= \boldsymbol{\frac{1-e^{-T(s-2)}}{(s-2)}}
\end{aligned}$$

Alternate method:

$$\begin{aligned}
\mathscr{L}\{f(t)\} = F(s) &\triangleq \int_0^{\infty} f(t)e^{-st}dt \\
&= \int_0^{T} e^{2t}e^{-st}dt \\
&= \boldsymbol{\frac{1-e^{-T(s-2)}}{(s-2)}}
\end{aligned}$$

R.P 5.6

Find the Laplace transform of $f(t)$ shown in Fig. R.P. 5.6.

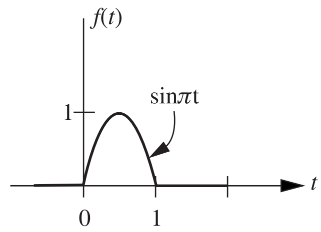


Figure R.P. 5.6

SOLUTION

$f(t)$ is a discrete pulse and can be expressed mathematically as:

$$\begin{aligned}
f(t) &= x(t)g(t)\\
&= \sin \pi t[u(t) - u(t-1)]\\
&= \sin \pi t u(t) - \sin \pi t u(t-1)\\
&= \sin \pi t u(t) - \sin[\pi(t-1+1)]u(t-1)\\
\Rightarrow \qquad f(t) &= \sin \pi t u(t) - \sin[\pi(t-1)]\cos \pi(t-1)\\
&\quad - \cos[\pi(t-1)]\sin \pi u(t-1)\\
&= \sin \pi t u(t) + \sin[\pi(t-1)]u(t-1)
\end{aligned}$$

Hence, $$F(s) = \mathscr{L}\{f(t)\} = \frac{\pi}{s^2+\pi^2} + \frac{e^{-1s}\pi}{s^2+\pi^2}$$

$$= \boldsymbol{\frac{\pi}{s^2+\pi^2}[1+e^{-s}]}$$

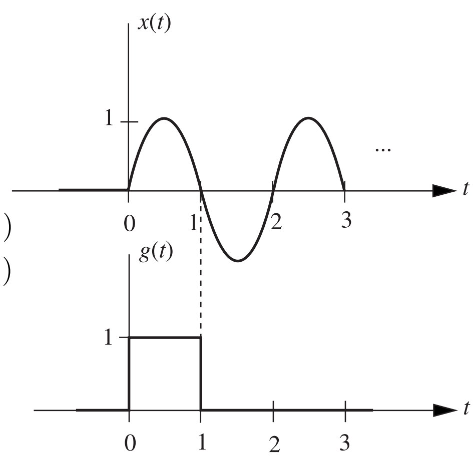


Figure R.P. 5.6(a)

R.P 5.7

Determine the Laplace transform of $f(t)$ shown in Fig. R.P. 5.7.

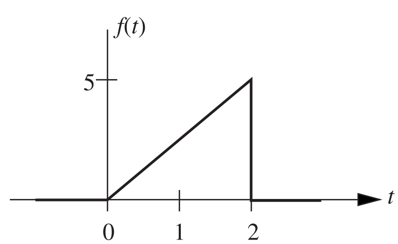


Figure R.P. 5.7

SOLUTION

We can write,

$$
\begin{aligned}
f(t) &= x(t)g(t) \\
&= \left[\frac{5}{2}t\right][u(t) - u(t-2)] \\
&= \frac{5}{2}tu(t) - \frac{5}{2}tu(t-2) \\
&= \frac{5}{2}tu(t) - \frac{5}{2}(t-2+2)u(t-2) \\
&= \frac{5}{2}tu(t) - \frac{5}{2}(t-2)u(t-2) - 5u(t-2)
\end{aligned}
$$

$$
\text{Hence, } \mathscr{L}\{f(t)\} = F(s) = \frac{5}{2}\left(\frac{1}{s^2}\right) - \frac{5}{2}\left(\frac{1}{s^2}\right)e^{-2s} - \frac{5}{s}e^{-2s}
$$

$$
= \boldsymbol{\frac{5}{2s^2}[1 - e^{-2s} - 2se^{-2s}]}
$$

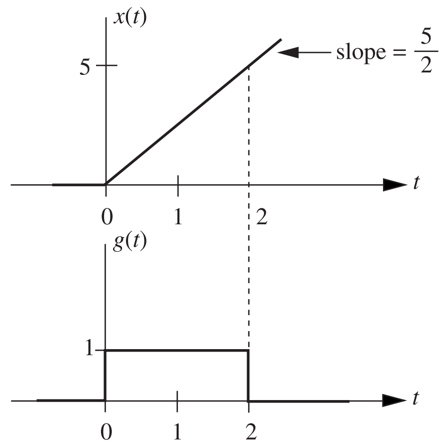


Figure R.P. 5.7(a)

R.P 5.8

Find the Laplace transform of $f(t)$ shown in Fig. R.P. 5.8.

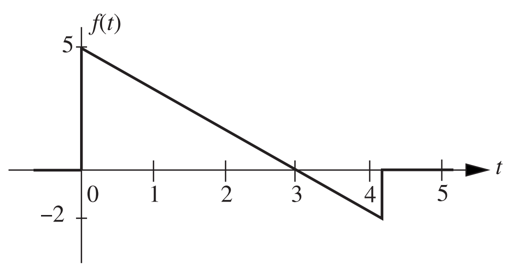


Figure R.P. 5.8

SOLUTION

The equation of a straight line is $y = mx + c$, where m = slope of the line and c = intercept on y-axis.

Hence, $f(t) = \frac{-5}{3}t + 5$

When $f(t) = -2$, let us find t.

That is, $\quad -2 = \dfrac{-5}{3}t + 5$

$\Rightarrow \qquad t = 4.2$ Seconds

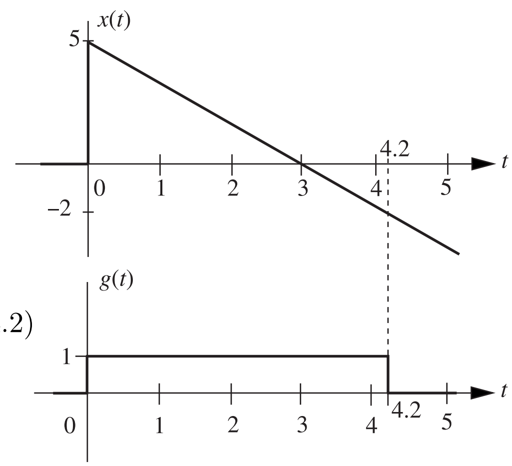


Figure R.P. 5.8(a)

Mathematically,

$$
\begin{aligned}
f(t) &= x(t)g(t)\\
&= \left[\frac{-5}{3}t + 5\right][u(t) - u(t-4.2)]\\
&= \frac{-5}{3}tu(t) + \frac{5}{3}tu(t-4.2) + 5u(t) - 5u(t-4.2)\\
&= \frac{-5}{3}tu(t) + \frac{5}{3}(t - 4.2 + 4.2)u(t-4.2)\\
&\quad +5u(t) - 5u(t-4.2)\\
&= \frac{-5}{3}tu(t) + \frac{5}{3}(t-4.2)u(t-4.2) + 7u(t-4.2)\\
&\quad +5u(t) - 5u(t-4.2)\\
&= \frac{-5}{3}tu(t) + \frac{5}{3}(t-4.2)u(t-4.2) + 2u(t-4.2) + 5u(t)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
F(s) &= \mathscr{L}\{f(t)\}\\
&= \frac{-5}{3s^2} + \frac{5}{3s^2}e^{-4.2s} + \frac{2}{s}e^{-4.2s} + \frac{5}{s}\\
&= \boldsymbol{\frac{-5 + 5e^{-4.2s} + 6se^{-4.2s} + 15s}{3s^2}}
\end{aligned}
$$

R.P 5.9

If $f(0^-) = -3$ and $15u(t) - 4\delta(t) = 8f(t) + 6f'(t)$, find $f(t)$ (hint: by taking the Laplace transform of the differential equation, solving for $F(s)$ and by inverting, find $f(t)$).

SOLUTION

Given, $\qquad 15u(t) - 4\delta(t) = 8f(t) + 6f'(t)$

Taking Laplace transform on both the sides, we get

$$\frac{15}{s} - 4 = 8F(s) + 6[sF(s) - f(0^-)]$$

$$\Rightarrow \qquad \frac{15}{s} - 4 = 8F(s) + 6sF(s) + 18$$

Therefore,

$$F(s)(6s+8) = -18 + \frac{15-4s}{s}$$

$$
\begin{aligned}
\Rightarrow \qquad F(s) &= \frac{-18}{(6s+8)} + \frac{15-4s}{s(6s+8)}\\
&= \frac{-22s+15}{6s\left(s+\dfrac{4}{3}\right)} = \frac{K_1}{s} + \frac{K_2}{s+\dfrac{4}{3}}
\end{aligned}
$$

The constants K_1 and K_2 are found using the theory of partial fractions.

$$K_1 = \left.\frac{-22s+15}{6\left(s+\frac{4}{3}\right)}\right|_{s=0} = 1.875$$

$$K_2 = \left.\frac{-22s+15}{6s}\right|_{s=\frac{-4}{3}} = -5.542$$

Hence, $$F(s) = \frac{1.875}{s} - \frac{5.542}{s+\frac{4}{3}}$$

Taking the inverse, we get $$\boldsymbol{f(t) = \left[1.875 - 5.542e^{\frac{-4}{3}t}\right] u(t)}$$

R.P 5.10

Find the inverse Laplace transform of the following functions:

(a) $F(s) = \dfrac{s+1}{s^2+4s+13}$

(b) $F(s) = \dfrac{3e^{-s}}{s^2+2s+17}$

SOLUTION

(a)
$$\begin{aligned} F(s) &= \frac{s+1}{(s+2)^2+9} \\ &= \frac{(s+2)-1}{(s+2)^2+9} \\ &= \frac{s+2}{(s+2)^2+3^2} - \frac{1}{(s+2)^2+3^2} \\ &= \frac{s+2}{(s+2)^2+3^2} - \frac{1}{3}\frac{3}{(s+2)^2+3^2} \end{aligned}$$

The determination of the Laplace inverse makes use of the following two Laplace transform pairs:

$$\mathcal{L}\{e^{-bt}\sin at\} = \frac{a}{(s+b)^2+a^2}$$

$$\mathcal{L}\{e^{-bt}\cos at\} = \frac{s+b}{(s+b)^2+a^2}$$

Hence,
$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\{F(s)\} \\ &= \boldsymbol{e^{-2t}\cos 3t - \frac{1}{3}e^{-2t}\sin 3t} \end{aligned}$$

(b)
$$F(s) = \frac{3e^{-s}}{s^2+2s+17}$$

Let $$F(s) = e^{-s}X(s)$$

where $$X(s) = \frac{3}{s^2+2s+17} = \frac{3}{(s+1)^2+4^2}$$

$$= \frac{3}{4}\left[\frac{4}{(s+1)^2+4^2}\right]$$

$$\Rightarrow \quad x(t) = \frac{3}{4}e^{-t}\sin 4t$$

Since $$F(s) = e^{-s}X(s)$$

we get, $$f(t) = x(t-1)$$

Therefore, $$f(t) = \frac{3}{4}e^{-(t-1)}\sin[4(t-1)], \qquad t > 1$$

$$\boldsymbol{f(t) = \frac{3}{4}e^{-(t-1)}\sin[4(t-1)]u(t-1)}$$

Laplace transform method for solving a set of differential equations:

1. Identify the circuit variables such as inductor currents and capacitor voltages.
2. Obtain the differential equations describing the circuit and keep a watch on the initial conditions of the circuit variables.
3. Obtain the Laplace transform of the various differential equations.
4. Using Cramer's rule or a similar technique, solve for one or more of the unknown variables, obtaining the solution in s domain.
5. Find the inverse transform of the unknown variables and thus obtain the solution in the time domain.

R.P 5.11

Referring to the RL circuit of Fig. R.P. 5.11, (a) write a differential equation for the inductor current $i_L(t)$. (b) Find $I_L(s)$, the Laplace transform of $i_L(t)$. (c) Solve for $i_L(t)$.

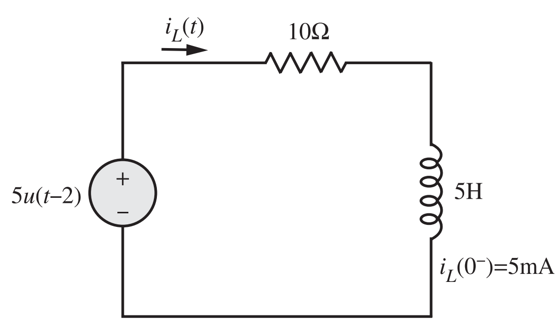


Figure R.P. 5.11

SOLUTION

(a) *Applying KVL clockwise*, we get

$$10i_L(t) + 5\frac{di_L}{dt} - 5u(t-2) = 0$$

(b) Taking Laplace transform of the above equation, we get

$$10I_L(s) + 5[sI_L(s) - i_L(0^-)] = \frac{5}{s}e^{-2s}$$

$$\Rightarrow \qquad I_L(s) = \frac{\frac{5}{s}e^{-2s} + 5i_L(0^-)}{5s+10}$$

$$= \frac{e^{-2s} + 5\times 10^{-3}s}{s(s+2)}$$

$$= e^{-2s}\left[\frac{K_1}{s} + \frac{K_2}{s+2}\right] + \frac{5\times 10^{-3}s}{s(s+2)}$$

where

$$K_1 = \left.\frac{1}{s+2}\right|_{s=0} = \frac{1}{2}$$

$$K_2 = \left.\frac{1}{s}\right|_{s=-2} = -\frac{1}{2}$$

Hence,

$$\boldsymbol{I_L(s) = \frac{1}{2}e^{-2s}\left[\frac{1}{s} - \frac{1}{s+2}\right] + \frac{5\times 10^{-3}}{(s+2)}}$$

(c) Taking Inverse Laplace transform, we get

$$i_L(t) = \frac{1}{2}\left[u(t) - e^{-2t}u(t)\right]_{t\to t-2} + 5\times 10^{-3}e^{-2t}u(t)$$

$$= \boldsymbol{\frac{1}{2}\left[u(t-2) - e^{-2t}u(t-2)\right] + 5\times 10^{-3}e^{-2t}u(t)}$$

R.P 5.12

Obtain a single integrodifferential equation in terms of i_C for the circuit of Fig. R.P. 5.12. Take the Laplace transform, solve for $I_C(s)$, and then find $i_C(t)$ by making use of inverse transform.

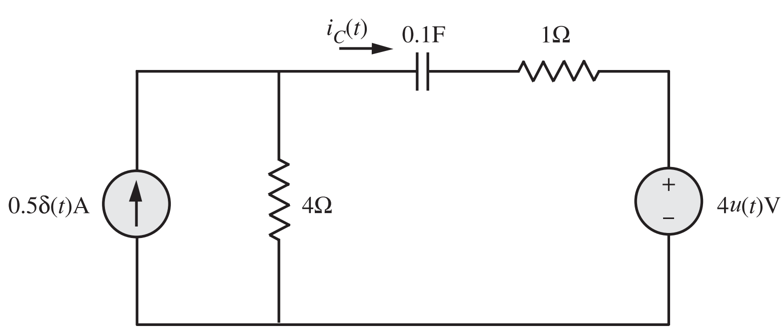


Figure R.P. 5.12

SOLUTION

Applying KVL clockwise to the right-mesh, we get

$$4u(t) + i_C + 10\int_0^\infty i_C dt + 4[i_C - 0.5\delta(t)] = 0$$

Taking Laplace transform, we get

$$4\frac{1}{s} + I_C(s) + \frac{10I_C(s)}{s} + 4I_C(s) - 2 = 0$$

$$\Rightarrow \qquad I_C(s) = \frac{2s-4}{5s+10}$$

$$= \mathbf{0.4} - \frac{\mathbf{1.6}}{\boldsymbol{s+2}}$$

Taking inverse Laplace transform, we get

$$\boldsymbol{i_C(t) = 0.4\delta(t) - 1.6e^{-2t}u(t)} \textbf{ Amps.}$$

R.P 5.13

Refer the circuit shown in Fig. R.P. 5.13. Find $i(0)$ and $i(\infty)$ using initial and final value theorems.

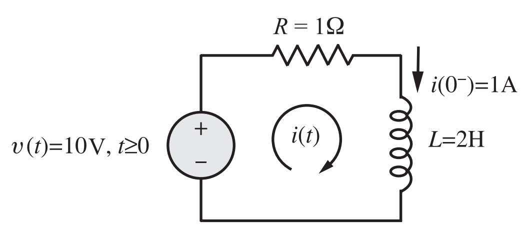


Figure R.P. 5.13

SOLUTION

Applying KVL we get

$$i + 2\frac{di}{dt} = 10$$

Taking Laplace transform, on both the sides, we get

$$I(s) + 2[sI(s) - i(0^-)] = \frac{10}{s}$$

$$\Rightarrow \qquad I(s) + 2[sI(s) - 1] = \frac{10}{s}$$

$$\Rightarrow \qquad I(s)[1+2s] = \frac{10}{s} + 2$$

$$\Rightarrow \qquad I(s) = \frac{10}{s(1+2s)} + \frac{2}{1+2s}$$
$$= \frac{10+2s}{s(1+2s)}$$
$$= \frac{5+s}{s\left(s+\frac{1}{2}\right)}$$

According to initial value theorem,

$$i(0) = \lim_{s\to\infty} sI(s)$$
$$= \lim_{s\to\infty} s\frac{(s+5)}{s\left(s+\frac{1}{2}\right)}$$
$$= \lim_{s\to\infty} \frac{1+\frac{5}{s}}{1+\frac{1}{2s}} = \mathbf{1}$$

We know from fundamentals for an inductor, $i(0^+) = i(0^-) = i(0)$. Hence, $i(0)$ found using initial value theorem verifies the initial value of $i(t)$ given in the problem.

From final value theorem,

$$i(\infty) = \lim_{s\to 0} sI(s)$$
$$= \lim_{s\to 0} \frac{s(s+5)}{s\left(s+\frac{1}{2}\right)} = \frac{5}{1/2} = \mathbf{10\ A}$$

R.P 5.14

Find $i(t)$ and $v_C(t)$ for the circuit shown in Fig. R.P. 5.14 when $v_C(0)$ = 10 V and $i(0)$ = 0 A. The input source is $v_i = 15u(t)$ V. Choose R so that the roots of the characteristic equation are real.

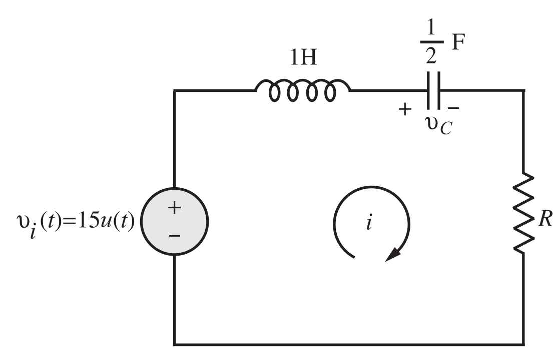


Figure R.P. 5.14

SOLUTION

Applying KVL clockwise, we get

$$L\frac{di}{dt} + v_C + R_i = v_i(t) \tag{5.15}$$

The differential equation describing the variable v_C is

$$C\frac{dv_C}{dt} = i \tag{5.16}$$

The Laplace transform of equation (5.15) is

$$L[sI(s) - i(0) + V_C(s) + RI(s) = V_i(s)] \tag{5.17}$$

The Laplace transform of equation (5.16) is

$$C[sV_C(s) - v_C(0) = I(s)] \tag{5.18}$$

Noting that $i(0) = 0$, substituting for C and L and rearranging equation (5.17) and (5.18), we get,

$$[R + s]I(s) + V_C(s) = V_i(s) = \frac{15}{s} \tag{5.19}$$

$$-I(s) + \frac{1}{2}sV_C(s) = 5 \tag{5.20}$$

Putting equations (5.19) and (5.20) in matrix form, we get

$$\begin{bmatrix} R+s & 1 \\ -1 & \frac{1}{2}s \end{bmatrix} \begin{bmatrix} I(s) \\ V_C(s) \end{bmatrix} = \begin{bmatrix} \frac{15}{s} \\ 5 \end{bmatrix} \tag{5.21}$$

Solving for $I(s)$ using Cramer's rule, we get

$$I(s) = \frac{5}{s^2 + Rs + 2}$$

The inverse Laplace transform of $I(s)$ will depend on the value of R. The equation $s^2 + Rs + 2 = 0$ is defined as the characteristic equation. For the roots of this equation to be real, it is essential that $b^2 - 4ac \geq 0^*$.

This means that, $\quad R^2 - 4 \times 1 \times 2 \geq 0$

$$\Rightarrow \quad R \geq 2\sqrt{2}$$

---

*The condition $b^2 - 4ac \geq 0$ is with respect to algebraic equaion $ax^2 + bx + c = 0$.

Let us choose the value of R as 3

Then $$I(s) = \frac{5}{s^2+3s+2} = \frac{5}{(s+1)(s+2)}$$

$\Rightarrow$ $$I(s) = \frac{K_1}{s+1} + \frac{K_2}{s+2}$$

where $$K_1 = \left.\frac{5}{s+2}\right|_{s=-1} = 5$$

$$K_2 = \left.\frac{5}{s+1}\right|_{s=-2} = -5$$

Hence, $$I(s) = \frac{5}{s+1} - \frac{5}{s+2}$$

Taking inverse Laplace transform, we get

$$\boldsymbol{i(t) = 5e^{-t}u(t) - 5e^{-2t}u(t)}$$

Please note that $t = 0$ gives $i(0) = 0$ and $t = \infty$ gives $i(\infty) = 0$.
Solving the matrix equation (5.21) for $V_C(s)$, using Cramer's rule, we get

$$V_C(s) = \frac{10s^2 + 10Rs + 30}{s(s^2 + Rs + 2)}$$

Substituting the value of R, we get

$$V_C(s) = \frac{10(s^2+3s+3)}{s(s+1)(s+2)}$$

Using partial fraction expansion, we can write,

$$V_C(s) = \frac{K_1}{s} + \frac{K_2}{s+1} + \frac{K_3}{s+2}$$

where, $K_1 = 15$, $K_2 = -10$, $K_3 = 5$

Hence, $$V_C(s) = \frac{15}{s} - \frac{-10}{s+1} + \frac{5}{s+2}$$

Taking inverse Laplace transform, we get

$$\boldsymbol{v_C(t) = 15u(t) - 10e^{-t}u(t) + 5e^{-2t}u(t)}$$

Verification:
Putting $t = 0$, we get

$$v_C(0) = 15 - 10 + 5 = 10 \text{ V}$$
$$v_C(\infty) = 15 - 0 + 0 = 15 \text{ V}$$

This checks the validity of results obtained.

R.P 5.15

For the circuit shown in Fig. R.P. 5.15, the steady state is reached with the 100 V source. At $t = 0$, switch K is opened. What is the current through the inductor at $t = \frac{1}{2}$ seconds ?

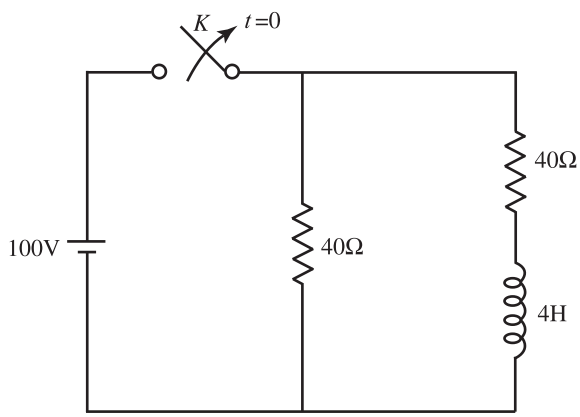


Figure R.P. 5.15

SOLUTION

At $t = 0^-$, the circuit is as shown in Fig. 5.15(a).

$$i_2(0^+) = i_2(0^-) = 2.5 \text{ A}$$

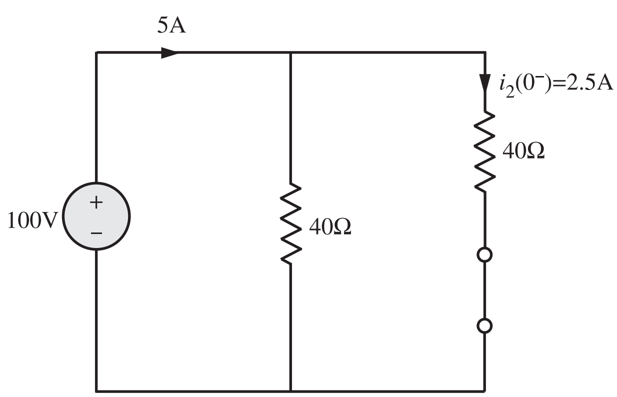


Figure R.P. 5.15(a)

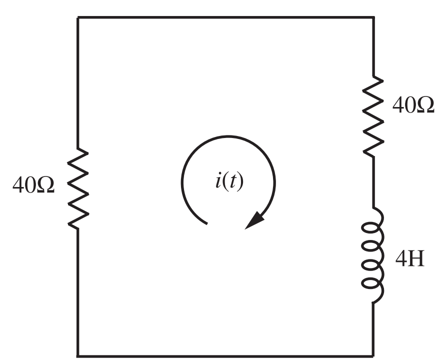


Figure R.P. 5.15(b)

For $t \geq 0^+$, the circuit diagram is as shown in Fig. 5.15(b). *Applying KVL clockwise* to the circuit, we get

$$80i(t) + 4\frac{di}{dt} = 0$$

Taking Laplace transform, we get

$$80I(s) + 4[sI(s) - i(0^-)] = 0$$
$$\Rightarrow \quad 80I(s) + 4sI(s) = 4 \times 2.5$$
$$\Rightarrow \quad [20 + s]I(s) = 2.5$$
$$I(s) = \frac{2.5}{s + 20}$$

Taking inverse Laplace transform, we get,

$$i(t) = 2.5e^{-20t}$$

At $t = 0.5$ sec, we get

$$\boldsymbol{i(0.5) = 2.5e^{-10} = 1.135 \times 10^{-4}\ \mathrm{A}}$$

R.P 5.16

Refer the circuit shown in Fig. R.P. 5.16. Find:

(a) $v_o(t)$ for $t \geq 0$

(b) $i_o(t)$ for $t \geq 0$

(c) Does your solution for $i_o(t)$ make sense when $t = 0$? Explain.

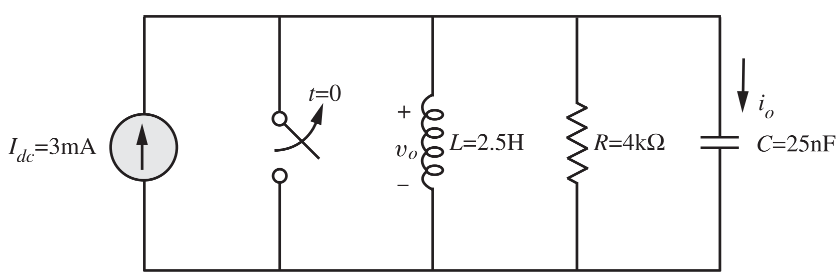


Figure R.P. 5.16

SOLUTION

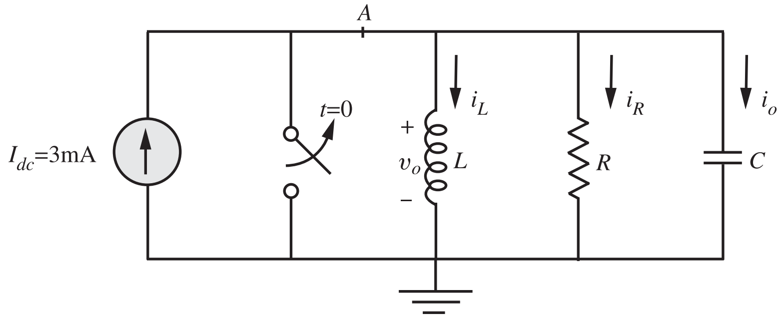


Figure R.P. 5.16(a)

KCL at node A: (for $t \geq 0$)

$$I_{dc} = \frac{1}{L}\int_0^t v_o dt + \frac{v_o}{R} + C\frac{dv_o}{dt}$$

Taking Laplace transform,

$$\frac{I_{dc}}{s} = \frac{V_o(s)}{sL} + \frac{V_o(s)}{R} + sCV_o(s)$$

Hence, $$V_o(s) = \frac{\dfrac{I_{dc}}{C}}{s^2 + \left(\dfrac{1}{RC}\right)s + \dfrac{1}{LC}}$$

Substituting the values of I_{dc}, R, L, and C, we find that

$$V_o(s) = \frac{120,000}{s^2 + 10,000s + 16 \times 10^6} = \frac{120,000}{(s+2000)(s+8000)}$$

Using partial fractions, we get

$$V_o(s) = \frac{K_1}{s+2000} + \frac{K_2}{s+8000}$$

where $K_1 = 20$, and $K_2 = -20$

Hence, $$V_o(s) = \frac{20}{s+2000} - \frac{20}{s+8000}$$

Taking inverse Laplace transform, we get

$$\boldsymbol{v_o(t) = 20e^{-2000t}u(t) - 20e^{-8000t}u(t)}$$

(b) $i_o(t) = C\dfrac{dv_o}{dt}$

Hence $$I_o(s) = C\left[sV_o(s) - v_o(0)\right]$$

For $t \leq 0^-$, since the switch was in closed state, the circuit was not activated by the source. This means that $v_o(0) = v_o(0^-) = v_o(0^+) = 0$ and $i_L(0^+) = i_L(0^-) = 0$.

Then,
$$\begin{aligned} I_o(s) &= CsV_o(s) \\ &= \frac{25 \times 10^{-9} \times s \times 120,000}{s^2 + 10,000s + 16 \times 10^6} \\ &= \frac{3 \times 10^{-3}s}{(s+2000)(s+8000)} \\ &= \frac{K_1}{s+2000} + \frac{K_2}{s+8000} \end{aligned}$$

We find that, $K_1 = -10^{-3}$, and $K_2 = 4 \times 10^{-3}$

Therefore, $$I_o(s) = \frac{-10^{-3}}{s+2000} + \frac{4 \times 10^{-3}}{s+8000}$$

Taking inverse Laplace transform, we get

$$\boldsymbol{i_o(t) = 4e^{-8000t}u(t) - e^{-2000t}u(t)\mathrm{mA}}$$

(c) $i_o(0^+) = 4 - 1 = 3\mathrm{mA}$

Yes. The initial inductor current is zero by hypothesis $\left(i_L(0^+) = I_L(0^-) = 0\right)$. Also, the initial resistor current is zero because $v_o(0^+) = v_o(0^-) = 0$. Thus at $t = 0^+$, the source current appears in the capacitor.

R.P 5.17

Refer the circuit shown in Fig. R.P. 5.17. The circuit parameters are $R = 10\mathrm{k}\Omega$, $L = 800$ mH and $C = 100\mathrm{nF}$, if $V_{dc} = 70\mathrm{V}$, find:

(a) $v_o(t)$ for $t \geq 0$

(b) $i_o(t)$ for $t \geq 0$

(c) Use initial and final value theorems to check the inital and final values of current and voltage.

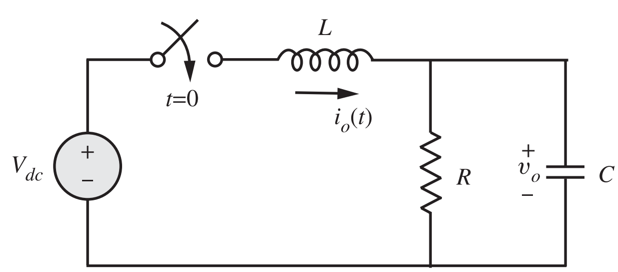


Figure R.P. 5.17

SOLUTION

At $t = 0^-$, switch is open and at $t = 0^+$, the switch is closed. Since at $t = 0^-$, the circuit is not energized by dc source, $i_o(0^-) = 0$ and $v_o(0^-) = 0$. Then by the hypothesis, that the current in an inductor and voltage across a capacitor cannot change instantaneously,

$$i_o(0^+) = i_o(0^-) = 0 \text{ and } \quad v_o(0^+) = v_o(0^-) = 0$$

The *KCL equation* when the switch is closed (for $t \geq 0^+$) is given by

$$C\frac{dv_o}{dt} + \frac{v_o}{R} + \frac{1}{L}\int_0^t (v_o - V_{dc})d\tau = 0$$

$$\Rightarrow \qquad C\frac{dv_o}{dt} + \frac{v_o}{R} + \frac{1}{L}\int_0^t v_o d\tau = \frac{1}{L}\int_0^t V_{dc} d\tau$$

$$\Rightarrow \qquad C\frac{dv_o}{dt} + \frac{v_o}{R} + \frac{1}{L}\int_0^t v_o d\tau = \frac{1}{L}V_{dc}t$$

Laplace transform of the above equation gives

$$C\left[sV_o(s) - v_o(0)\right] + \frac{V_o(s)}{R} + \frac{1}{L}\frac{V_o(s)}{s} = \frac{1}{L}\frac{V_{dc}}{s^2}$$

Since $v_o(0)$ is same as $v_o(0^-)$, we get

$$CsV_o(s) + \frac{V_o(s)}{R} + \frac{1}{L}\frac{V_o(s)}{s} = \frac{V_{dc}}{Ls^2}$$

$$\Rightarrow \qquad V_o(s) = \frac{\dfrac{V_{dc}}{LC}}{s\left(s^2 + \dfrac{1}{RC}s + \dfrac{1}{LC}\right)}$$

Substituting the values of V_{dc}, R, L, and C, we get

$$V_o(s) = \frac{875 \times 10^6}{s\left[s^2 + 1000s + 1250 \times 10^4\right]}$$

$$= \frac{875 \times 10^6}{s(s - s_1)(s - s_2)}$$

where $\qquad s_1, s_2 = -500 \pm \sqrt{25 \times 10^4 - 1250 \times 10^4}$

$$= -500 \pm j3500$$

Hence, $\qquad V_o(s) = \dfrac{875 \times 10^6}{s(s + 500) - j3500)(s + 500 + j3500)}$

Using partial fractions, we get

$$V_o(s) = \frac{K_1}{s} + \frac{K_2}{s + 500 - j3500} + \frac{K_2^*}{s + 500 + j3500}$$

We find that $\qquad K_1 = \dfrac{875 \times 10^6}{125 \times 10^5} = 70$

$$K_2 = \frac{875 \times 10^6}{(-500 + j3500)(j7000)}$$

$$= 5\sqrt{50}\,\underline{/171.87^\circ}$$

$$V_o(s) = \frac{70}{s} + \frac{5\sqrt{50}\,\underline{/171.87^\circ}}{s+500-j3500} + \frac{5\sqrt{50}\,\underline{/-171.87^\circ}}{\mathrm{s}+500+j3500}$$

Taking inverse Laplace transform, we get

$$v_o(t) = \left[70 + 5\sqrt{50}\,\underline{/171.87^\circ}\ e^{-(500-j3500)t} + 5\sqrt{50}\,\underline{/-171.87^\circ}\ e^{-(500+j3500)t}\right] u(t)$$

The inverse of $V_o(s)$ can be expressed in a better form by following the technique described below:

Let us consider a transformed function

$$F(s) = \frac{C+jd}{s+a-j\omega} + \frac{C-jd}{s+a+j\omega}$$

$$= \frac{m\,\underline{/\theta}}{s+a-j\omega} + \frac{m\,\underline{/-\theta}}{s+a+j\omega}$$

where $$m = \sqrt{c^2+d^2} \text{ and } \theta = \tan^{-1}\left[\frac{d}{c}\right]$$

The inverse transform of $F(s)$ is given by

$$f(t) = 2me^{-at}\cos(\omega t + \theta)u(t)$$

(For the proof see R.P. 5.19)

In the present context,

$$m = 5\sqrt{50}, \theta = 171.87^\circ$$

$$\omega = 3500 \text{ and } a = 500$$

This means that, $$\mathcal{L}^{-1}\left\{\frac{5\sqrt{50}\,\underline{/171.87^\circ}}{s+500-j3500} + \frac{5\sqrt{50}\,\underline{/-171.87^\circ}}{s+500+j3500}\right\}$$

$$= 2 \times 5\sqrt{50}e^{-500t}\cos(3500t + 171.87^\circ)$$

$$= 10\sqrt{50}e^{-500t}\cos(3500t + 171.87^\circ)$$

Hence, $\boldsymbol{v_o(t) = \left[70 + 10\sqrt{50}e^{-500t}\cos(3500t + 171.87^\circ)\right] u(t)}$

(b) $i_o(t) = \dfrac{v_o}{R} + C\dfrac{dv_o}{dt}$

Taking Laplace transforms on both the sides, we get

$$I_o(s) = \frac{V_o(s)}{R} + C\left[sV_o(s) - v_o(0^-)\right]$$

$$\Rightarrow \qquad I_o(s) = \frac{V_o(s)}{R} + CsV_o(s)$$

$$\Rightarrow \qquad I_o(s) = CV_o(s)\left[s + \frac{1}{RC}\right]$$

$$= \left(\frac{V_{dc}}{L}\right)\left[\frac{s + \dfrac{1}{RC}}{s\left(s^2 + \dfrac{1}{RC}s + \dfrac{1}{LC}\right)}\right]$$

Substituting the values of V_{dc}, R, L, and C, we get

$$I_o(s) = \frac{87.5(s+1000)}{s(s+500-j3500)(s+500+j3500)}$$
$$= \frac{K_1}{s} + \frac{K_2}{s+500-j3500} + \frac{K_2^*}{s+500+j3500}$$

We find that,

$$K_1 = \frac{87.5 \times 1000}{1250 \times 10^4} = 7\text{mA}$$
$$K_2 = \frac{87.5(500+j3500)}{(-500+j3500)(j7000)}$$
$$= 12.5\,\underline{/-106.26^\circ}\ \text{mA}$$
$$I_o(s) = \frac{7}{s} + \frac{12.5\,\underline{/-106.26^\circ}}{s+500-j3500} + \frac{12.5\,\underline{/106.26^\circ}}{s+500+j3500}$$

The inverse Laplace transform yields,

$$i_o(t) = \left[7 + 12.5\,\underline{/-106.26^\circ}\ e^{-(500-j3500)t} + 12.5\,\underline{/106.26^\circ}\ e^{-(500+j3500)t}\right]u(t)$$
$$= \boldsymbol{\left[7 + 25e^{-500t}\cos(3500t - 106.26^\circ)\right]u(t)\text{mA}}$$

(c) $V_o(s) = \dfrac{\dfrac{V_{dc}}{LC}}{s\left(s^2 + \left(\dfrac{1}{RC}\right)s + \dfrac{1}{LC}\right)}$

From Final Value theorem: $v_o(\infty) = \lim_{t\to\infty} v_o(t) = \lim_{s\to 0} sV_o(s) = \dfrac{V_{dc} \times LC}{LC}$ **= 70V**

The same result may be obtained by putting $t = \infty$ in the expression for $v_o(t)$.

From initial value theorem : $\quad v_o(0) = \lim_{s\to\infty} sV_o(s)$

$$= \lim_{s\to\infty} \frac{\cancel{s}\dfrac{V_{dc}}{LC}}{\cancel{s}\left(s^2 + \dfrac{1}{RC}s + \dfrac{1}{LC}\right)}$$

$$= 0$$

This verifies our beginning analysis that $v_o(0^+) = v_o(0^-) = 0$. The same result may be obtained by putting $t = 0$ in the expression for $v_o(t)$.

We know that, $\quad I_o(s) = \dfrac{V_{dc}}{L} \dfrac{\left(s + \dfrac{1}{RC}\right)}{s\left(s^2 + \dfrac{1}{RC}s + \dfrac{1}{LC}\right)}$

From final value theorem : $\quad I_o(\infty) = \lim_{s\to 0} sI_o(s)$

$$= \lim_{s\to 0} \cancel{s}\frac{V_{dc}}{L} \frac{\left(s + \dfrac{1}{RC}\right)}{\cancel{s}\left(s^2 + \dfrac{1}{RC}s + \dfrac{1}{LC}\right)}$$

$$= \frac{V_{dc}}{L} \frac{\dfrac{1}{RC}}{\dfrac{1}{LC}}$$

$$= \frac{V_{dc}}{R} = \frac{70}{10 \times 10^3} = \mathbf{7\ mA}$$

The same result may be obtained by putting $t = \infty$ in the expression for $i_o(t)$.

From initial value theorem : $\quad i_o(0) = \lim_{s\to\infty} sI_o(s)$

$$= \lim_{s\to\infty} s\left[\frac{V_{dc}}{L} \frac{\left(s + \dfrac{1}{RC}\right)}{s\left(s^2 + \dfrac{1}{RC}s + \dfrac{1}{LC}\right)}\right]$$

$$= \mathbf{0}$$

This agrees with our initial analysis that the initial current through the inductor is zero. The same result can be obtained by putting $t = 0$ in the expression for $i_o(t)$.

R.P 5.18

Apply the initial and final value theorems to each of the functions given below:

(a) $F(s) = \dfrac{s^2 + 5s + 10}{s + 6}$ (b) $F(s) = \dfrac{s^2 + 5s + 10}{5(s^2 + 6s + 8)}$

SOLUTION

Since in $F(s)$ referred in (a) and (b) are improper [1] fractions, the corresponding time domain counterparts, $f(t)$ contain impulses.

Thus, neither the initial value theorem nor the final value theorems may be applied to these transformed functions.

R.P 5.19

Find the inverse Laplace transform of $F(s) = \dfrac{c + jd}{s + a - j\omega} + \dfrac{c - jd}{s + a + j\omega}$

SOLUTION

Expressing $c + jd$ and $c - jd$ in the exponenetial from, we get,

$$F(s) = \frac{me^{j\theta}}{s + a - j\omega} + \frac{me^{-j\theta}}{s + a + j\omega}$$

where $$m = \sqrt{c^2 + d^2} \text{ and } \theta = \tan^{-1}\left[\frac{d}{c}\right]$$

Hence,
$$\begin{aligned}
f(t) &= \mathscr{L}^{-1}\{F(s)\} \\
&= me^{j\theta}e^{-(a-j\omega)t}u(t) + me^{-j\theta}e^{-(a+j\omega)t}u(t) \\
&= me^{-at}e^{j(\theta+\omega t)}u(t) + me^{-at}e^{-j(\theta+\omega t)}u(t) \\
&= 2me^{-at}\left[\frac{e^{j(\theta+\omega t)} + e^{-j(\theta+\omega t)}}{2}\right]u(t) \\
&= \boldsymbol{2me^{-at}\cos(\theta + \omega t)u(t)}
\end{aligned}$$

R.P 5.20

Find the initial and final values of $f(t)$ when $F(s) = \dfrac{60}{s^2 - 2s + 1}$

SOLUTION

Initial value theorem

$$\begin{aligned}
f(0) &= \lim_{s\to\infty} sF(s) \\
&= \lim_{s\to\infty} s\frac{60}{s^2 - 2s + 1} = \mathbf{0}
\end{aligned}$$

[1] If the degree of the numerator polynomial is greater than or equal to the degree of the denominator polynomial, the fraction is said to be improper.

Final value theorem:

The poles of $F(s)$ are given by finding the roots of the denominator polynomial. That is,

$$s^2 - 2s + 1 = 0 \quad \Rightarrow \quad (s-1)^2 = 0 \quad \Rightarrow \quad s = 1, 1$$

Since both the poles of $F(s)$ lie to the right of the s plane, final value theorem cannot be used to find $f(\infty)$.

R.P 5.21

Find $i(t)$ for the circuit of Fig.R.P. 5.21, when $i_1(t) = 7e^{-6t}$ A for $t \geq 0$ and $i(0) = 0$. Also find $i(\infty)$.

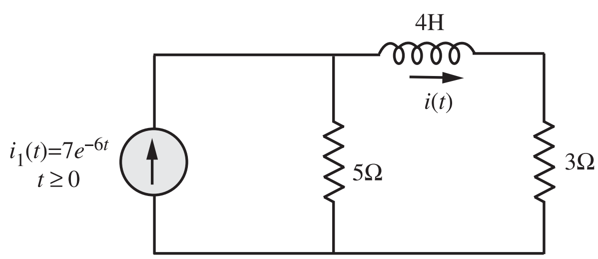


Figure R.P. 5.21

SOLUTION

Refer Fig. R.P. 5.21(a).

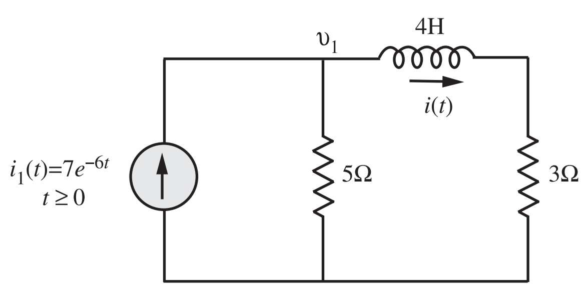


Figure R.P. 5.21(a)

KCL at node v_1: $\dfrac{v_1}{5} + i = 7e^{-6t}$

Also, $\quad v_1 = 3i + 4\dfrac{di}{dt}$

Hence, $\quad \dfrac{1}{5}\left[3i + 4\dfrac{di}{dt}\right] + i = 7e^{-6t}$

$$\Rightarrow \qquad \frac{4}{5}\frac{di}{dt} + \frac{8}{5}i = 7e^{-6t}$$

$$\Rightarrow \qquad \frac{di}{dt} + 2i = \frac{35}{4}e^{-6t}$$

Taking Laplace transform of the differential equation, we get

$$[sI(s) - i(0)] + 2I(s) = \frac{35}{4}\frac{1}{s+6}$$

$$\Rightarrow \qquad I(s) = \frac{35}{4}\frac{1}{(s+2)(s+6)}$$

Using partial fraction expansion, we get

$$I(s) = \frac{K_1}{s+2} + \frac{K_2}{s+6}$$

and find that $$K_1 = \frac{35}{16} \text{ and } K_2 = \frac{-35}{16}$$

Hence, $$I(s) = \frac{35}{16}\left[\frac{1}{s+2}\right] - \frac{35}{16}\left[\frac{1}{s+6}\right]$$

$$\Rightarrow \qquad \boldsymbol{i(t) = \frac{35}{16}\left[e^{-2t} - e^{-6t}\right] u(t)}$$

5.7 Circuit element models and initial conditions

In the analysis of a circuit, the Laplace transform can be carried one step further by transforming the circuit itself rather than the differential equation. Earlier we have seen how to represent a circuit in time domain by differential equations and then use Laplace transform to transform the differential equations into algebraic equations. In this section, we will see how we can represent a circuit in s domain using the Laplace transform and then analyze it using algebraic equations.

5.7.1 Resistor

The voltage-current relationship for a resistor R is given by Ohm's law:

$$v(t) = i(t)R \tag{5.22}$$

Taking Laplace transform on both the sides, we get

$$V(s) = I(s)R \tag{5.23}$$

Fig. 5.19 (a) shows the representation of a resistor in time domain and Fig. 5.19(b) in frequency domain using Laplace transform.

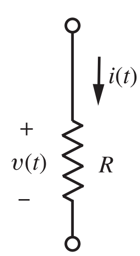


Figure 5.19(a) Resistor represented in time domain

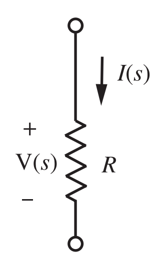


Figure 5.19(b) Resistor represented in the frequency domain using Laplace transform

The impedance of an element is defined as

$$Z(s) = \frac{V(s)}{I(s)}$$

provided all *initial conditions are zero*. Please note that the impedance is a concept defined only in frequency domain and not in time domain. In the case of a resistor, there is no initial condition to be set to zero. Comparision of equations (5.22) and (5.23) reveals that, resistor R has same representation in both time and frequency domains.

5.7.2 Capacitor

For a capacitor with capacitance C, the time-domain voltage-current relationship is

$$v(t) = \frac{1}{C}\int_{0}^{t} i(\tau)d\tau + v(0) \tag{5.24a}$$

The s domain characterization is obtained by taking the Laplace transform of the above equation. That is,

$$V(s) = \frac{1}{Cs}I(s) + \frac{v(0)}{s} \tag{5.24b}$$

To find the impedance of a capacitor, set the initial condition $v(0)$ to zero. Then from equation (5.24b), we get $Z(s) = \frac{V(s)}{I(s)} = \frac{1}{Cs}$ as the impedance of the capacitor. With the help of equation (5.24b), we can draw the frequency domain representation of a Capacitor and the same is shown in Fig. 5.20(b). This equivalent circuit is drawn so that the *KVL equation* represented by equation (5.24 b) is satisfied. Performing source transformation on the equivalent s domain circuit for a capacitor which is shown in Fig. 5.20(b), we get an alternate frequency domain representation as shown in Fig. 5.20(c).

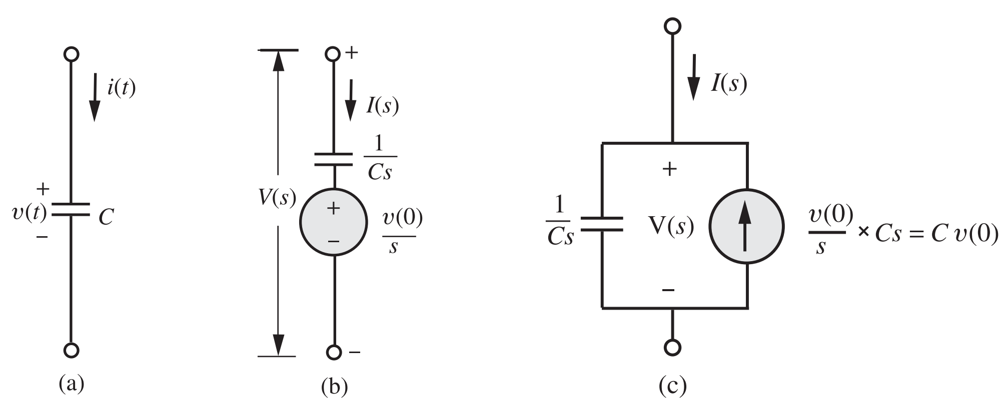


Figure 5.20(a) A capacitor represented in time domain
(b) A capacitor represented in the frequency domain
(c) Alternate frequency domain representation for a capacitor

5.7.3 Inductor

For an inductor with inductance L, the time domain voltage-current relation is

$$v(t) = L\frac{di(t)}{dt} \tag{5.25}$$

The Laplace transform of equation (5.25) yields,

$$V(s) = LsI(s) - Li(0) \tag{5.26}$$

To find impedance of an inductor, set the initial condition $i(0)$ to zero. Then from equation (5.26), we get

$$Z(s) = \frac{V(s)}{I(s)} = Ls \tag{5.27}$$

which represents the impedance of the inductor. Equation (5.26) is used to get the frequency domain representation of an inductor and the same is shown in Fig. 5.21(b). The series connection of elements corresponds to sum of the voltages in equation (5.26). Converting the voltage source in Fig.5.21(b) into an equivalent current source, we get an alternate representation for the inductor in frequency domain which is as shown in Fig. 5.21(c).

To find the frequency domain representation of a circuit, we replace the time domain representation of each element in the circuit by its frequency domain representation.

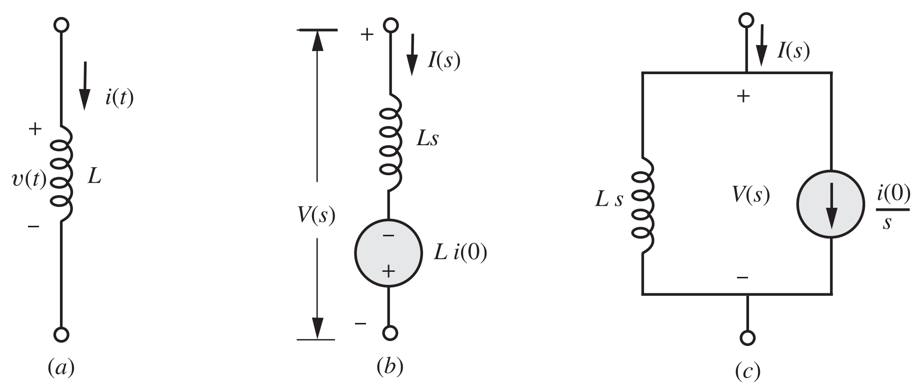


Figure 5.21(a) An inductor represented in time domain
(b) An inductor represented in the frequency domain
(c) An alternate frequency domain representation

To find the complete response of a circuit, we first get its frequency domain representation. Next, using KVL or KCL, we find the variables of interest in s doamin. Finally, we use the inverse Laplace transform to represent the variables of interest in time domain.

EXAMPLE 5.16

Determine the voltage $v_C(t)$ and the current $i_C(t)$ for $t \geq 0$ for the circuit shown in Fig. 5.22.

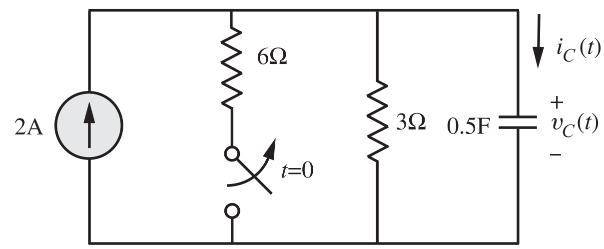


Figure 5.22

SOLUTION

We shall analyze this circuit using nodal technique. Hence we represent the capacitor in frequency domain by a parallel circuit since it is easier to account for current sources than voltage sources while handling nodal equations.

The symbol for switch indicates that at $t = 0^-$ it is closed and at $t = 0^+$, it is open. The circuit at $t = 0^-$ is shown in Fig. 5.23(a). Let us assume that at $t = 0^-$, the circuit is in steady state. Under steady state condition, capacitor acts as on open circuit as shown in Fig. 5.23(a).

$$i_1(0^-) = \frac{2 \times 6}{6+3} = \frac{12}{9} = \frac{4}{3}$$

$$v_C(0^-) = \frac{4}{3} \times 3 = 4\mathbf{V}$$

Hence, $\quad v_C(0) = v_C(0^+) = v_C(0^-) = 4\mathbf{V}$

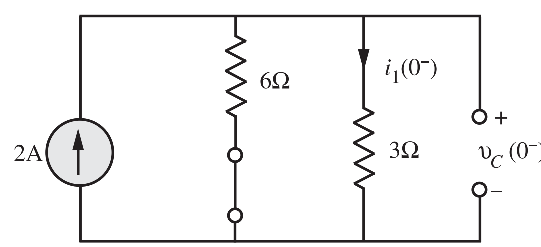


Figure 5.23(a)

Fig. 5.23(b) represents the frequency domain representation of the circuit shown in Fig. 5.22.

KCL at top node:

$$\frac{V_C(s)}{3} + \frac{s}{2}V_C(s) = 2 + \frac{2}{s}$$

$$\Rightarrow \qquad V_C(s) = \frac{6}{s} - \frac{2}{s + \dfrac{2}{3}}$$

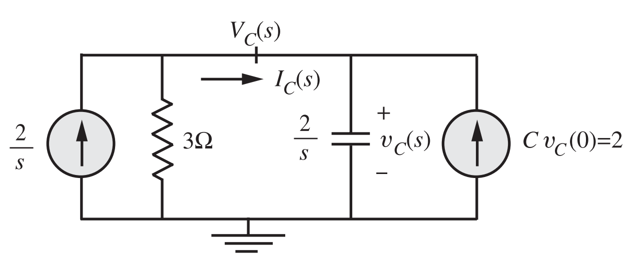


Figure 5.23(b)

Inverse Laplace transform yields

$$v_C(t) = \left[6 - 2e^{-\frac{2}{3}t}\right] u(t)\text{V}$$

Also,
$$I_C(s) = \frac{V_C(s)}{\frac{2}{s}} - 2 = \frac{\frac{2}{3}}{s + \frac{2}{3}}$$

$$\Rightarrow \qquad i_C(t) = \frac{2}{3}e^{-\frac{2}{3}t}u(t)\text{A}$$

EXAMPLE 5.17

Determine the current $i_L(t)$ for $t \geq 0$ for the circuit shown in Fig. 5.24.

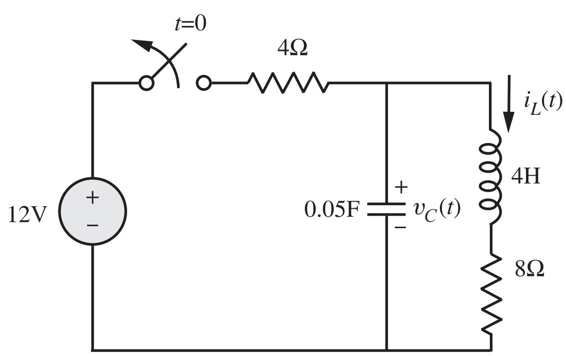


Figure 5.24

SOLUTION

At $t = 0^-$, switch is closed and at $t = 0^+$, it is open. Let us assume that at $t = 0^-$, the circuit is in steady state. In steady state, capacitor is open and inductor is short. The equivalent circuit at $t = 0^-$ is as shown in Fig. 5.25(a).

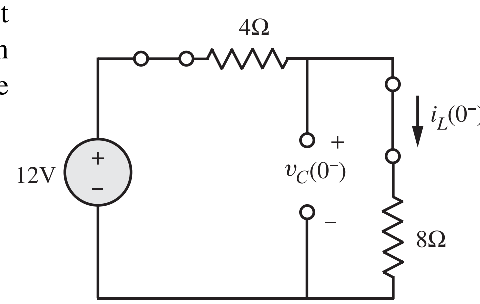


Figure 5.25(a)

$$i_L(0^-) = \frac{12}{8+4} = 1\text{A}$$

$$v_C(0^-) = 1 \times 8 = 8\text{V}$$

Therefore,
$$i_L(0) = i_L(0^+) = i_L(0^-) = 1\text{A}$$

$$v_C(0) = v_C(0^+) = v_C(0^-) = 8\text{V}$$

For $t \geq 0^+$, the circuit in frequency domain is as shown in Fig. 5.25(b). We will use *KVL* to find $i_L(t)$. Hence, we use series circuits to represent both the capacitor and inductor in the frequency domain. These series circuits contain voltage sources rather than current sources. It is easier to account for voltage sources than current sources when writing mesh equations. This justifies the selection of series representation for both the capacitor and inductor.

Applying *KVL* clockwise to the right mesh, we get

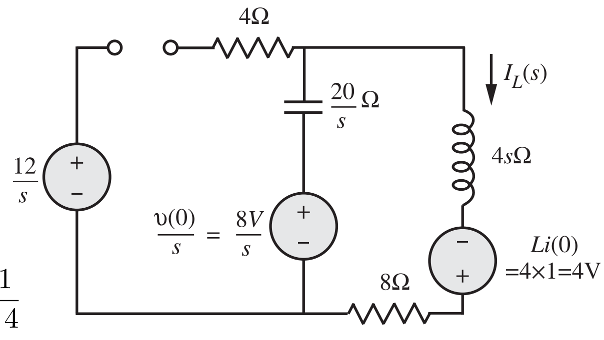


Figure 5.25(b)

$$\frac{-8}{s}+\frac{20}{s}I_L(s)+4sI_L(s)-4+8I_L(s)=0$$

$$\Rightarrow \quad \frac{8}{s}+4=\left[\frac{20}{s}+8+4s\right]I_L(s)$$

$$\Rightarrow \quad I_L(s)=\frac{2+s}{s^2+2s+5}=\frac{(s+1)+1}{(s+1)^2+4}$$

$$\Rightarrow \quad I_L(s)=\frac{s+1}{(s+1)^2+2^2}+\frac{1}{2}\left[\frac{2}{(s+1)^2\times 2^2}\right]$$

We know the Laplace transform pairs:

$$\mathscr{L}\left\{e^{-at}\cos bt\right\}=\frac{s+a}{(s+a)^2+b^2}$$

and
$$\mathscr{L}\left\{e^{-at}\sin bt\right\}=\frac{b}{(s+a)^2+b^2}$$

Hence,
$$\boldsymbol{i_L(t)=\left[e^{-t}\cos 2t+\frac{1}{2}e^{-t}\sin 2t\right]u(t)\mathrm{A}}$$

EXAMPLE 5.18

Find $v_o(t)$ of the circuit shown in Fig. 5.26.

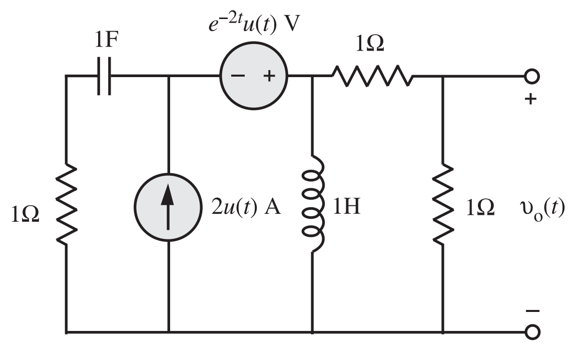


Figure 5.26

SOLUTION

The unit step function $u(t)$ is defined as follows:

$$u(t)=\begin{cases}1, & t\geq 0^+\\ 0, & t\leq 0^-\end{cases}$$

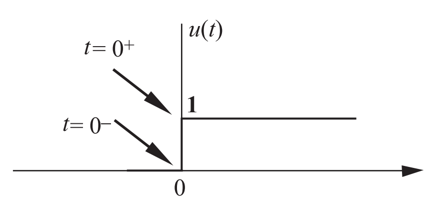


Figure 5.27(a)

Since the circuit has two independent sources with $u(t)$ associated with them, the circuit is not energized for $t \leq 0^-$. Hence the initial current through the inductor is zero. That is, $i_L(0^-) = 0$. Since current through an inductor cannot change instantaneously,

$$i_L(0) = i_L(0^+) = i_L(0^-) = 0$$

Also,
$$v_C(0) = v_C(0^+) = v_C(0^-) = 0$$

The equivalent circuit for $t \geq 0^+$ in frequency domain is as shown in Fig. 5.27(b).

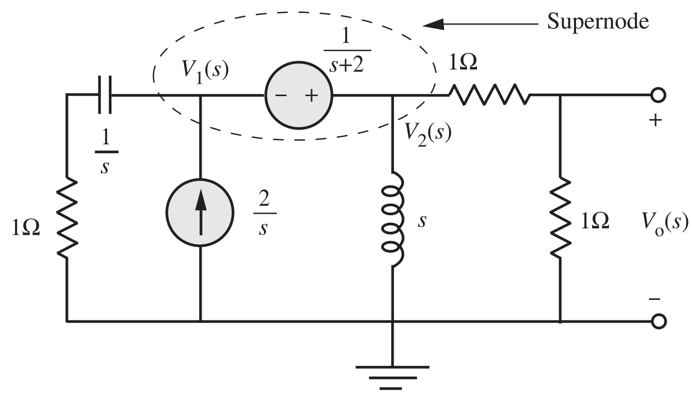


Figure 5.27(b)

KCL at supernode:

$$\frac{V_1(s)}{1+\dfrac{1}{s}} + \frac{V_2(s)}{s} + \frac{V_2(s)}{2} = \frac{2}{s}$$

$$\Rightarrow \quad V_1(s)\left[\frac{1}{1+\dfrac{1}{s}}\right] + V_2(s)\left[\frac{1}{s}+\frac{1}{2}\right] = \frac{2}{s}$$

$$\Rightarrow \quad V_1(s)\left[\frac{s}{s+1}\right] + V_2(s)\left[\frac{2+s}{2s}\right] = \frac{2}{s}$$

The constraint equation:

Applying KVL to the path comprising of current source → voltage source → inductor,

we get,
$$-V_1(s) - \frac{1}{s+2} + V_2(s) = 0$$

$$V_2(s) - V_1(s) = \frac{1}{s+2}$$

$$\Rightarrow \quad V_1(s) - V_2(s) = -\frac{1}{s+2}$$

Putting the above two equations in matrix form, we get

$$\begin{bmatrix} \dfrac{s}{s+1} & \dfrac{2+s}{2s} \\ 1 & -1 \end{bmatrix} \begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix} = \begin{bmatrix} \dfrac{2}{s} \\ \dfrac{-1}{s+2} \end{bmatrix}$$

Solving for $V_2(s)$ and then applying the principle of voltage division, we get

$$V_o(s) = \frac{1}{2}V_2(s) = \frac{2(3s^2+6s+4)}{2(s+2)(3s^2+3s+2)}$$

$$\Rightarrow \qquad V_o(s) = \frac{\left(s^2+2s+\dfrac{4}{3}\right)}{(s+2)(s+0.5-j0.646)(s+0.5+j0.646)}$$

Using partial fractions, we can write

$$V_o(s) = \frac{K_1}{s+2} + \frac{K_2}{s+0.5-j0.646} + \frac{K_2^*}{s+0.5+j0.646}$$

We find that $K_1 = 0.5$

$$K_2 = 0.316\,\underline{/-37.76}$$

Hence, $$V_o(s) = \frac{0.5}{s+2} + \frac{0.316\,\underline{/-37.76}}{s+0.5-j0.646} + \frac{0.316\,\underline{/37.76}}{s+0.5+j0.646}$$

We know that, $$\mathscr{L}^{-1}\left[\frac{1}{s+a}\right] = e^{-at}u(t)$$

$$\mathscr{L}^{-1}\left[\frac{m\,\underline{/\theta}}{s+a-j\omega} + \frac{m\,\underline{/-\theta}}{s+a+j\omega}\right]$$
$$= 2me^{-at}\cos(\omega t+\theta)u(t)$$

Hence, $$\boldsymbol{v_o(t) = 0.5e^{-2t}u(t) + 0.632e^{-0.5t}\cos\,[0.646t - 37.76°]\,u(t)}$$

EXAMPLE 5.19

For the network shown in Fig. 5.28, find $v_o(t)$, $t > 0$, using mesh equations.

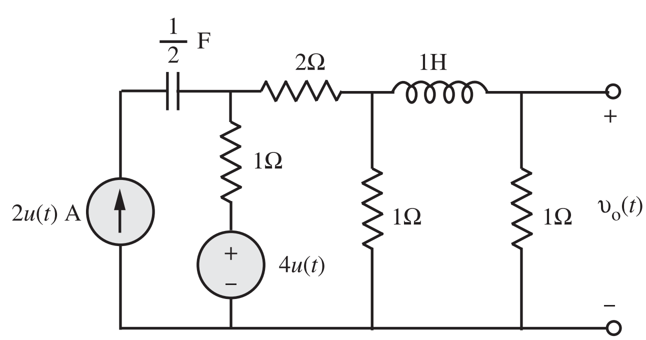


Figure 5.28

SOLUTION

The step function $u(t)$ is defined as follows.

$$u(t) = \begin{cases} 1, & t \geq 0^+ \\ 0, & t \leq 0^- \end{cases}$$

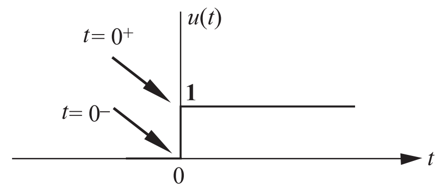


Figure 5.29(a)

Since the circuit is not energized for $t \leq 0^-$, there are no initial conditions in the circuit. For $t \geq 0^+$, the frequency domain equivalent circuit is shown in Fig. 5.29(b).

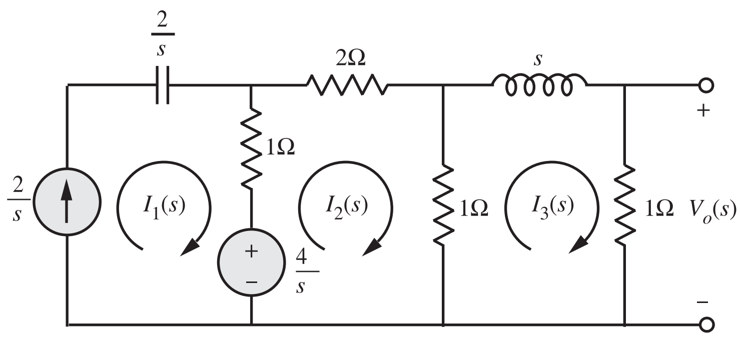


Figure 5.29(b)

By inspection, we find that $I_1(s) = \dfrac{2}{s}$

KVL clockwise for mesh 2:

$$\frac{-4}{s} + 1\left[I_2(s) - I_1(s)\right] + 2I_2(s) + 1\left[I_2(s) - I_3(s)\right] = 0$$

$$\Rightarrow \qquad \frac{-4}{s} - I_1(s) + I_2(s)\left[1 + 2 + 1\right] - I_3(s) = 0$$

Substituting the value of $I_1(s)$, we get

$$\frac{-4}{s} + 4I_2(s) - I_3(s) = \frac{2}{s}$$

$$\Rightarrow \qquad 4I_2(s) - I_3(s) = \frac{6}{s}$$

KVL clockwise for mesh 3:

$$1\left[I_3(s) - I_2(s)\right] + sI_3(s) + 1I_3(s) = 0$$

$$\Rightarrow \qquad -I_2(s) + I_3(s)\left[s + 2\right] = 0$$

Putting the *KVL equations* for mesh 2 and mesh 3 in matrix form, we get

$$\begin{bmatrix} 4 & -1 \\ -1 & s+2 \end{bmatrix} \begin{bmatrix} I_2(s) \\ I_3(s) \end{bmatrix} = \begin{bmatrix} \dfrac{6}{s} \\ 0 \end{bmatrix}$$

Solving for $I_3(s)$, using Cramer's rule, we get

$$I_3(s) = \frac{1.5}{s\left(s + \frac{7}{4}\right)}$$

$$\Rightarrow \qquad V_o(s) = I_3(s) \times 1 = \frac{1.5}{s\left(s + \frac{7}{4}\right)}$$

Using partial fractions, we can write

$$V_o(s) = \frac{K_1}{s} + \frac{K_2}{s + \frac{7}{4}}$$

We find that, $$K_1 = \frac{6}{7}, \text{ and } K_2 = \frac{-6}{7}$$

Hence, $$V_o(s) = \frac{6}{7}\left[\frac{1}{s} - \frac{1}{s + \frac{7}{4}}\right]$$

$$\Rightarrow \qquad \boldsymbol{v_o(t) = \frac{6}{7}\left[1 - e^{-\frac{7}{4}t}\right] u(t)}$$

EXAMPLE 5.20

Use mesh analysis to find $v_o(t)$, $t > 0$ in the network shown in Fig. 5.30.

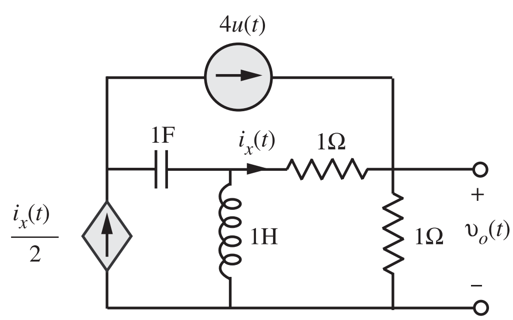


Figure 5.30

SOLUTION

The circuit is not energized for $t \leq 0^-$ because the independent current source is associated with $u(t)$. This means that there are no initial conditions in the circuit. The frequency domain circuit for $t \geq 0^+$ is shown in Fig. 5.31.

By inspection we find that:

$$I_1(s) = \frac{4}{s}, \quad I_2(s) = \frac{I_x(s)}{2}$$

$$I_x(s) = I_3(s) - \frac{4}{s} \quad \Rightarrow \quad 2I_2(s) = I_3(s) - \frac{4}{s} \quad \Rightarrow \quad I_2(s) = \frac{1}{2}\left[I_3(s) - \frac{4}{s}\right]$$

KVL for mesh 3 gives

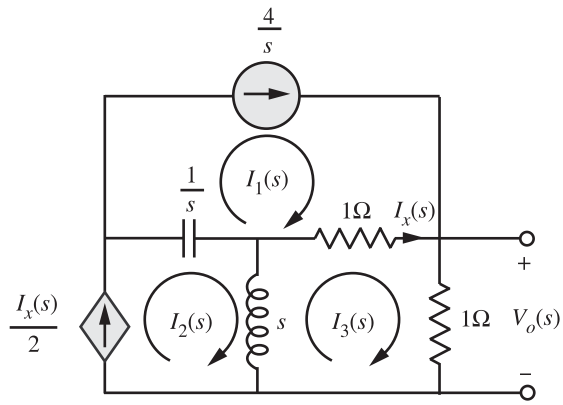


Figure 5.31

$$s\left[I_3(s)-I_2(s)\right]+1\left[I_3(s)-\frac{4}{s}\right] +1\times I_3(s)=0$$

$$\Rightarrow \qquad s\left[I_3(s)-\frac{1}{2}\left(I_3(s)-\frac{4}{s}\right)\right] +\left[I_3(s)-\frac{4}{s}\right]+I_3(s)=0$$

$$\Rightarrow \qquad I_3(s)=\frac{-4(s-2)}{s(s+4)}$$

and
$$V_o(s)=1[I_3(s)] =\frac{-4(s-2)}{s(s+4)}$$

By partial fractions, we can write

$$V_o(s)=\frac{K_1}{s}+\frac{K_2}{s+4}$$

We find that
$$K_1=2,\ K_2=-6$$

Hence,
$$V_o(s)=\frac{2}{s}-\frac{6}{s+4}$$

Taking inverse Laplace transform, we get

$$\boldsymbol{v_o(t)=2u(t)-6e^{-4t}u(t)}$$

EXAMPLE 5.21

Using the principle of superposition, find $v_o(t)$ for $t>0$. Refer the circuit shown in Fig. 5.32.

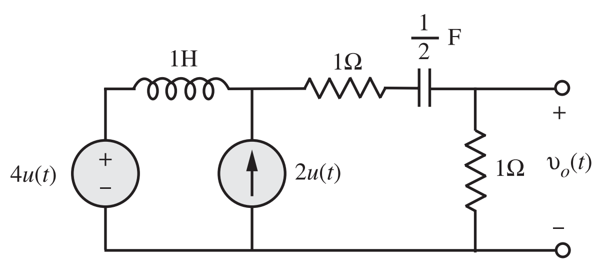


Figure 5.32

SOLUTION

Since both the independent sources are associated with $u(t)$, which is zero for $t\le 0^-$, the circuit will not have any initial conditions. The frequency domain circuit for $t\ge 0^+$ is shown in Fig. 5.33(a).

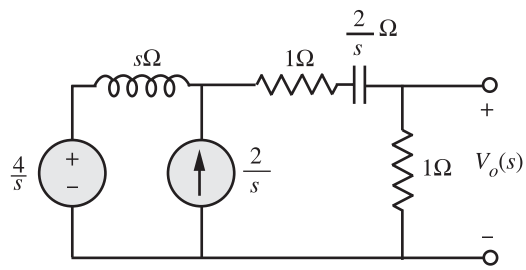


Figure 5.33(a)

As a first step, let us find the contribution to $V_o(s)$ due to voltage source alone. This needs the deactivation of the current source.

Referring to Fig. 5.33(b), we find that

$$I(s) = \frac{\frac{4}{s}}{s + 1 + \frac{2}{s} + 1}$$

$$\Rightarrow \qquad V_{o_1}(s) = I(s)[1] = \frac{4}{s^2 + 2s + 2}$$

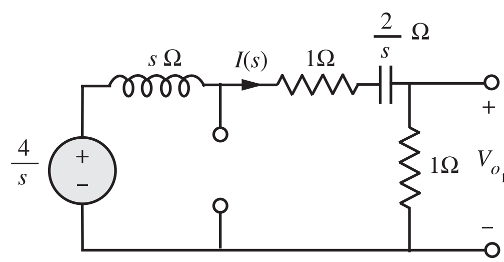


Figure 5.33(b)

Next let us find the contribution to the output due to current source alone.

Refer to Fig. 5.33(c). Using the principle of current division,

$$I_1(s) = \frac{\frac{2}{s} \times s}{s + 1 + \frac{2}{s} + 1}$$

$$\Rightarrow \qquad V_{o_2}(s) = 1\,[I_1(s)] = \frac{2s}{s^2 + 2s + 2}$$

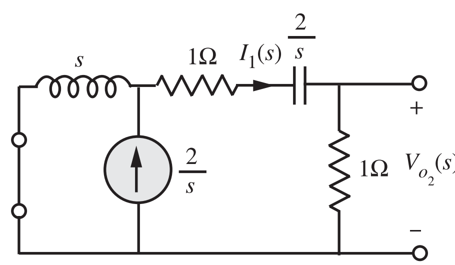


Figure 5.33(c)

Finally adding the two contributions, we get

$$\begin{aligned} V_o(s) &= V_{o_1}(s) + V_{o_2}(s) \\ &= \frac{4}{s^2 + 2s + 2} + \frac{2s}{s^2 + 2s + 2} = \frac{2s + 4}{s^2 + 2s + 2} \\ &= \frac{K_1}{s + 1 - j1} + \frac{K_1^*}{s + 1 + j1} \end{aligned}$$

We find that, $\qquad K_1 = \sqrt{2}\,\underline{/-45^\circ}$

Hence, $\qquad V_o(s) = \dfrac{\sqrt{2}\,\underline{/-45^\circ}}{s + 1 - j1} + \dfrac{\sqrt{2}\,\underline{/+45^\circ}}{s + 1 + j1}$

We know that: $\mathscr{L}^{-1}\left\{\dfrac{m\,\underline{/\theta}}{s+a-jb}+\dfrac{m\,\underline{/-\theta}}{s+a+jb}\right\}=2me^{-at}\cos(bt+\theta)u(t)$

Hence, $\boldsymbol{v_o(t)=2\sqrt{2}e^{-t}\cos(t+45^\circ)u(t)}$

EXAMPLE 5.22

(a) Convert the circuit in Fig. 5.34 to an appropriate s domain representation.

(b) Find the Thevein equivalent seen by 1Ω resistor.

(c) Analyze the simplified circuit to find an expression for $i(t)$.

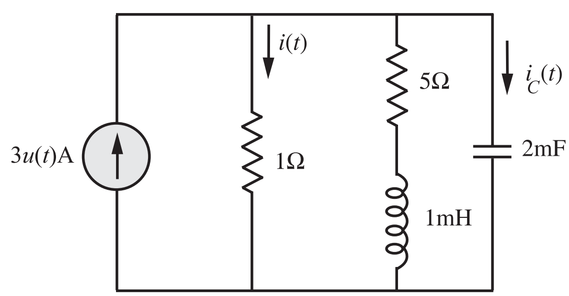


Figure 5.34

SOLUTION

(a) Since the independent current source has $u(t)$ in it, the circuit is not activated for $t \le 0^-$. In otherwords, all the initial conditions are zero. Fig.5.35 (a) shows the s domain equivalent circuit for $t \ge 0^+$.

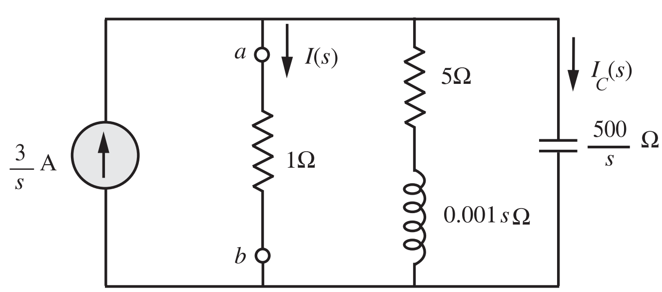


Figure 5.35(a)

(b) Sine we are interested in the current in 1Ω using the Thevenin theorem, remove the 1Ω resistor from the circuit shown in Fig. 5.35(a). The resulting circuit thus obtained is shown in Fig. 5.35(b).

$Z_t(s)$ is found by deactivating the independent current source.

$$Z_t(s) = (5 + 0.001s)||\frac{500}{s}$$
$$= \frac{2500 + 0.5s}{0.001s^2 + 5s + 500}\Omega$$

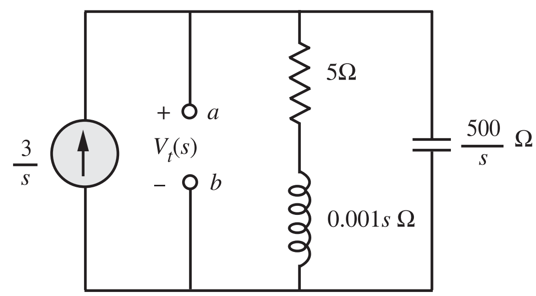


Figure 5.35(b)

Referring to Fig. 5.35 (b),

$$V_t(s) = \frac{3}{s}\,[Z_t(s)]$$
$$= \frac{7.5 \times 10^6 + 1500s}{s(s^2 + 5000s + 5 \times 10^5)}\text{Volts}$$

The Thevenin equivalent circuit along with 1Ω resistor is shown in Fig. 5.35 (c).

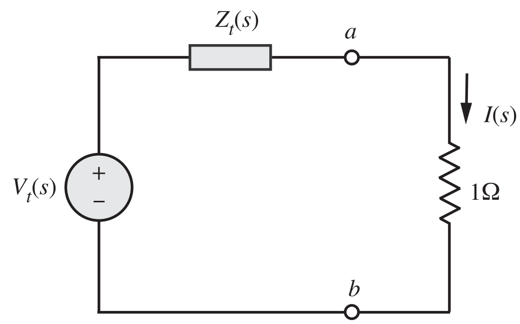


Figure 5.35(c)

$$I(s) = \frac{V_t(s)}{Z_t(s) + 1}$$
$$= \frac{7.5 \times 10^6 + 1500s}{s(s^2 + 5500s + 3 \times 10^6)}$$
$$= \frac{7.5 \times 10^6 + 1500s}{s(s + 4886)(s + 614)}$$

Using partial fractions, we get

$$I(s) = \frac{2.5}{s} + \frac{0.008}{s + 4886} - \frac{2.508}{s + 614}$$

Taking inverse Laplace transforms, we get

$$\boldsymbol{i(t) = \left(2.5 + 0.008e^{-4886t} - 2.508e^{-614t}\right) u(t)\mathbf{A}}$$

Check:

$$i(0) = 2.5 + 0.008 - 2.508 = 0$$

and

$$i(\infty) = 2.5.$$

These could be verified by evaluating $i(t)$ at $t = 0$ and $t = \infty$ using the concepts explained in Chapter 4.

EXAMPLE 5.23

Refer the RLC circuit shown in Fig. 5.36. Find the complete response for $v(t)$ if $t \geq 0^+$. Take $v(0) = 2$V.

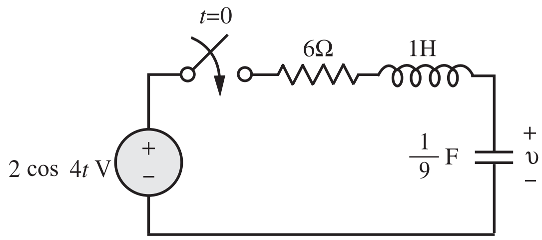


Figure 5.36

SOLUTION

Since we wish to analyze the circuit given in Fig. 5.36 using KVL, we shall represent L and C in frequency domain using series circuits to accomodate the initial conditions. Accordingly, we get the frequency domain circuit shown in Fig. 5.36 (a).

Applying KVL clockwise to the circuit shown in Fig. 5.36 (a), we get

$$\frac{-2s}{s^2+16} + \left(6 + s + \frac{9}{s}\right) I(s) + \frac{2}{s} = 0$$

$$\Rightarrow \quad I(s) = \frac{-32}{(s^2+6s+9)(s^2+16)}$$

Hence, $V(s) = I(s)\left[\frac{9}{s}\right] + \frac{2}{s}$

$$= \frac{2}{s} + \frac{-288}{s(s+3)^2(s^2+16)}$$

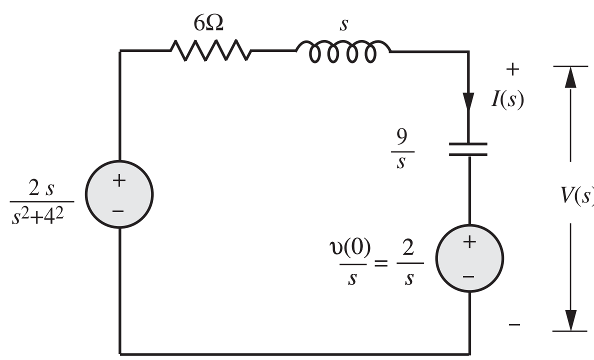


Figure 5.36(a)

Using partial fraction, we get

$$V(s) = \frac{2}{s} + \left[\frac{K_1}{s} + \frac{K_2}{s+3} + \frac{K_3}{(s+3)^2} + \frac{K_4}{s-j4} + \frac{K_4^*}{s+j4}\right]$$

Solving for K_1, K_2, K_3, and K_4, we get

$$K_1 = \left.\frac{-288}{(s+3)^2(s^2+16)}\right|_{s=0} = -2$$

$$K_2 = \frac{d}{ds}\left[\frac{-288}{s(s^2+16)}\right]_{s=-3} = 2.2$$

$$K_3 = \left.\frac{-288}{s(s^2+16)}\right|_{s=-3} = 3.84$$

$$K_4 = \left.\frac{-288}{s(s+3)^2(s+j4)}\right|_{s=j4} = 0.36\,\underline{/-106.2}$$

Therefore

$$V(s) = \frac{2}{s} - \frac{2}{s} + \frac{2.2}{s+3} + \frac{3.84}{(s+3)^2} + \frac{0.36\,\underline{/-106.2}}{s-j4} + \frac{0.36\,\underline{/106.2}}{s+j4}$$

Taking inverse Laplace Transform we get,

$$\boldsymbol{v(t) = 2.2e^{-3t} + 3.84te^{-3t} + 0.72\cos(4t - 106.2^\circ)}$$

Verification:
Putting $t = 0$ in the above equation

$$\begin{aligned} v(0) &= 2.2 + 0 + 0.72\cos(-106.2^\circ) \\ &= 2.2 - 0.2 = 2\text{V} \end{aligned}$$

(The same quantity is given in the problem)

5.8 Waveform synthesis

The three important singularity functions explained in section 5.3 are very useful as building blocks in constructing other waveforms. In this section, we illustrate the concept of waveform synthesis with a number of exmaples, and also determine expressions for these waveforms.

EXAMPLE 5.24

Express the voltage pulse shown in Fig.5.37 in terms of unit step function and then find $V(s)$. Also find $\mathscr{L}\left\{\dfrac{dv(t)}{dt}\right\}$.

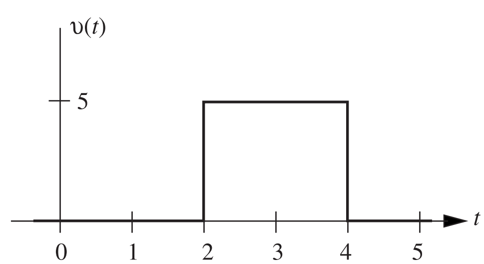


Figure 5.37

SOLUTION

The pulse shown in Fig. 5.37 is the gate function. This function may be regarded as a step function that switches on at $t = 2$ secs and switches off at $t = 4$ secs.

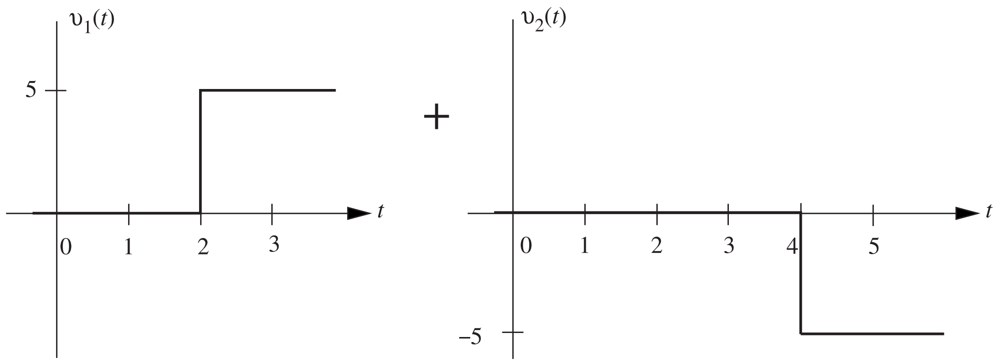


Figure 5.37(a)

Referring to Figs. 5.37 and 5.37 (a), we can write

$$v(t) = v_1(t) + v_2(t)$$

$$\Rightarrow \quad v(t) = 5u(t-2) - 5u(t-4)$$

Hence, $$V(s) = \frac{5}{s}e^{-2s} - \frac{5}{s}e^{-4s}$$

$$= \boldsymbol{\frac{5}{s}\left[e^{-2s} - e^{-4s}\right]}$$

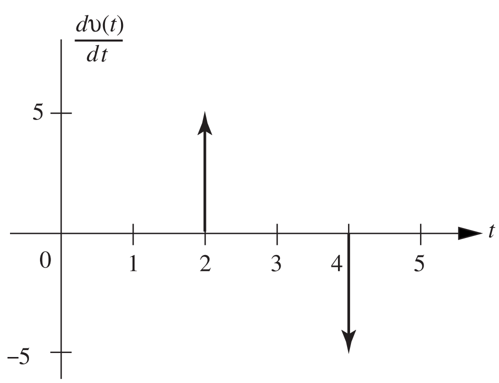


Figure 5.37(b)

Taking the derivative of $v(t)$, we get

$$\frac{dv(t)}{dt} = 5\left[\delta(t-2) - \delta(t-4)\right]$$

Fig. 5.37(b) shows the graph of $\frac{dv(t)}{dt}$.

We can obtain Fig. 5.37(b) directly from Fig. 5.36 by observing that at $t = 2$ seconds, there is a sudden rise of 5V leading to $5\delta(t-2)$. Similarly, at $t = 4$ seconds, a sudden fall of 5V leading to $-5\delta(t-4)$.

We know the Laplace trasnform pair

$$\mathscr{L}\{\delta(t-a)\} = e^{-as}\,\mathscr{L}\{\delta(t)\}$$

$$= e^{-as}$$

Hence, $$\boldsymbol{\mathscr{L}\left\{\frac{dv(t)}{dt}\right\} = 5\left[e^{-2s} - e^{-4s}\right]}$$

EXAMPLE 5.25

Express the current pulse in Fig.5.38 in terms of the unit step.

Find: (i) $\mathscr{L}\{i(t)\}$ (ii) $\mathscr{L}\{\int i(t)dt\}$.

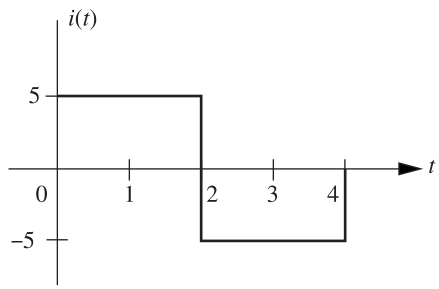


Figure 5.38

SOLUTION

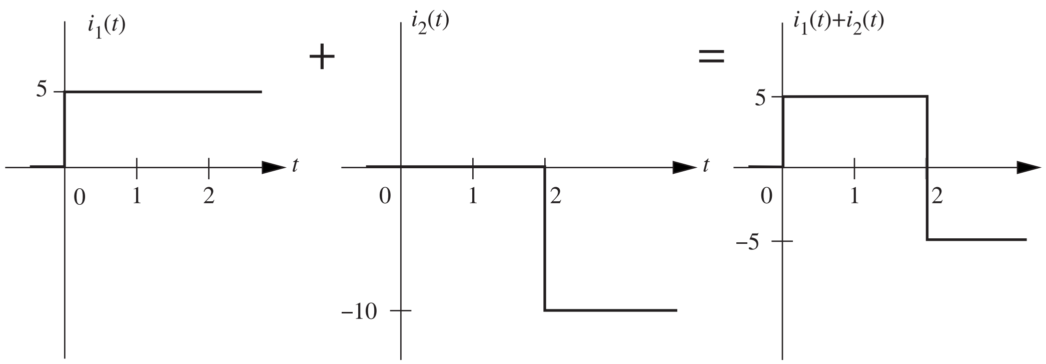


Figure 5.39(a)

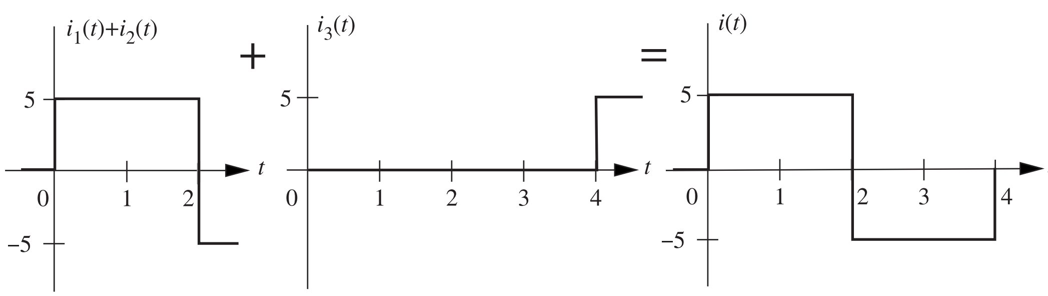


Figure 5.39(b)

Referring to Figs. 5.39 (a) and (b), using the principle of synthesis, we can write

$$\begin{aligned} i(t) &= i_1(t) + i_2(t) + i_3(t) \\ &= 5u(t) - 10u(t-2) + 5u(t-4) \end{aligned}$$

The Laplace transform of the above equation yields

$$I(s) = \frac{5}{s} - \frac{10}{s}e^{-2s} + \frac{5}{s}e^{-4s}$$
$$= \frac{5}{s}\left[1 - 2e^{-2s} + e^{-4s}\right]$$
$$= \boldsymbol{\frac{5}{s}\left[1 - e^{-2s}\right]^2}$$

Let $$f(t) = \int i(t)dt$$

then, $$f(t) = \int [5u(t) - 10u(t-2) + 5u(t-4)]dt$$
$$= 5r(t) - 10r(t-2) + 5r(t-4)$$
$$= f_1(t) + f_2(t) + f_3(t)$$

The function $f_1(t)$ is a ramp of slope = 5 as shown in Fig. 5.39 (c). To this, if we add a ramp of slope = -10, the effect of this addition is, we get a ramp of slope = 5 $-10 = -5$ for $t \geq 2$ secs till we encounter the next ramp. At $t = 4$ seconds, if we add a ramp with a slope of +5, the net slope beyond $t = 4$ seconds is $-5 + 5 = 0$. Thus figure $f(t)$ is drawn as shown in Fig. 5.39 (d).

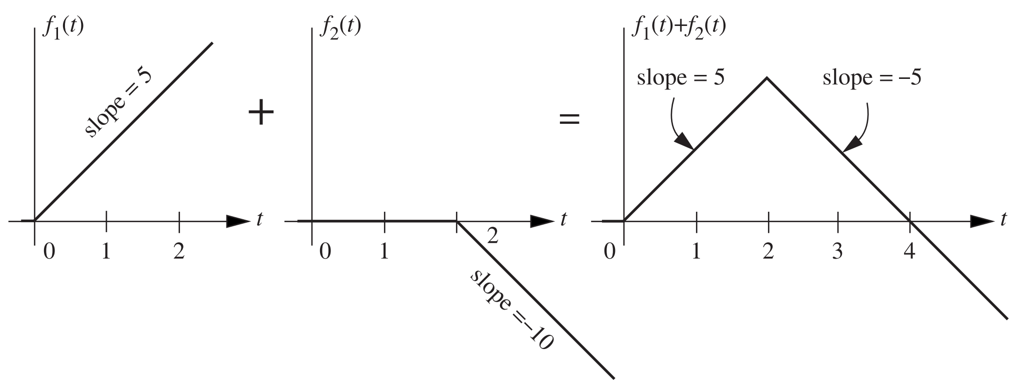


Figure 5.39(c)

$$\mathscr{L}\{f(t)\} = F(s)$$
$$= \mathscr{L}\{5r(t) - 10r(t-2) + 5r(t-4)\}$$
$$= \frac{5}{s^2} - \frac{10}{s^2}e^{-2s} + \frac{5}{s^2}e^{-4s}$$
$$= \boldsymbol{\frac{5}{s^2}\left[1 - 2e^{-2s} + e^{-4s}\right]}$$

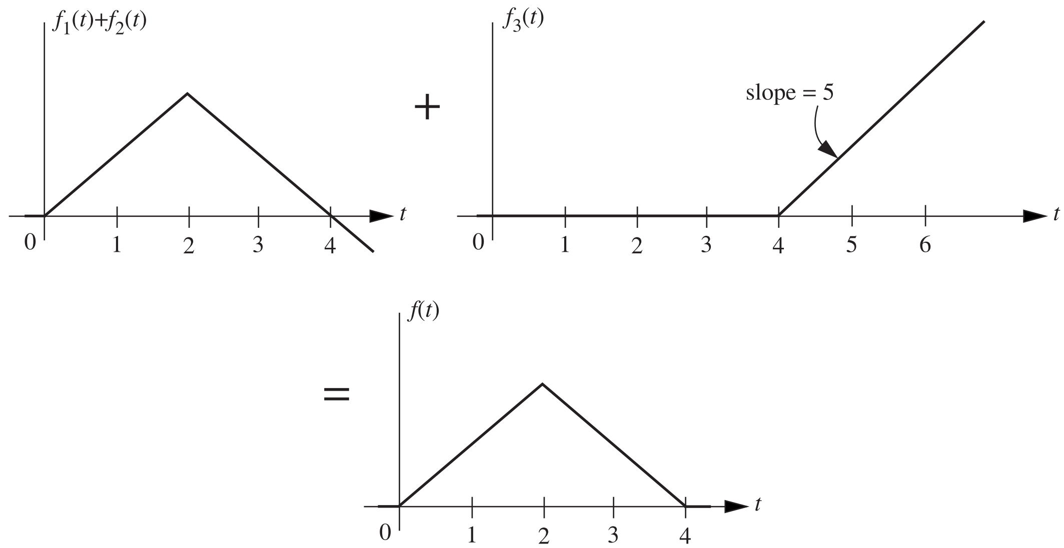


Figure 5.39(d)

EXAMPLE 5.26

Express the sawtooth function in terms of singularity functions. Then find $\mathscr{L}\{v(t)\}$.

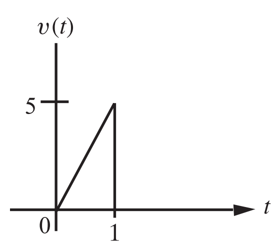


Figure 5.40

SOLUTION

There are three methods to solve this problem.

Method 1:

The function $v_1(t)$ is a ramp function of slope = +5. This slope +5 should continue till $t = 1$ second. Hence at $t = 1$ second, a ramp of slope $t = -5$ is added to $v_1(t)$. The graph of $v_1(t) + v_2(t)$ is shown in Fig. 5.41(a). Next, to $v_1(t) + v_2(t)$, a step of -5V is added at $t = 1$ second.

Hence,
$$\begin{aligned}
v(t) &= v_1(t) + v_2(t) + v_3(t) \\
&= 5r(t) - 5r(t-1) - 5u(t-1) \\
V(s) = \mathscr{L}\{f(t)\} &= \frac{5}{s^2} - \frac{5}{s^2}e^{-s} - \frac{5}{s}e^{-s} \\
&= \frac{5}{s^2}\left[1 - e^{-s} - se^{-s}\right]
\end{aligned}$$

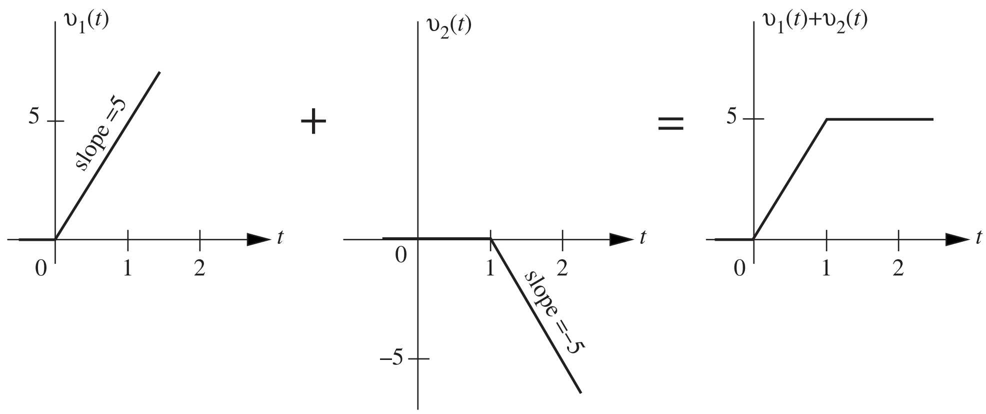


Figure 5.41(a)

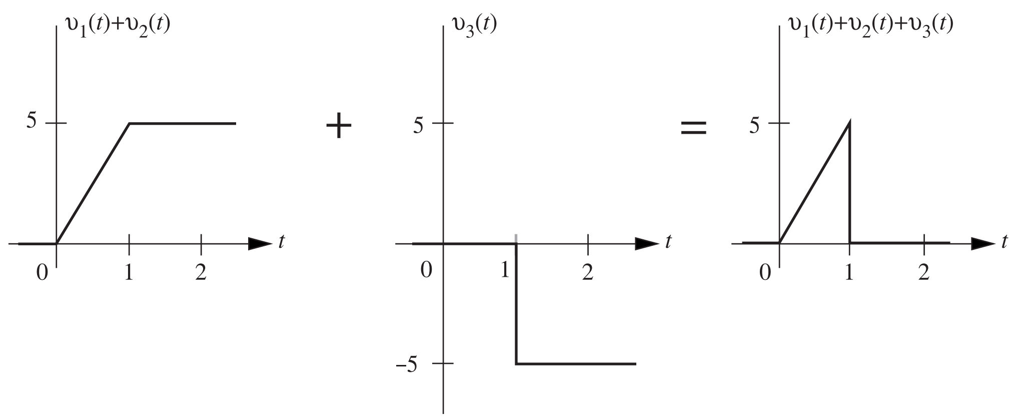


Figure 5.41(b)

Method 2:

This method involves graphical manipulation.

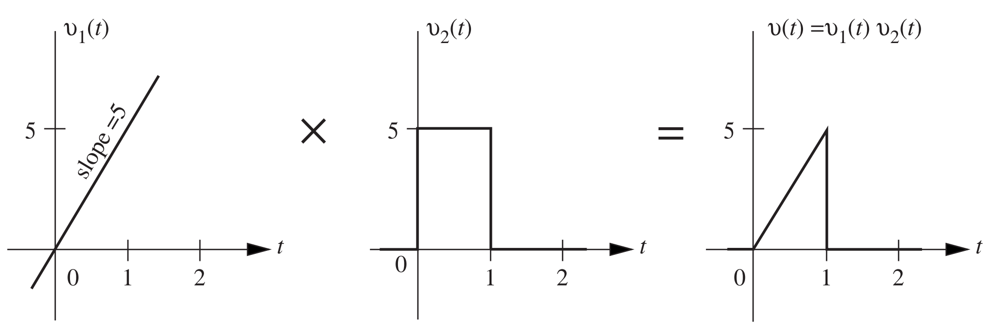


Figure 5.41(c)

The equation of a straight line passing through the origin is $y = mx$, where $m =$ slope of the line. This allows us to write $v_1(t) = 5t$. From Fig. 5.41(c), we can write

$$\begin{aligned} v(t) &= v_1(t)v_2(t) \\ &= 5t\left[u(t) - u(t-1)\right] \\ &= 5tu(t) - 5tu(t-1) \\ &= 5tu(t) - 5(t-1+1)u(t-1) \\ &= 5tu(t) - 5(t-1)u(t-1) - 5u(t-1) \\ &= 5r(t) - 5r(t-1) - 5u(t-1) \end{aligned}$$

Hence, $$\boldsymbol{V(s) = \frac{5}{s^2}\left[1 - e^{-s} - se^{-s}\right]}$$

Method 3:

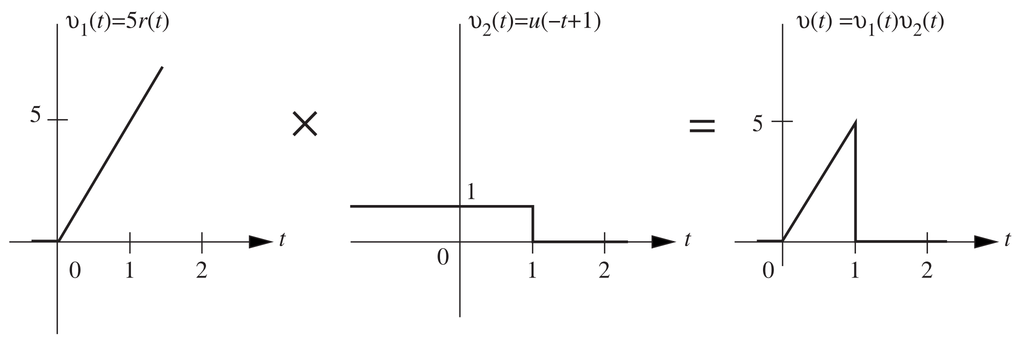


Figure 5.41(d)

This method also involves graphical manipulation. We observe from Fig. 5.41(d) that $v(t)$ is a multiplication of a ramp function and a unit step function.

Thus, $$\begin{aligned} v(t) &= v_1(t)v_2(t) \\ &= 5r(t)\left[u(-t+1)\right] \end{aligned}$$

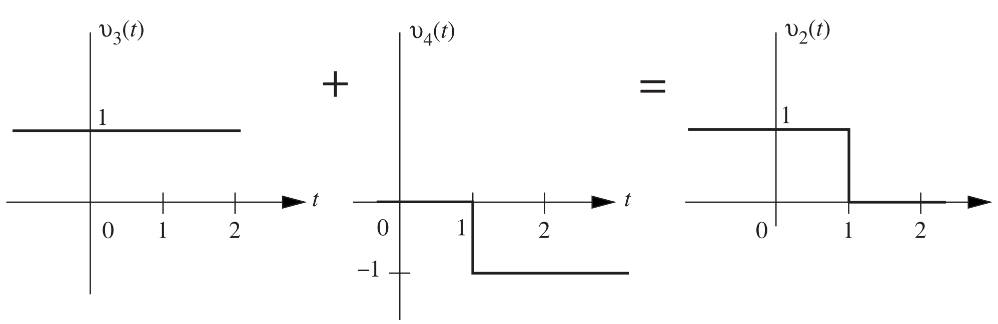


Figure 5.41(e)

From Fig. 5.41(e), we can write

$$v_2(t) = v_3(t) + v_4(t)$$

$$\Rightarrow \qquad v_2(t) = 1 - u(t-1)$$

$$\Rightarrow \qquad u(-t+1) = 1 - u(t-1)$$

Hence, $$v(t) = 5r(t)\,[1 - u(t-1)]$$
$$= 5r(t) - 5r(t)u(t-1)$$

We know that, $$r(t) = tu(t)$$

Hence, $$v(t) = 5r(t) - 5tu(t)u(t-1)$$
$$= 5r(t) - 5(t-1+1)u(t)u(t-1)$$
$$= 5r(t) - 5(t-1)u(t)u(t-1) - 5u(t)u(t-1)$$

Please note that, $u(t)u(t-1) = u(t-1)$ [Refer Fig. 5.41(f)]

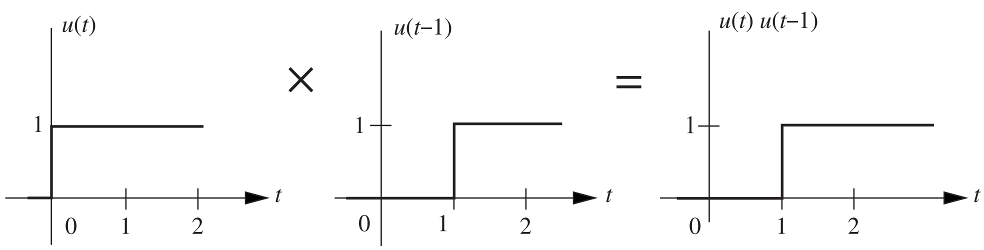


Figure 5.41(f)

Thus, $$v(t) = 5r(t) - 5(t-1)u(t-1) - 5u(t-1)$$
$$= 5r(t) - 5r(t-1) - 5u(t-1)$$

Hence, $$\boldsymbol{V(s) = \frac{5}{s^2}\left[1 - e^{-s} - se^{-s}\right]}$$

EXAMPLE 5.27

Given the signal

$$x(t) = \begin{cases} 3, & t < 0 \\ -2, & 0 < t < 1 \\ 2t-4, & t > 1 \end{cases}$$

Express $x(t)$ in terms of singularity functions. Also find $\mathscr{L}\{x(t)\}$.

SOLUTION

The signal $x(t)$ may be viewed as follows:

(i) in the interval, $t < 0$, $x(t)$ may be regarded as $3u(-t)$

(ii) in the interval, $0 < t < 1$, $x(t)$ may be viewed as $-2[u(t) - u(t-1)]$ and

(iii) for $t > 1$, $x(t)$ may be viewed as $(2t-4)u(t-1)$

$$\text{Thus,} \qquad x(t) = 3u(-t) - 2\left[u(t) - u(t-1)\right] + (2t-4)u(t-1)$$

$$\begin{aligned}\Rightarrow \qquad x(t) &= 3\left[1-u(t)\right] - 2u(t) + 2u(t-1) + 2tu(t-1) - 4u(t-1)\\ &= 3 - 5u(t) - 2u(t-1) + 2(t-1+1)u(t-1)\\ &= 3 - 5u(t) - 2u(t-1) + 2(t-1)u(t-1) + 2u(t-1)\\ &= \mathbf{3 - 5u(t) + 2r(t-1)}\end{aligned}$$

$\mathcal{L}\{x(t)\}$ cannot be found because $x(t)$ contains a constant 3 for $-\infty < t < 0$ (a noncausal signal).

EXAMPLE 5.28

Express $f(t)$ in terms of singularity functions and then find $F(s)$.

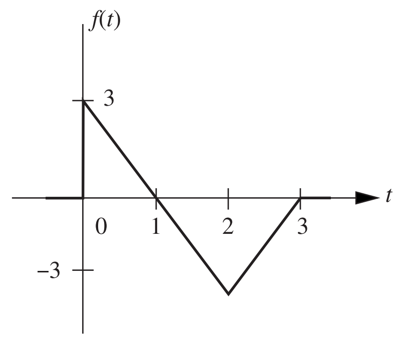


Figure 5.42

SOLUTION

To find $f(t)$ for $0 < t < 2:$
Equation of the straight line 1 is

$$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$$

Here, y is $f(t)$ and x is t.
Hence,

$$\frac{f\,(t)-3}{t-0} = \frac{-3-3}{2-0}$$

$$\Rightarrow \qquad 2f(t) - 6 = -6t$$

$$\Rightarrow \qquad f(t) = 3 - 3t$$

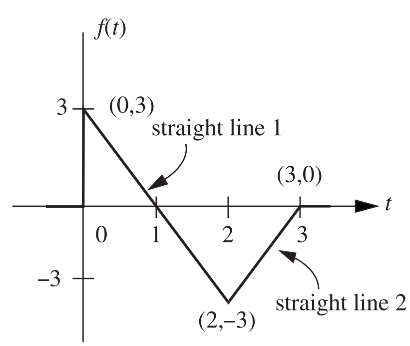


Figure 5.43

To find $f(t)$ for $2 < t < 3$:

Here,

$$\frac{f(t)+3}{t-2} = \frac{0+3}{3-2}$$

$$\Rightarrow \qquad f(t)+3 = 3t-6$$

$$\Rightarrow \qquad f(t) = 3t-9$$

Hence

$$f(t) = \begin{cases} 3-3t, & 0 < t < 2 \\ 3t-9, & 2 < t < 3 \\ 0, & \text{otherwise} \end{cases}$$

The above equation may also be written as :

$$\begin{aligned} f(t) &= [3-3t][u(t)-u(t-2)] + [3t-9][u(t-2)-u(t-3)] \\ &= 3u(t) - 3u(t-2) - 3tu(t) + 3tu(t-2) + 3tu(t-2) \\ &\quad -3tu(t-3) - 9u(t-2) + 9u(t-3) \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad f(t) &= 3u(t) - 12u(t-2) - 3tu(t) + 6tu(t-2) - 3tu(t-3) + 9u(t-3) \\ &= 3u(t) - 12u(t-2) - 3tu(t) + 6(t-2+2)u(t-2) \\ &\quad -3(t-3+3)u(t-3) + 9u(t-3) \\ &= 3u(t) - 12u(t-2) - 3tu(t) + 6(t-2)u(t-2) \\ &\quad +12u(t-2) - 3(t-3)u(t-3) - 9u(t-3) + 9u(t-3) \end{aligned}$$

$$\boldsymbol{f(t) = 3u(t) - 3tu(t) + 6(t-2)u(t-2) - 3(t-3)u(t-3)}$$

Hence,

$$\begin{aligned} F(s) &= \mathscr{L}\{f(t)\} \\ &= \boldsymbol{\frac{3}{s} - \frac{3}{s^2} + \frac{6}{s^2}e^{-2s} - \frac{3}{s^2}e^{-3s}} \end{aligned}$$

EXAMPLE 5.29

Express the function $f(t)$ shown in Fig. 5.44 using singularity functions and then find $F(s)$.

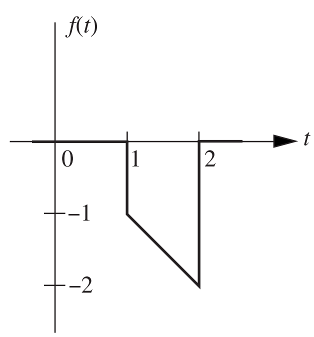


Figure 5.44

SOLUTION

Equation of the straight line shown in Fig. 5.45(a) is

$$\frac{f_1(t)+1}{t-1} = \frac{-2+1}{2-1}$$

$$\Rightarrow \qquad f_1(t)+1 = -t+1$$

$$\Rightarrow \qquad f_1(t) = -t$$

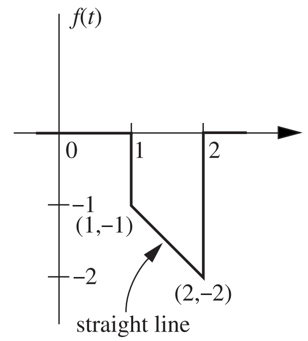


Figure 5.45(a)

The above equation is for the values t lying between 1 and 2. This could be expressed, by writing

$$f(t) = f_1(t)\,g(t)$$

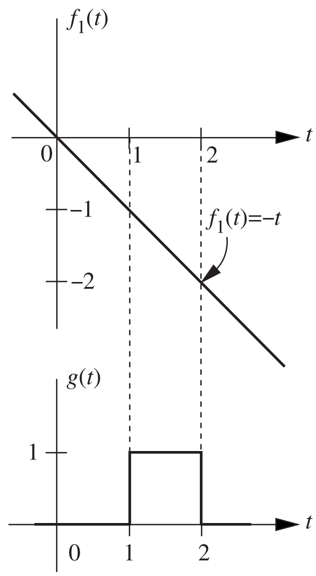


Figure 5.45(b)

$$\begin{aligned}\Rightarrow \quad f(t) &= -t\,[u(t-1)-u(t-2)] \\ &= -(t-1+1)\,u(t-1)+(t-2+2)\,u(t-2) \\ &= -(t-1)\,u(t-1)-u(t-1)+(t-2)\,u(t-2)+2u(t-2) \\ &= -r(t-1)-u(t-1)+r(t-2)+2u(t-2)\end{aligned}$$

Hence,

$$\begin{aligned} F(s) &= \mathcal{L}\{f(t)\} \\ &= \boldsymbol{-\frac{1}{s^2}e^{-s}-\frac{1}{s}e^{-s}+\frac{1}{s^2}e^{-2s}+\frac{2}{s}e^{-2s}}\end{aligned}$$

EXAMPLE 5.30

Find the Laplace transform of the function $f(t)$ shown in Fig. 5.46.

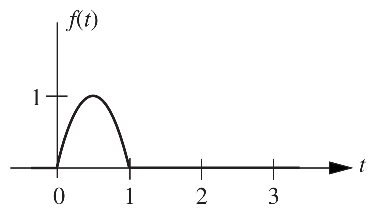


Figure 5.46

SOLUTION

Method 1:

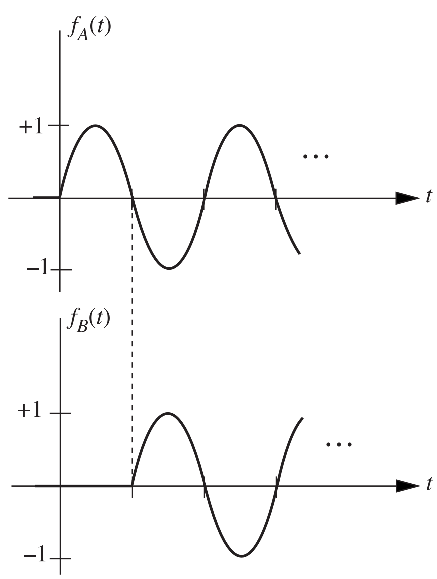


Figure 5.47(a)

We can write,

$$f(t) = f_A(t) + f_B(t)$$

$$= \sin \pi t u(t) + \sin \pi (t-1) u(t-1)$$

Hence,

$$F(s) = \mathscr{L}\{f(t)\} = \frac{\pi}{s^2+\pi^2} + \frac{\pi}{s^2+\pi^2} e^{-s}$$

$$= \boldsymbol{\frac{\pi}{s^2+\pi^2}\left[1+e^{-s}\right]}$$

Method 2 :

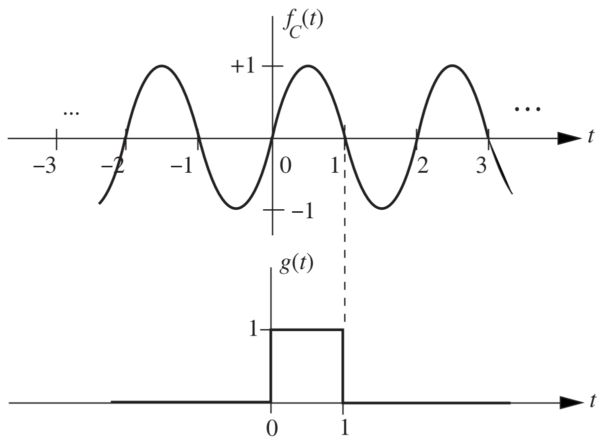


Figure 5.47(b)

Graphically, we can manipulate $f(t)$ as

$$\begin{aligned} f(t) &= f_C(t)\,g(t) \\ &= \sin \pi t\,[u(t) - u(t-1)] \\ &= \sin \pi t u(t) - \sin \pi t u(t-1) \\ &= \sin \pi t u(t) - \sin \pi t\,(t-1+1)\,[u(t-1)] \\ &= \sin \pi t u(t) - \sin(\pi(t-1)+\pi)\,u(t-1) \\ &= \sin \pi t u(t) + \sin \pi(t-1)\,u(t-1) \end{aligned}$$

Hence,
$$\begin{aligned} F(s) = \mathscr{L}\{f(t)\} &= \frac{\pi}{s^2+\pi^2} + \frac{s}{s^2+\pi^2}e^{-s} \\ &= \boldsymbol{\frac{\pi}{s^2+\pi^2}\left[1+e^{-s}\right]} \end{aligned}$$

EXAMPLE 5.31

Find the Laplace transform of the signal $x(t)$ shown in Fig. 5.48.

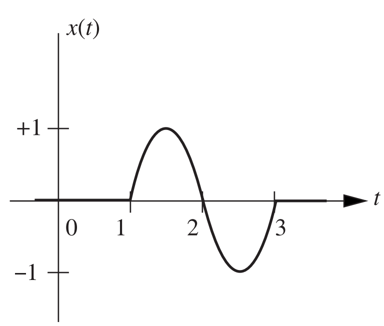


Figure 5.48

SOLUTION

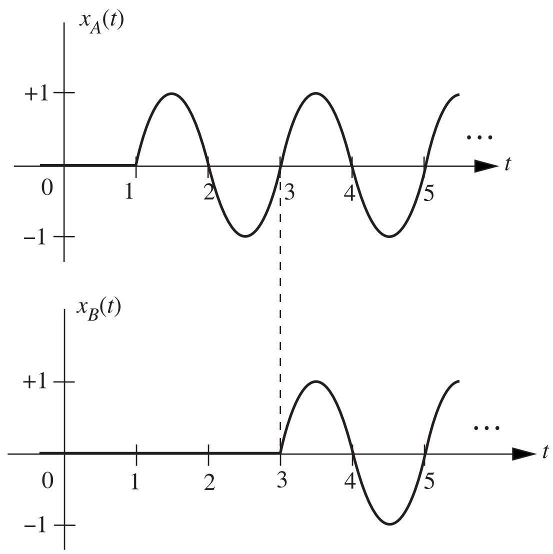


Figure 5.49

Mathematically, we can write $x(t)$ as

$$x(t) = x_A(t) - x_B(t)$$
$$= \sin \pi (t-1)\, u(t-1) - \sin \pi (t-3)\, u(t-3)$$

Hence,
$$\mathcal{L}\{x(t)\} = X(s) = \frac{\pi}{s^2+\pi^2} e^{-s} - \frac{\pi}{s^2+\pi^2} e^{-3s}$$
$$= \boldsymbol{\frac{\pi}{s^2+\pi^2}\left[e^{-s} - e^{-3s}\right]}$$

EXAMPLE 5.32

Refer the waveform shown in Fig. 5.50. The equation for the waveform is $\sin t$ from 0 to π, $-\sin t$ from π to 2π. Show that the Lapalce transform of this waveform is $F(s) = \dfrac{1}{s^2+1} \coth\left(\dfrac{\pi s}{2}\right)$.

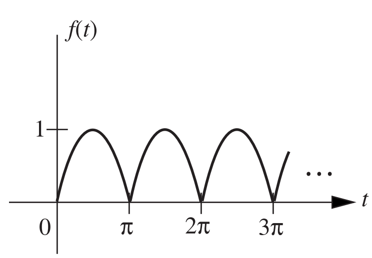


Figure 5.50

SOLUTION

$f(t)$ is a periodic waveform with a period $T = \pi$ seconds. Let $f_1(t)$ be the waveform $f(t)$ described over only one period. The Laplace transform of $f(t)$ and $f_1(t)$ are related as

$$F(s) = \frac{F_1(s)}{1 - e^{-sT}}$$

Let us now proceed to find $F_1(s)$. From Fig. 5.51 (b), we can write

$$\begin{aligned} f_1(t) &= f_A(t) + f_B(t) \\ &= \sin t u(t) + \sin(t - \pi) u(t - \pi) \\ \Rightarrow \quad F_1(s) &= \frac{1}{s^2+1} + \frac{1}{s^2+1} e^{-\pi s} \\ &= \frac{(1 + e^{-\pi s})}{s^2+1} \end{aligned}$$

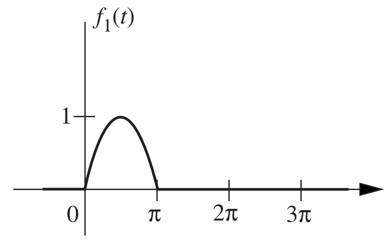


Figure 5.51(a)

$$\text{Hence,} \quad F(s) = \frac{F_1(s)}{1 - e^{-sT}} = \frac{F_1(s)}{1 - e^{-\pi s}}$$

$$\begin{aligned} \Rightarrow \quad F(s) &= \frac{(1 + e^{-\pi s})}{(s^2+1)(1 - e^{-\pi s})} \\ &= \frac{1}{s^2+1} \, \frac{\cancel{e^{-\pi s/2}} \, \dfrac{\left[e^{\pi s/2} + e^{-\pi s/2}\right]}{2}}{\cancel{e^{-\pi s/2}} \, \dfrac{\left[e^{\pi s/2} - e^{-\pi s/2}\right]}{2}} \\ \Rightarrow \quad F(s) &= \frac{1}{s^2+1} \, \frac{\cosh\left(\dfrac{\pi s}{2}\right)}{\sinh\left(\dfrac{\pi s}{2}\right)} \\ &= \boldsymbol{\frac{1}{s^2+1} \coth\left(\frac{\pi s}{2}\right)} \end{aligned}$$

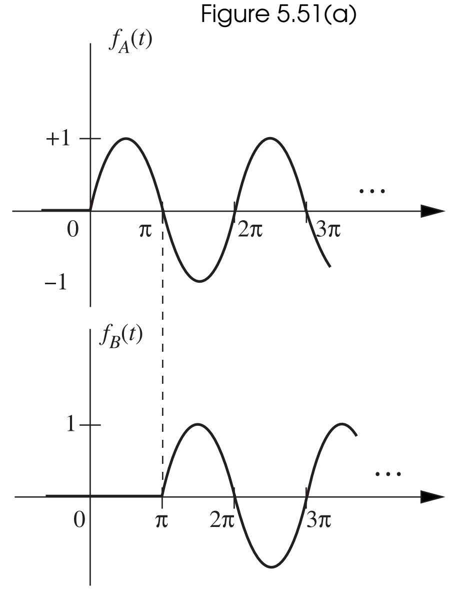


Figure 5.51(b)

EXAMPLE 5.33

Find the Laplace transform of the pulse shown in Fig. 5.52.

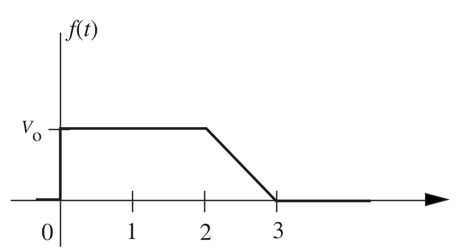


Figure 5.52

SOLUTION

We can describe Fig. 5.52 mathematically as

$$f(t) = \begin{cases} V_o, & 0 < t < 2 \\ -V_o t + 3V_o, & 2 < t < 3 \end{cases}$$

The expression for $f(t)$ for $2 < t < 3$ is obtianed as follows :
Equation of a straight line between two points is given by

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

In the present context, $y = f(t), x = t, (x_1, y_1) = (2, V_o)$ and $(x_2, y_2) = (3, 0)$

Hence, $$\frac{f(t) - V_o}{t - 2} = \frac{0 - V_o}{3 - 2}$$

$$\Rightarrow \quad f(t) = -V_o t + 3V_o$$

The time domain expression for $f(t)$ between $t = 0$ and 3 could be written using graphical manipulation as

$$f(t) = V_o [u(t) - u(t-2)] + [-V_o t + 3V_o][u(t-2) - u(t-3)]$$

The first term on the right-side of the above equation defines $f(t)$ for $0 < t < 2$ and the second term on the right-side defines $f(t)$ for $2 < t < 3$.

$$\begin{aligned} f(t) &= V_o u(t) - V_o u(t-2) - V_o t u(t-2) + V_o t u(t-3) + 3V_o u(t-2) - 3V_o u(t-3) \\ &= V_o u(t) - V_o u(t-2) - V_o (t-2+2) u(t-2) \\ &\quad + V_o (t-3+3) u(t-3) + 3V_o u(t-2) - 3V_o u(t-3) \\ &= V_o u(t) - V_o u(t-2) - V_o (t-2) u(t-2) - 2V_o u(t-2) + V_o (t-3) u(t-3) \\ &\quad + 3V_o u(t-3) + 3V_o u(t-2) - 3V_o u(t-3) \\ &= V_o u(t) - V_o (t-2) u(t-2) + V_o (t-3) u(t-3) \end{aligned}$$

$$\Rightarrow \quad f(t) = V_o u(t) - V_o r(t-2) + V_o r(t-3)$$

Hence, $$\begin{aligned} F(s) &= \mathscr{L}\{f(t)\} \\ &= \boldsymbol{\frac{V_o}{s} - \frac{V_o}{s^2} e^{-2s} + \frac{V_o}{s^2} e^{-3s}} \end{aligned}$$

EXAMPLE 5.34

Consider a staircase waveform which extends to infinity and at $t = nt_0$ jumps to the value $n + 1$, being a superposition of unit step functions. Determine the Laplace transform of this waveform.

SOLUTION

We can write,

$$f(t) = u(t) + u(t - t_0) + u(t - 2t_0) + \cdots$$

$$F(s) = \mathcal{L}\{f(t)\} = \frac{1}{s} + \frac{1}{s}e^{-t_0 s} + \frac{1}{s}e^{-2t_0 s} + \cdots$$

$$= \frac{1}{s}\left[1 + e^{-t_0 s} + e^{-2t_0 s} + \cdots\right]$$

Let $e^{-t_0 s} = x$

then $F(s) = \dfrac{1}{s}\left[1 + x + x^2 + \cdots\right]$

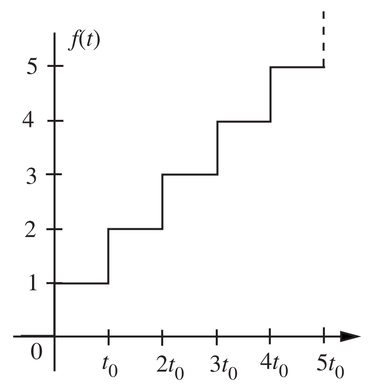


Figure 5.53

From Binomial theorem, we have

$$(1 - x)^{-1} = 1 + x + x^2 + \cdots$$

Hence, $$F(s) = \frac{1}{s(1 - x)}$$

$$= \boldsymbol{\frac{1}{s\left(1 - e^{-t_0 s}\right)}}$$

EXAMPLE 5.35

(a) Find the Laplace transform of the staircase waveform shown in Fig. 5.54. (b) If this voltage were applied to an RL series circuit with $R = 1\Omega$ and $L = 1H$, find the current $i(t)$.

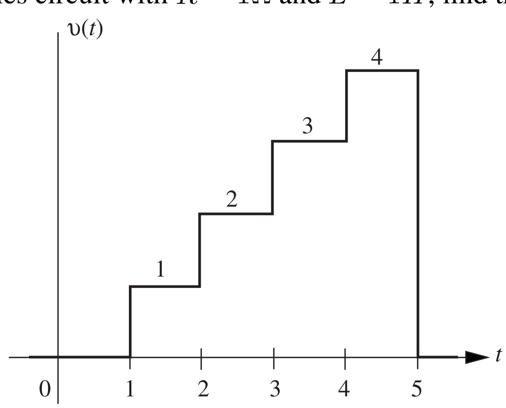


Figure 5.54

SOLUTION

(a) We can express mathematically, the voltage waveform shown in Fig. 5.54 as,

$$v(t) = \begin{cases} 1, & 1 < t < 2 \\ 2, & 2 < t < 3 \\ 3, & 3 < t < 4 \\ 4, & 4 < t < 5 \\ 0, & \text{elsewhere} \end{cases}$$

or $$v(t) = [u(t-1) - u(t-2)] + 2[u(t-2) - u(t-3)]$$
$$+3[u(t-3) - u(t-4)] + 4[u(t-4) - u(t-5)]$$
$$= u(t-1) + u(t-2) + u(t-3) + u(t-4) - 4u(t-5)$$

Taking the Laplace transform, we get

$$\boldsymbol{V(s) = \frac{1}{s}\left[e^{-s} + e^{-2s} + e^{-3s} + e^{-4s} - 4e^{-5s}\right]}$$

(b) Assuming all initial conditions to be zero, the time domian circuit shown in Fig. 5.55 gets transformed to a circuit as shown in Fig. 5.56.

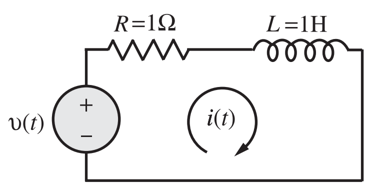


Figure 5.55 Time Domain Circuit

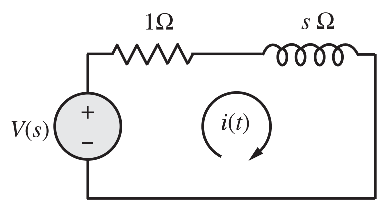


Figure 5.56 Frequency Domain Circuit

From Fig. 5.56, we can write

$$I(s) = \frac{V(s)}{s+1}$$

$$\Rightarrow\ I(s) = \frac{1}{s(s+1)}e^{-s} + \frac{1}{s(s+1)}e^{-2s} + \frac{1}{s(s+1)}e^{-3s} + \frac{1}{s(s+1)}e^{-4s} - \frac{4}{s(s+1)}e^{-5s}$$

$$\Rightarrow\ I(s) = \left[\left(\frac{1}{s} - \frac{1}{s+1}\right)e^{-s} + \left(\frac{1}{s} - \frac{1}{s+1}\right)e^{-2s} + \left(\frac{1}{s} - \frac{1}{s+1}\right)e^{-3s}\right.$$
$$\left. + \left(\frac{1}{s} - \frac{1}{s+1}\right)e^{-4s} - 4\left(\frac{1}{s} - \frac{1}{s+1}\right)e^{-5s}\right]$$

Taking the inverse Laplace transform, we get

$$i(t) = \left[u(t) - e^{-t}u(t)\right]_{t\to t-1} + \left[u(t) - e^{-t}u(t)\right]_{t\to t-2} + \left[u(t) - e^{-t}u(t)\right]_{t\to t-3}$$
$$+ \left[u(t) - e^{-t}u(t)\right]_{t\to t-4} - 4\left[u(t) - e^{-t}u(t)\right]_{t\to t-5}$$

$$\boldsymbol{\Rightarrow i(t) = \left[1 - e^{-(t-1)}\right]u(t-1) + \left[1 - e^{-(t-2)}\right]u(t-2) + \left[1 - e^{-(t-3)}\right]u(t-3)}$$
$$\boldsymbol{+ \left[1 - e^{-(t-4)}\right]u(t-4) - 4\left[1 - e^{-(t-5)}\right]u(t-5)}$$

EXAMPLE 5.36

A voltage pulse of 10 V magnitude and 5 μsec duration is applied to the RC network shown in Fig. 5.57. Find the current $i(t)$ if $R = 10\Omega$ and $C = 0.05\mu F$.

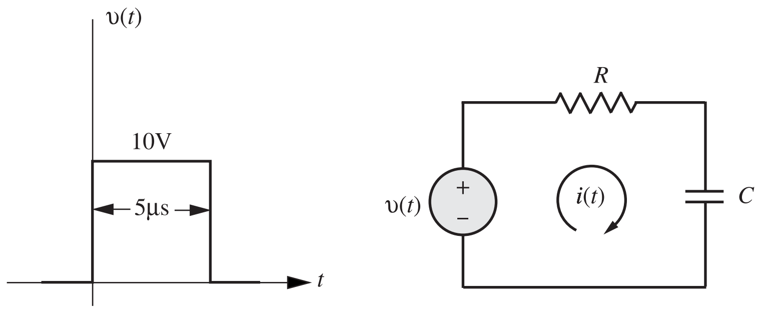


Figure 5.57

SOLUTION

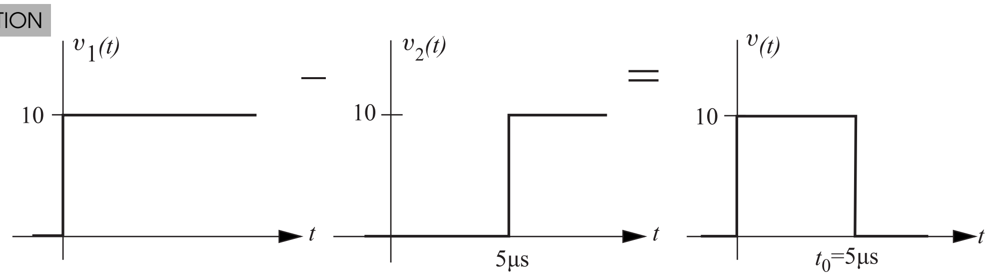


Figure 5.58(a)

Mathematically, we can express $v(t)$ as follows :

$$v(t) = v_1(t) - v_2(t)$$
$$= 10u(t) - 10u(t - t_0)$$

Hence,
$$V(s) = \mathscr{L}\{v(t)\}$$
$$= \frac{10}{s}\left[1 - e^{-t_0 s}\right]$$

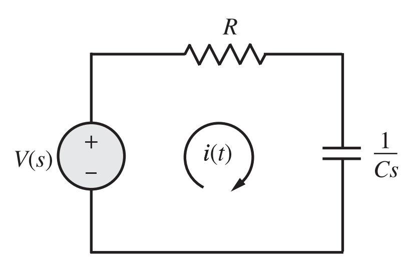


Figure 5.58(b)

Assuming all initial conditions to be zero, the Laplace transformed network is as shown in Fig. 5.58(b).

$$I(s) = \frac{V(s)}{R + \frac{1}{Cs}}$$
$$= \frac{10\left(1 - e^{-t_0 s}\right)}{s\left[R + \frac{1}{Cs}\right]}$$

$$I(s) = \frac{10Cs}{s(RCs+1)}\left(1 - e^{-t_0 s}\right)$$
$$= \frac{10}{R}\frac{1}{s + \dfrac{1}{RC}}\left(1 - e^{-t_0 s}\right)$$
$$= \frac{10}{R}\left[\frac{1}{s + \dfrac{1}{RC}} - \frac{1}{s + \dfrac{1}{RC}}e^{-t_0 s}\right]$$

Taking inverse Laplace transform yields

$$i(t) = \frac{10}{R}e^{\frac{-t}{RC}}u(t) - \left.\frac{10}{R}e^{\frac{-t}{RC}}u(t)\right|_{t\to t-t_0}$$
$$= \frac{10}{R}e^{\frac{-t}{RC}}u(t) - \frac{10}{R}e^{\frac{-(t-t_0)}{RC}}u(t-t_0)$$
$$\boldsymbol{i(t) = e^{\frac{-t}{0.5\times10^{-6}}}u(t) - e^{\frac{-\left(t-5\times10^{-6}\right)}{0.5\times10^{-6}}}u\left(t - 5\times10^{-6}\right)}$$

EXAMPLE 5.37

Find the Laplace transform of the waveform shown in Fig. 5.59.

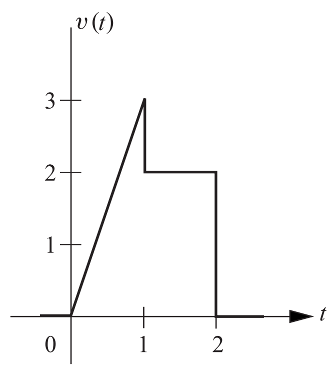


Figure 5.59

SOLUTION

$$v(t) = \begin{cases} 3t, & 0 < t < 1 \\ 2, & 1 < t < 2 \end{cases}$$

or
$$v(t) = 3t\left[u(t) - u(t-1)\right] + 2\left[u(t-1) - u(t-2)\right]$$
$$= 3tu(t) - 3tu(t-1) + 2u(t-1) - 2u(t-2)$$
$$= 3tu(t) - 3(t-1+1)u(t-1) + 2u(t-1) - 2u(t-2)$$
$$= 3tu(t) - 3(t-1)u(t-1) - 3u(t-1) + 2u(t-1) - 2u(t-2)$$

$$v(t) = 3tu(t) - 3(t-1)u(t-1) - u(t-1) - 2u(t-2)$$
$$= 3r(t) - 3r(t-1) - u(t-1) - 2u(t-2)$$

Hence,
$$V(s) = \mathscr{L}\{v(t)\}$$
$$= \boldsymbol{\frac{3}{s^2} - \frac{3}{s^2}e^{-s} - \frac{1}{s}e^{-s} - \frac{2}{s}e^{-2s}}$$

5.9 The System function

The system function or transfer function of a linear time-invariant system is defined as the ratio of Laplace transform of the output to Laplace transform of the input under the assumption that all initial conditions are zero.

Hence, for relaxed LTI system, the response $Y(s)$ to an input $X(s)$ is $H(s)\,X(s)$, where $H(s)$ is the system function. The system function $H(s)$ may be found in several ways:

1. For a system defined by a linear differential equation, by taking Laplace transform of the differential equation and then finding the ratio $\dfrac{Y(s)}{X(s)}$.
2. From the Laplace transform of impulse response $h(t)$.
3. From the s domain model of a physical system like an electrical system.

EXAMPLE 5.38

The output $y(t)$ of an LTI system is found to be $e^{-3t}u(t)$ when the input $x(t)$ is $0.5u(t)$.

(a) Find the impulse response $h(t)$ of the system.

(b) Find the output when the input is $x(t) = e^{-t}u(t)$.

SOLUTION

(a) Taking Laplace transforms of $x(t)$ and $y(t)$, we get

$$Y(s) = \frac{1}{s+3},\; X(s) = \frac{0.5}{s}$$

Hence
$$H(s) = \frac{Y(s)}{X(s)} = \frac{2s}{s+3}$$

$\Rightarrow$
$$H(s) = \frac{2(s+3)-6}{(s+3)} = 2 - \frac{6}{s+3}$$

Taking inverse Laplace transform, we get

$$\boldsymbol{h(t) = 2\delta(t) - 6e^{-3t}u(t)}$$

(b) $$x(t) = e^{-t}u(t)$$

$$\Rightarrow \qquad X(s) = \frac{1}{s+1}$$

Thus,
$$Y(s) = X(s)H(s)$$
$$= \frac{2s}{(s+1)(s+3)}$$
$$= \frac{K_1}{s+1} + \frac{K_2}{s+3}$$

where
$$K_1 = \left.\frac{2s}{s+3}\right|_{s=-1} = -1$$
$$K_2 = \left.\frac{2s}{s+1}\right|_{s=-3} = 3$$

Therefore,
$$Y(s) = \frac{-1}{s+1} + \frac{3}{s+3}$$

Taking inverse Laplace transform of $Y(s)$, we get

$$y(t) = -e^{-t} + 3e^{-3t},\ t \geq 0$$

or
$$y(t) = \left(-e^{-t} + 3e^{-3t}\right), u(t)$$

EXAMPLE 5.39

Determine the output $v(t)$ for the circuit shown in Fig. 5.60.

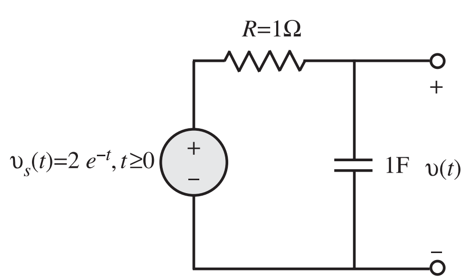


Figure 5.60

SOLUTION

The transformed network of Fig. 5.60 with the assumption that all initial conditions are zero is shown in Fig. 5.61(a).

$$V(s) = \frac{1}{s} I(s)$$

$$= \frac{1}{s}\left[\frac{V_s(s)}{1+\dfrac{1}{s}}\right]$$

$$\Rightarrow \qquad H(s) = \frac{V(s)}{V_s(s)} = \frac{1}{s+1}$$

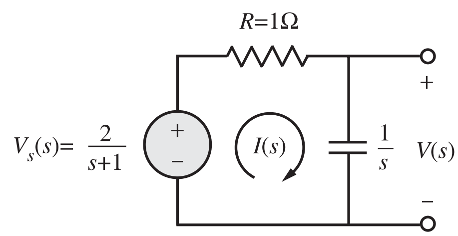


Figure 5.61(a)

The inverse Laplace transform of $H(s)$ is called the impulse response of the circuit and is denoted by $h(t)$.

$$h(t) = e^{-t} u(t)$$

I method :

From Convolution theorem, we have,

$$v(t) = h(t) * v_s(t)$$

$$= \int_0^\infty h(\tau)\, v_s(t-\tau)\, d\tau$$

$$= \int_0^\infty e^{-\tau} u(\tau) \times 2e^{-(t-\tau)} u(t-\tau)\, d\tau$$

$$= 2e^{-t} \int_0^\infty u(\tau)\, u(t-\tau)\, d\tau$$

Let us compute the product $u(\tau)\, u(t-\tau)$ for different values of τ

$$u(\tau) = \begin{cases} 1, & \tau < 0 \\ 0, & \tau > 0 \end{cases}$$

$$u(t-\tau) = \begin{cases} 1, & t-\tau > 0 \quad \text{or} \quad \tau < t \\ 0, & t-\tau < 0 \quad \text{or} \quad \tau > t \end{cases}$$

Hence, $$u(\tau)\, u(t-\tau) = \begin{cases} 1, & 0 < \tau < t, \;\; t > 0 \\ 0, & \quad \text{otherwise} \end{cases}$$

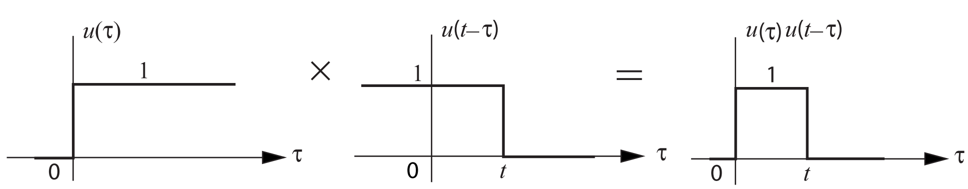


Figure 5.61(b)

Therefore, $$v\,(t) = 2e^{-t}\int_{0}^{t} d\tau = 2te^{-t},\ t \geq 0$$
$$= \boldsymbol{2te^{-t}u\,(t)}$$

II method :

In the frequency domain, convolution operation is transformed into a multiplicative operation.

That is, $$V\,(s) = H\,(s)\,V_s\,(s)$$
$$= \frac{1}{(s+1)} \times \frac{2}{(s+1)}$$
$$= \frac{2}{(s+1)^2}$$

Inverse Laplace transform yields,

$$\boldsymbol{v(t) = 2te^{-t}u(t)\mathbf{Volts}}$$

Reinforcement problems

R.P 5.22

(a) Find $H\,(s) = \dfrac{V_o\,(s)}{V_i\,(s)}$ for the circuit shown in Fig. R.P. 5.22. (b) Determine $v_o(t)$ when the intital current in the inductor is zero.

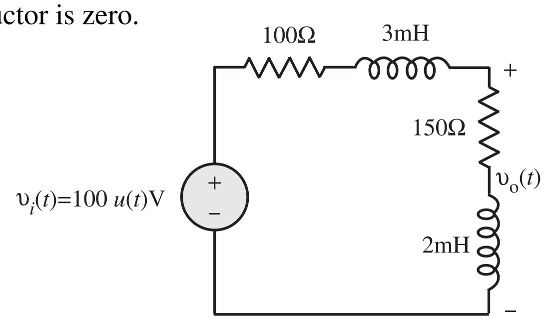


Figure R.P.5.22

SOLUTION

The Laplace transformed network with all initial conditions set to zero is shown in Fig. R.P. 5.22(a).

$$V_o\,(s) = I\,(s)\left[150 + 2\times10^{-3}s\right]$$
$$= \frac{V_i\,(s)\left[150 + 2\times10^{-3}s\right]}{100 + 3\times10^{-3}s + 150 + 2\times10^{-3}s}$$
$$\Rightarrow \qquad H\,(s) = \frac{V_o\,(s)}{V_i\,(s)} = \boldsymbol{\frac{1.5\times10^5 + 2s}{2.5\times10^5 + 5s}}$$

(b) $V_o(s) = H(s)\, V_i(s)$

$$= \frac{1.5 \times 10^5 + 2s}{2.5 \times 10^5 + 5s} \times \frac{100}{s}$$

$$= \frac{40\left[s + 0.75 \times 10^5\right]}{s\left[s + 0.5 \times 10^5\right]}$$

$$= \frac{K_1}{s} + \frac{K_2}{s + 0.5 \times 10^5}$$

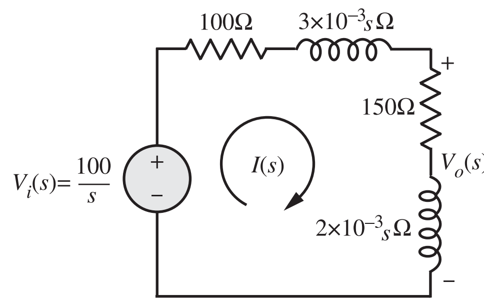


Figure R.P. 5.22(a)

where $$K_1 = \left.\frac{40\left[s + 0.75 \times 10^5\right]}{\left[s + 0.5 \times 10^5\right]}\right|_{s=0} = 60$$

$$K_2 = \left.\frac{40\left[s + 0.75 \times 10^5\right]}{s}\right|_{s=-0.5\times 10^5} = -20$$

Hence, $$V_o(s) = \frac{60}{s} - \frac{20}{s + 0.5 \times 10^5}$$

Taking inverse Laplace transform, we get

$$\boldsymbol{v_o(t) = \left[60 - 20e^{-0.5\times 105t}\right] u(t)\ \textbf{Volts}}$$

R.P 5.23

Refer the circuit shown in Fig. R.P. 5.23. The switch closes at $t = 0$. Determine the voltage $v(t)$ after the switch closes.

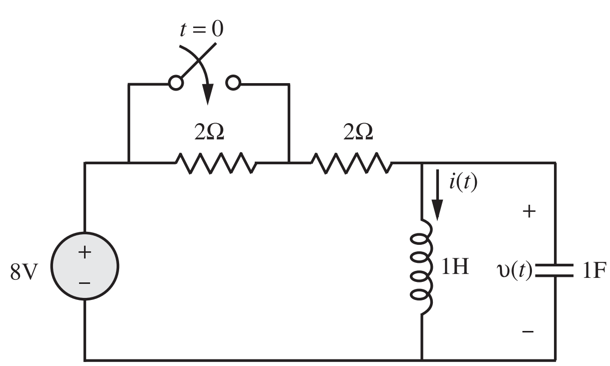


Figure R.P. 5.23

SOLUTION

The switch is open at $t = 0^-$ and closed at $t = 0^+$. Let us assume that at $t = 0^-$, the circuit is in steady state. The circuit at $t = 0^-$ is shown in Fig. R.P. 5.23(a).

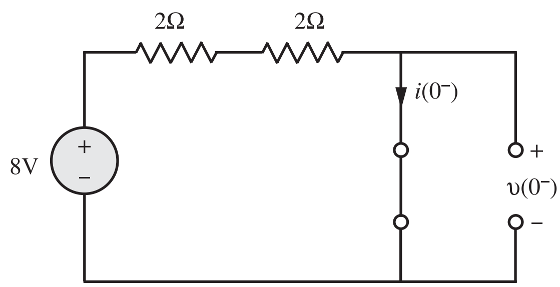


Figure R.P. 5.23(a)

Referring to Fig. R.P. 5.23(a), we get

$$i\left(0^-\right) = \frac{8}{2+2} = 2\text{A}$$
$$v\left(0^-\right) = 0$$

From switching principles, we know that the current through an inductor and the voltage across a capacitor cannot change instantaneously. Therefore,

$$i\left(0\right) = i\left(0^+\right) = i\left(0^-\right) = 2\text{A}$$

and
$$v\left(0\right) = v\left(0^+\right) = v\left(0^-\right) = 0\text{V}$$

We shall solve this probelm using nodal technique. Hence, in the frequency domain, we will use the parallel models for the capacitor and inductor because the parallel models contain current sources rather than voltage sources. The frequency domain circuit is shown in Fig. R.P. 5.23(b).

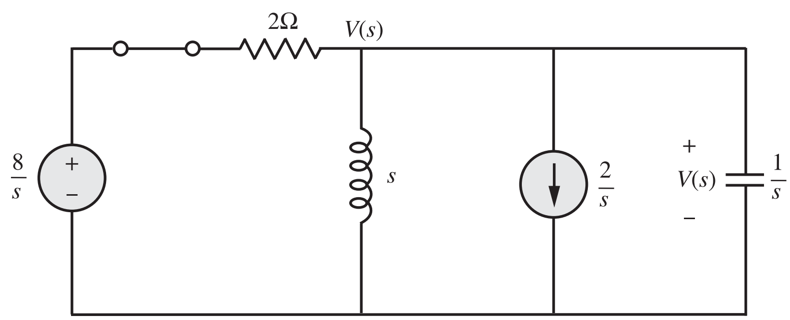


Figure R.P.5.23(b)

KCL at node $V(s)$:

$$\frac{V(s) - \dfrac{8}{s}}{2} + \frac{V(s)}{s} + \frac{V(s)}{\dfrac{1}{s}} + \frac{2}{s} = 0$$

$$\Rightarrow \qquad V\left(s\right)\left[\frac{1}{2} + \frac{1}{s} + s\right] = \frac{4}{s} - \frac{2}{s}$$

$$\Rightarrow \qquad V\left(s\right)\left[\frac{s+2+2s^2}{2s}\right] = \frac{2}{s}$$

$$\Rightarrow \qquad V(s) = \frac{4}{2s^2 + s + 2}$$

$$= \frac{2}{s^2 + 0.5s + 1}$$

$$= \frac{2}{s^2 + 0.5s + (0.25)^2 - (0.25)^2 + 1}$$

$$= \frac{2}{(s + 0.25)^2 + (0.96824)^2}$$

$$= \frac{2}{0.96824} \times \frac{0.96824}{(s + 0.25)^2 + (0.96824)^2}$$

$$= 2.066 \times \frac{0.96824}{(s + 0.25)^2 + (0.96824)^2}$$

We know that,

$$\mathscr{L}^{-1}\left\{\frac{a}{(s+b)^2 + a^2}\right\} = e^{-bt} \sin at\, u(t)$$

Hence,

$$\boldsymbol{v(t) = 2.066e^{-0.25t} \sin(0.96824t)\, u(t)} \textbf{ Volts}$$

R.P 5.24

Find the impulse response of the circuit shown in Fig. R.P. 5.24.

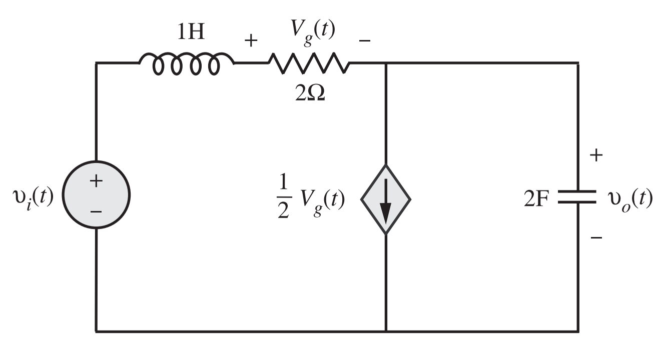


Figure R.P. 5.24

SOLUTION

The frequency domain representation of the circuit is shown in Fig. R.P. 5.24(a) by assuming that all initial conditions to be zero.

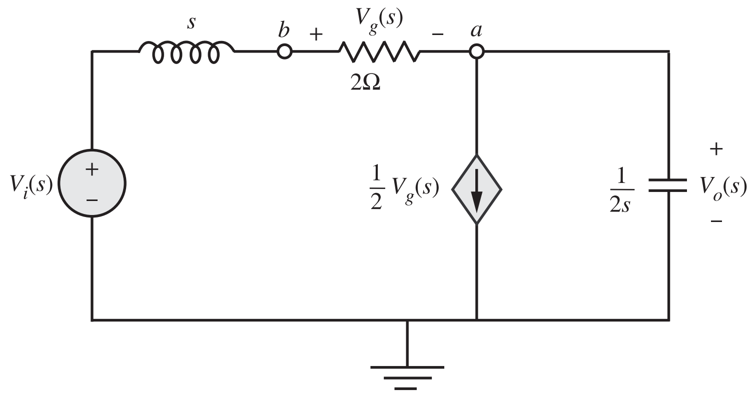


Figure R.P. 5.24(a)

KCL at node a:

$$\frac{V_a(s)-V_b(s)}{2}+\frac{1}{2}V_g(s)+\frac{V_a(s)}{\frac{1}{2s}}=0$$

$$\Rightarrow \qquad \frac{V_a(s)-V_b(s)}{2}+\frac{V_a(s)-V_b(s)}{2}+2sV_a(s)=0$$

$$\Rightarrow \qquad V_a(s)\left[\frac{1}{2}+\frac{1}{2}+2s\right]-V_b(s)\left[\frac{1}{2}+\frac{1}{2}\right]=0$$

$$\Rightarrow \qquad V_a(s)\left[1+2s\right]-V_b(s)=0$$

KCL at node b:

$$\frac{V_b(s)-V_i(s)}{s}+\frac{V_b(s)-V_a(s)}{2}=0$$

$$\Rightarrow \qquad V_a(s)\left[\frac{-1}{2}\right]+V_b(s)\left[\frac{1}{s}+\frac{1}{2}\right]=\frac{V_i(s)}{s}$$

$$\Rightarrow \qquad \frac{-V_a(s)}{2}+\frac{(2+s)}{2s}V_b(s)=\frac{V_i(s)}{s}$$

$$\Rightarrow \qquad -sV_a(s)+(2+s)V_b(s)=2V_i(s)$$

Putting the above nodal equations in matrix form, we get

$$\begin{bmatrix} 1+2s & -1 \\ -s & 2+s \end{bmatrix}\begin{bmatrix} V_a(s) \\ V_b(s) \end{bmatrix}=\begin{bmatrix} 0 \\ 2V_i(s) \end{bmatrix}$$

Solving, we get

$$V_a(s)=\frac{2V_i(s)}{2+s+4s+2s^2-s}$$

$$\Rightarrow \qquad \frac{V_a(s)}{V_i(s)}=\frac{1}{(s+1)^2}$$

Given $\qquad v_i(t)=\delta(t) \quad \Rightarrow \quad V_i(s)=1$

Hence,
$$\frac{V_a(s)}{1} = \frac{1}{(s+1)^2}$$
$$\Rightarrow \qquad V_a(s) = \frac{1}{(s+1)^2}$$

Taking inverse Laplace transform, we get

$$\boldsymbol{v_a(t) = h(t) = te^{-t}u(t)}$$

R.P 5.25

Find the convolution of $h(t) = t$ and $f(t) = e^{-\alpha t}$ for $t > 0$, using the inverse transform of $H(s)F(s)$.

SOLUTION

$$h(t) * f(t) = \mathscr{L}^{-1}\{H(s)F(s)\}$$

where
$$H(s) = \mathscr{L}\{h(t)\} = \frac{1}{s^2}$$
$$F(s) = \mathscr{L}\{f(t)\} = \frac{1}{s+\alpha}$$

Hence,
$$H(s)F(s) = \frac{1}{s^2(s+\alpha)}$$
$$= \frac{K_1}{s} + \frac{K_2}{s^2} + \frac{K_3}{s+\alpha}$$

Solving the partial fractions yields

$$K_1 = -\frac{1}{\alpha^2}, \quad K_2 = \frac{1}{\alpha}, \quad K_3 = \frac{1}{\alpha^2}$$

Hence,
$$H(s)F(s) = \frac{-1}{\alpha^2}\left(\frac{1}{s}\right) + \frac{1}{\alpha}\left(\frac{1}{s^2}\right) + \frac{1}{\alpha^2}\left(\frac{1}{s+\alpha}\right)$$
$$\Rightarrow \qquad h(t) * f(t) = \mathscr{L}^{-1}\{H(s)F(s)\}$$
$$= -\frac{1}{\alpha^2}u(t) + \frac{1}{\alpha}tu(t) + \frac{1}{\alpha^2}e^{-\alpha t}u(t)$$
$$= \boldsymbol{\left[-\frac{1}{\alpha^2} + \frac{t}{\alpha} + \frac{1}{\alpha^2}e^{-\alpha t}\right]u(t)}$$

R.P 5.26

Consider a pulse of amplitude 5V for a duration of 4 seconds with its starting point $t = 0$. Find the convolution of this pulse with itself and draw the convolution $x(t) * x(t)$ versus time.

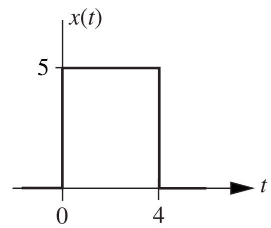


Figure R.P. 5.26(a)

SOLUTION

$$x(t) = 5u(t) - 5u(t-4)$$

$$\Rightarrow \quad X(s) = \frac{5}{s} - \frac{5}{s}e^{-4s}$$

Let $\quad y(t) = x(t) * x(t)$

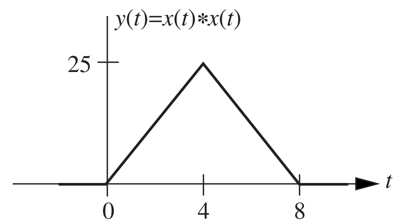


Figure R.P. 5.26(b)

Taking Laplace transform, we get

$$\begin{aligned} Y(s) &= X(s)\,X(s) \\ &= \frac{25}{s^2} - \frac{50}{s^2}e^{-4s} + \frac{25}{s^2}e^{-8s} \end{aligned}$$

Taking inverse Laplace transform, we get

$$y(t) = 25tu(t) - 50(t-4)\,u(t-4) + 25(t-8)\,u(t-8)$$

Hence, $\quad \boldsymbol{y(t) = 25r(t) - 50r(t-4) + 25r(t-8)}$

R.P 5.27

Show that $\quad \mathscr{L}\left\{\frac{Kt^{r-1}e^{-at}}{(r-1)!}\right\} = \frac{K}{(s+a)^r}$

SOLUTION

Let $\quad f(t) = 1$

then $\quad F(s) = \frac{1}{s}$

Thus, $\quad \frac{d^nF(s)}{ds^n} = \frac{(-1)^n\,n!}{s^{n-1}}$

We know that, $\quad \mathscr{L}\{t^nf(t)\} = (-1)^n\,\frac{d^nF(s)}{ds^n}$

With $f(t) = 1$, we get

$$\begin{aligned} \mathscr{L}\{t^n\} &= (-1)^n\left[\frac{(-1)^n\,n!}{s^{n+1}}\right] \\ &= \frac{n!}{s^{n+1}} \end{aligned}$$

Putting $n = r - 1$, we get

$$\mathscr{L}\left\{t^{r-1}\right\} = \frac{(r-1)!}{s^r}$$

and
$$\mathscr{L}\left\{t^{r-1}e^{-at}\right\} = \frac{(r-1)!}{(s+a)^r}$$

Therefore,
$$\frac{K}{(r-1)!}\mathscr{L}\left\{t^{r-1}e^{-at}\right\} = \frac{K}{(s+a)^r}$$

R.P 5.28

Tests conducted on a certain network revealed that the current was $i(t) = -2e^{-t} + 4e^{-3t}$ when a unit step voltage was suddenly applied to the input terminals of the network at $t = 0$. What voltage must be applied to get an output current of $i(t) = 2e^{-t}$ if the network remains unchanged?

SOLUTION

Given,
$$i(t) = -2e^{-t} + 4e^{-3t},\ t \geq 0 \text{ when } v(t) = u(t)$$

Hence,
$$I(s) = \frac{-2}{s+1} + \frac{4}{s+3}$$

and
$$V(s) = \frac{1}{s}$$

$$\text{System function} = H(s) = \frac{\text{Laplace transform of the output}}{\text{Laplace transform of the input}}$$

$$\Rightarrow \qquad H(s) = \frac{I(s)}{V(s)}$$

$$= \frac{2s\,(s-1)}{(s+1)\,(s+3)}$$

We have to find $v(t)$ when $i(t) = 2e^{-t}$.

First we will find $V(s)$ when $I(s) = \dfrac{2}{s+1}$ using the relation $H(s) = \dfrac{I(s)}{V(s)}$.

Hence,
$$V\,(s) = \frac{I\,(s)}{H\,(s)}$$

$$= \frac{\dfrac{2}{s+1}}{\dfrac{2s\,(s-1)}{(s+1)\,(s+3)}}$$

$$= \frac{(s+3)}{s\,(s-1)}$$

$$= \frac{K_1}{s} + \frac{K_2}{s-1}$$

Using partial fractions, we find that $K_1 = -3$ and $K_2 = 4$

Hence, $$V(s) = \frac{-3}{s} + \frac{4}{s-1}$$

$$\Rightarrow \quad \boldsymbol{v(t) = -3u(t) + 4e^{t}u(t)}\ \textbf{Volts}$$

R.P 5.29

Find the Laplace transform of the periodic waveform shown in Fig. R.P. 5.29.

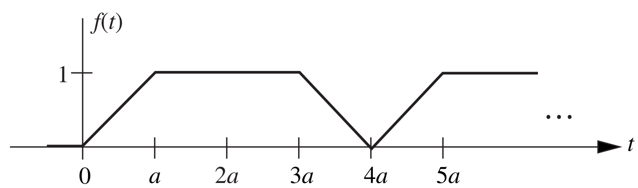


Figure R.P. 5.29

SOLUTION

The Laplace transform of a periodic waveform is found using the relation

$$F(s) = \frac{F_1(s)}{1 - e^{-sT}}$$

where $F_1(s) = \mathscr{L}\{f_1(t)\}$ = Laplace transform of $f(t)$ over $0 < t < T$. Where T = fundamental period of $f(t)$.

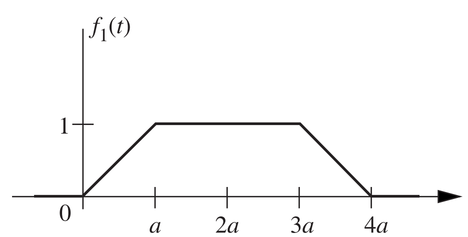


Figure R.P. 5.29(a)

Referring to Fig. R.P. 5.29(a) we can write:

$$f_1(t) = \begin{cases} \dfrac{t}{a}, & 0 < t < a \\ 1, & a < t < 3a \\ \dfrac{-1}{a}t + 4, & 3a < t < 4a \end{cases}$$

$$\Rightarrow \quad f_1(t) = \frac{1}{a}t\,[u(t) - u(t-a)] + [u(t-a) - u(t-3a)] + \left[-\frac{1}{a}t + 4\right][u(t-3a) - u(t-4a)]$$

$$= \frac{1}{a}tu(t) - \frac{1}{a}tu(t-a) + u(t-a) - u(t-3a) - \frac{1}{a}tu(t-3a) + \frac{1}{a}tu(t-4a) + 4u(t-3a) - 4u(t-4a)$$

$$\Rightarrow \quad f_1(t) = \frac{1}{a}tu(t) - \frac{1}{a}(t-a+a)\,u(t-a) + u(t-a) - u(t-3a)$$
$$-\frac{1}{a}(t-3a+3a)\,u(t-3a) + \frac{1}{a}(t-4a+4a)\,u(t-4a)$$
$$+\,4u(t-3a) - 4u(t-4a)$$
$$= \frac{1}{a}tu(t) - \frac{1}{a}(t-a)\,u(t-a) - u(t-a) + u(t-a) - u(t-3a)$$
$$-\frac{1}{a}(t-3a)\,u(t-3a) - 3u(t-3a) + \frac{1}{a}(t-4a)\,u(t-4a) + 4u(t-4a)$$
$$+4u(t-3a) - 4u(t-4a)$$
$$= \frac{1}{a}tu(t) - \frac{1}{a}(t-a)\,u(t-a) - \frac{1}{a}(t-3a)\,u(t-3a) + \frac{1}{a}(t-4a)\,u(t-4a)$$
$$= \frac{1}{a}r(t) - \frac{1}{a}r(t-a) - \frac{1}{a}r(t-3a) + \frac{1}{a}r(t-4a)$$

Hence, $$F_1(s) = \mathcal{L}\{f_1(t)\}$$
$$= \frac{1}{as^2} - \frac{1}{as^2}e^{-as} - \frac{1}{as^2}e^{-3as} + \frac{1}{as^2}e^{-4as}$$
$$= \frac{1}{as^2}\left(1 - e^{-as} - e^{-3as} + e^{-4as}\right)$$

Alternate method for finding $F_1(s)$:

From Figs. R.P. 5.29(b), (c), (d), we can write

$$f_1(t) = f_A(t) + f_B(t) + f_C(t) + f_D(t)$$
$$= \frac{1}{a}tu(t) - \frac{1}{a}(t-a)\,u(t-a) - \frac{1}{a}(t-3a)\,u(t-3a)$$
$$+\ \frac{1}{a}(t-4a)\,u(t-4a)$$

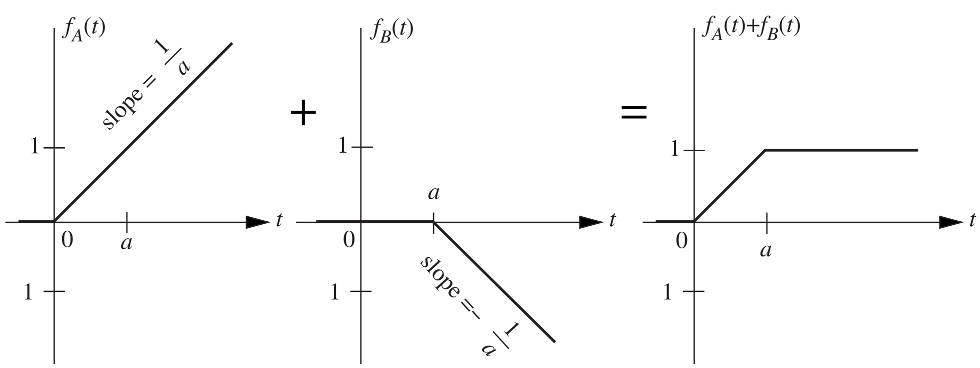


Figure R.P. 5.29(b)

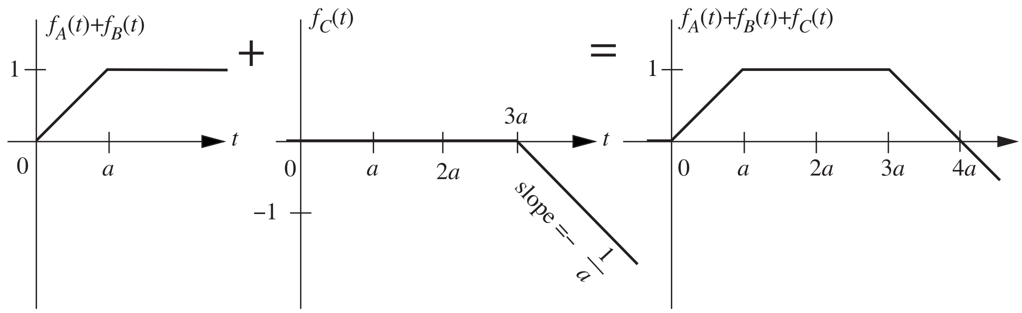


Figure R.P. 5.29(c)

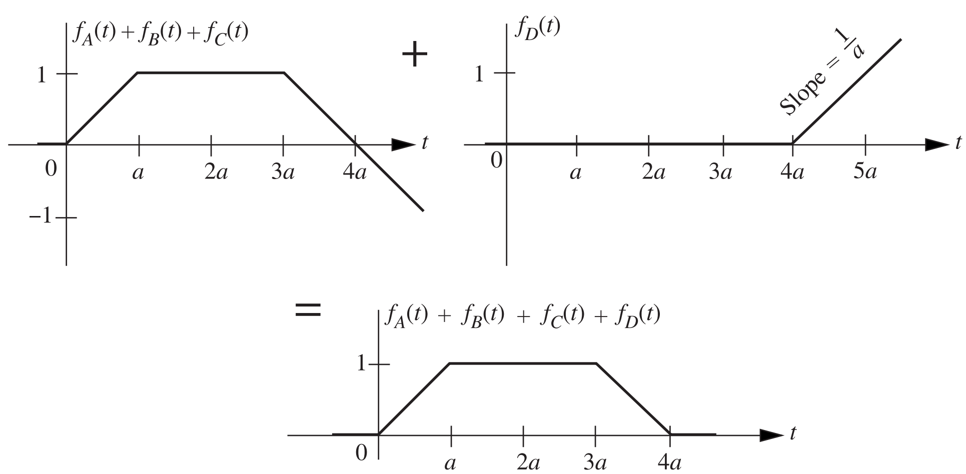


Figure R.P. 5.29(d)

Hence, $$F_1(s) = \mathscr{L}\{f_1(t)\} = \frac{1}{as^2} - \frac{1}{as^2}e^{-as} - \frac{1}{as^2}e^{-3as} + \frac{1}{as^2}e^{-4as}$$

Finally, $$F(s) = \mathscr{L}\{f(t)\} = \frac{F_1(s)}{1-e^{-sT}}$$

where $T = 4a$

$$F(s) = \boldsymbol{\frac{1}{as^2}\frac{\left(1-e^{-as}-e^{-3as}+e^{-4as}\right)}{\left(1-e^{-4as}\right)}}$$

R.P 5.30

Find the Laplace transform of the function $f(t)$ shown in Fig. R.P. 5.30.

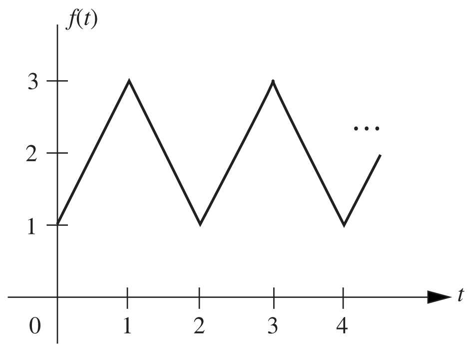


Figure R.P. 5.30

SOLUTION

Let $f(t) = x(t) + u(t)$, where $x(t)$ is a periodic triangular wave and is as shown in Fig. R.P. 5.30(a).

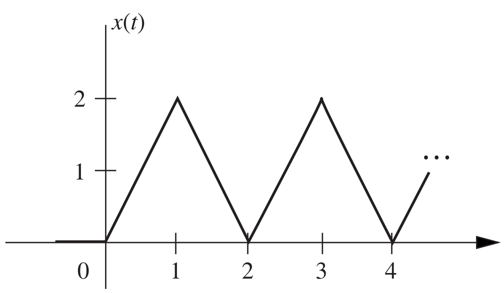


Figure R.P.5.30(a)

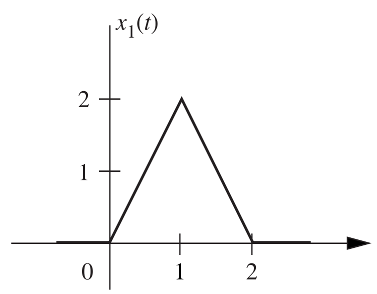


Figure R.P.5.30(b)

Let $x_1(t)$ be $x(t)$ within its first period as shown in Fig. R.P.5.30(b).
Referring to Fig. R.P. 5.30(b), we can write

$$x_1(t) = \begin{cases} 2t, & 0 < t < 1 \\ 4 - 2t, & 1 < t < 2 \end{cases}$$

$$\begin{aligned}
\Rightarrow \quad x_1(t) &= 2t\left[u(t) - u(t-1)\right] + (4-2t)\left[u(t-1) - u(t-2)\right] \\
&= 2tu(t) - 2tu(t-1) + 4u(t-1) - 4u(t-2) - 2tu(t-1) + 2tu(t-2) \\
&= 2tu(t) - 2(t-1+1)u(t-1) + 4u(t-1) - 4u(t-2) \\
&\quad -2(t-1+1)u(t-1) + 2(t-2+2)u(t-2) \\
&= 2tu(t) - 2(t-1)u(t-1) - 2u(t-1) + 4u(t-1) - 4u(t-2) \\
&\quad -2(t-1)u(t-1) - 2u(t-1) + 2(t-2)u(t-2) + 4u(t-2)
\end{aligned}$$

$$\Rightarrow \quad x_1(t) = 2tu(t) - 4(t-1)u(t-1) + 2(t-2)u(t-2)$$

$$\Rightarrow \quad x_1(t) = 2r(t) - 4r(t-1) + 2r(t-2)$$

Hence, $$X_1(s) = \mathscr{L}\{x_1(t)\} = \frac{2}{s^2} - \frac{4}{s^2}e^{-s} + \frac{2}{s^2}e^{-2s} = \frac{2}{s^2}\left(1 - 2e^{-s} + e^{-2s}\right) = \frac{2}{s^2}\left(1 - e^{-s}\right)^2$$

Since $x(t)$ is periodic,

$$X(s) = \mathscr{L}\{x(t)\} = \frac{X_1(s)}{1 - e^{-sT}}$$

where $T = 2$ seconds

Hence, $$X(s) = \frac{2}{s^2}\frac{\left(1 - e^{-s}\right)^2}{\left(1 - e^{-2s}\right)}$$

We know that, $$f(t) = x(t) + u(t)$$

Applying linearity property,

$$F(s) = X(s) + U(s) = \boldsymbol{\frac{2}{s}\frac{\left(1 - e^{-s}\right)^2}{\left(1 - e^{-2s}\right)} + \frac{1}{s}}$$

R.P 5.31

Find $f(t)$ using convolution integral for the function,

$$F(s) = \frac{4s}{(s+1)(s^2+4)}$$

SOLUTION

Let $$F(s) = F_1(s)F_2(s)$$

where $$F_1(s) = \frac{4}{s+1} \quad \Rightarrow \quad f_1(t) = 4e^{-t}u(t)$$

$$F_2(s) = \frac{s}{s^2+4} \quad \Rightarrow \quad f_2(t) = \cos 2t u(t)$$

$$f(t) = \mathscr{L}^{-1}[F_1(s)F_2(s)] = \int_0^{\infty} f_1(\lambda) f_2(t-\lambda)\, d\lambda$$

We know that $$u(\tau)\,u(t-\tau) = \begin{cases} 1, & 0 < \tau < t, \quad t > 0 \\ 0, & \text{otherwise} \end{cases}$$

Hence, $$f(t) = \int_0^t \cos 2\lambda 4e^{-(t-\lambda)} d\lambda$$

$$= 4e^{-t} \int_0^t e^{\lambda} \cos 2\lambda \, d\lambda$$

Using the standard integral formula

$$\int e^{ax} \cos\, bx\, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx)$$

we get $$f(t) = 4e^{-t} \left[\frac{e^{\lambda}}{1+4} (\cos 2\lambda + 2 \sin 2\lambda) \right]_{\lambda=0}^{t}$$

$$= \frac{4}{5} e^{-t} \left[e^{t} (\cos 2t + 2 \sin 2t - 1) \right]$$

$$= \frac{4}{5} \cos 2t + \frac{8}{5} \sin 2t - \frac{4}{5} e^{-t}, \; t \geq 0$$

$$f(t) = \left[\frac{4}{5} \cos 2t + \frac{8}{5} \sin 2t - \frac{4}{5} e^{-t} \right] u(t)$$

R.P 5.32

If $h(t) = 2e^{-3t}u(t)$ and $x(t) = u(t) - \delta(t)$. Find $y(t) = h(t) * x(t)$ by (a) using convolution in the time-domain (b) Finding $H(s)$ and $X(s)$ and then obtaining $\mathscr{L}^{-1}[H(s)X(s)]$

SOLUTION

Given $$h(t) = 2e^{-3t}u(t)$$

and $$x(t) = u(t) - \delta(t)$$

(a) $$y(t) = x(t) * h(t)$$

$$= \int_0^{\infty} x(\lambda)\, h(t-\lambda)\, d\lambda$$

$$= \int_0^{\infty} |u(\lambda) - \delta(\lambda)|\, 2e^{-3(t-\lambda)} u(t-\lambda)\, d\lambda$$

$$= \int_0^{\infty} 2e^{-3(t-\lambda)} u(t-\lambda)\, u(\lambda)\, d\lambda - 2 \int_0^{\infty} e^{-3(t-\lambda)} u(t-\lambda)\, \delta(\lambda)\, d\lambda$$

We know that, $$u\left(t-\lambda\right)u\left(\lambda\right)=\begin{cases}1, & 0<\lambda<t, \quad t>0\\ 0, & \text{otherwise}\end{cases}$$

The second integral on the right-hand side is evaluated using the sifting property for an impulse function.

Hence, $$y\left(t\right)=\int_{0}^{t}2e^{-3t}e^{3\lambda}d\lambda-2e^{-3(t-\lambda)}\ u\left(t-\lambda\right)|_{\lambda=0}$$

$$\Rightarrow \quad y\left(t\right)=2e^{-3t}\left[\frac{e^{3\lambda}}{3}\right]_{0}^{t}-2e^{-3t}u\left(t\right)$$
$$=\frac{2}{3}\left(1-e^{-3t}\right)-2e^{-3t}u\left(t\right)$$

Since $t>0$, we associate $u(t)$ in the first component on the right hand side of $y(t)$.

Then, $$y\left(t\right)=\frac{2}{3}\left(1-e^{-3t}\right)u\left(t\right)-2e^{-3t}u\left(t\right)$$
$$=\left[\boldsymbol{\frac{2}{3}-\frac{8}{3}e^{-3t}}\right]\boldsymbol{u\left(t\right)}$$

(b) Verification :

$$H\left(s\right)=\frac{2}{s+3},\ X\left(s\right)=\frac{1}{s}-1$$
$$\Rightarrow \quad Y\left(s\right)=X\left(s\right)H\left(s\right)$$
$$=\frac{2\left(1-s\right)}{s\left(s+3\right)}$$
$$=\frac{K_1}{s}+\frac{K_2}{s+3}$$

Using partial fractions, we find that

$$K_1=\frac{2}{3},\ K_2=\frac{-8}{3}$$

Hence, $$Y\left(s\right)=\frac{2}{3}\left(\frac{1}{s}\right)-\frac{8}{3}\left(\frac{1}{s+3}\right)$$

$$\Rightarrow \quad y\left(t\right)=\frac{2}{3}u\left(t\right)-\frac{8}{3}e^{-3t}u\left(t\right)$$
$$=\left[\boldsymbol{\frac{2}{3}-\frac{8}{3}e^{-3t}}\right]\boldsymbol{u\left(t\right)}$$

R.P 5.33

When an impulse $\delta(t)$ V is applied to a certain network, the ouput voltage is $v_o(t)=4u(t)-4u(t-2)$ V. Find and sketch $v_o(t)$ if the imput voltage is $2u\left(t-1\right)$ V.

SOLUTION

When $v_i(t) = \delta(t)$, it is given that $v_o(t) = 4u(t) - 4u(t-2)$.

From this data, we can find the transfer function $H(s)$ as follows:

$$H(s) = \frac{\mathscr{L}\{v_o(t)\}}{\mathscr{L}\{v_i(t)\}}$$
$$= \frac{\mathscr{L}\{4u(t) - 4u(t-2)\}}{\mathscr{L}\{\delta(t)\}}$$
$$= \frac{4}{s}\left[1 - e^{-2s}\right]$$

The transfer function $H(s)$ can be used to find $v_o(t)$ when $v_i(t) = 2u(t-1)$ V. This procedure is as follows:

$$H(s) \triangleq \frac{V_o(s)}{V_i(s)}$$

$$\Rightarrow \qquad V_o(s) = V_i(s)\,H(s)$$
$$= \frac{2}{s}e^{-s}\left[\frac{4}{s} - \frac{4}{s}e^{-2s}\right]$$
$$= \frac{8}{s^2}e^{-s} - \frac{8}{s^2}e^{-3s}$$

Taking inverse Laplace transform, we get

$$v_o(t) = 8(t-1)\,u(t-1) - 8(t-3)\,u(t-3)$$
$$= \boldsymbol{8r(t-1) - 8r(t-3)}$$

The corresponding wave form for $v_o(t)$ is sketched in Fig. R.P. 5.33

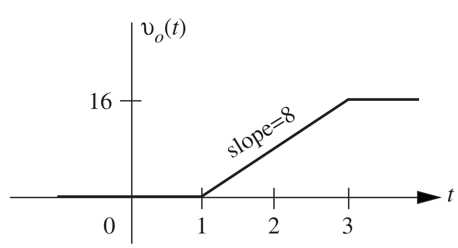


Figure R.P. 5.33

R.P 5.34

Refer the two circuits shown in Fig. R.P. 5.34(a) and (b). Given that $v_1(t) = \sin 10^3 t$ and $v_2(t) = e^{-1000t}$ for $t \geq 0$ and $c = 1\ \mu$F.

(a) Show that it is possible to have $i_1(t) = i_2(t)$ for all $t \geq 0$.

(b) Determine the required values of R and L for the condition in part (a) to hold good.

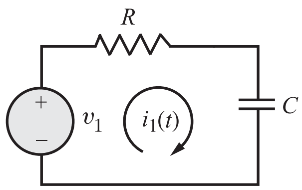


Figure R.P. 5. 34 (a)

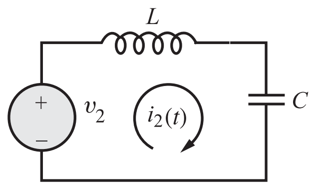


Figure R.P.5.34 (b)

SOLUTION

Referring Fig. R.P. 5.34(a) we can write in Laplace domain

$$I_1(s) = \frac{V_1(s)}{R + \dfrac{1}{Cs}}$$

Similarly, referring Fig. R.P. 5.34(b), we can write in Laplace domain

$$I_2(s) = \frac{V_2(s)}{sL + \dfrac{1}{Cs}}$$

$i_1(t) = i_2(t)$ means that $I_1(s) = I_2(s)$

Also,
$$V_1(s) = \mathscr{L}\left\{\sin 10^3 t\right\} = \frac{10^3}{s^2 + (10^3)^2}$$
$$V_2(s) = \mathscr{L}\left\{e^{-1000t}\right\} = \frac{1}{s + 10^3}$$

Hence, the condition $I_1(s) = I_2(s)$ gives,

$$\frac{10^3}{s^2 + 10^6} \frac{1}{R + \dfrac{10^6}{s}} = \frac{1}{s + 10^3} \frac{1}{sL + \dfrac{10^6}{s}}$$

$$\Rightarrow \qquad \frac{10^3}{R\left(s + \dfrac{10^6}{R}\right)(s^2 + 10^6)} = \frac{1}{L\,(s + 10^3)\left(s^2 + \dfrac{10^6}{L}\right)}$$

If the above equation is satisfied, then it is possible to have $i_1(t) = i_2(t)$. For this to happen, it is required that

$$\frac{R}{10^3} = L; \qquad \frac{10^6}{R} = 10^3 \quad \text{and} \quad 10^6 = \frac{10^6}{L}$$

The above conditions give $L = 1\text{H}$ and $R = 10^3\ \Omega$

R.P 5.35

For the circuit shown in Fig. R.P. 5.35 has zero initial conditions. At $t = 0$, the switch K is closed. Find the value of R such that the response $v(t) = 0.5 \sin \sqrt{2}t$ volts. Take the excitation as $i(t) = te^{-\sqrt{2}t}$ A.

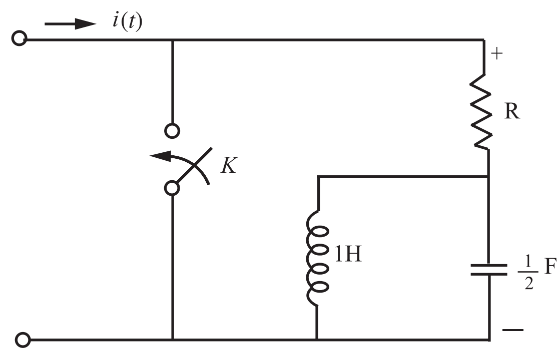


Figure R.P.5.35

SOLUTION

Given $i(t) = te^{-\sqrt{2}t}$

Taking Laplace transform of $i(t)$ gives

$$I(s) = \frac{1}{(s+\sqrt{2})^2}$$

Laplace transform of the response $v(t) = 0.5 \sin \sqrt{2}t$ is

$$V(s) = \frac{1}{2}\left[\frac{\sqrt{2}}{s^2+2}\right]$$

Hence,

$$Z(s) = \frac{V(s)}{I(s)}$$

$$= \frac{1}{\sqrt{2}}\frac{(s+\sqrt{2})^2}{s^2+2} \tag{5.29}$$

For the circuit shown in Fig. R.P. 5.35 we can write

$$Z(s) = R + \frac{1}{\frac{s}{2}+\frac{1}{s}}$$

$$\Rightarrow \qquad Z(s) = R + \frac{2s}{s^2+2} \tag{5.30}$$

Equating equations 5.29 and 5.30, we get

$$\frac{1}{\sqrt{2}} \frac{\left(s+\sqrt{2}\right)^2}{s^2+2} = R + \frac{2s}{s^2+2}$$

$$\Rightarrow \qquad \left(s^2+2\right) R - 2s = \frac{1}{\sqrt{2}} \left(s+\sqrt{2}\right)^2$$

$$= \frac{s^2}{\sqrt{2}} + 2s + \frac{2}{\sqrt{2}}$$

Equating the like powers of s, we get $\quad R = \frac{1}{\sqrt{2}} \Omega$

Exercise Problems

E.P 5.1

Find the Laplace transform of the following functions :

(a) $f_1(t) = \sin(\omega t + \theta)$

(b) $f_2(t) = \sin^2 t$

(c) $f_3(t) = \frac{1}{2a^3} [\sinh(at) - \sin(at)]$

Ans: $\boldsymbol{F_1(s) = \frac{s \sin\theta + \omega\cos\theta}{s^2+\omega^2}, \; F_2(s) = \frac{2}{s(s^2+4)}, \; F_3(s) = \frac{1}{(s^2-a^2)(s^2+a^2)}}$

E.P 5.2

In the network shown in Fig. E.P. 5.2, the switch K is moved from position a to position b at $t = 0$, a steady state having previously been established at position a. Solve for $i(t)$, using the Laplace transformation method.

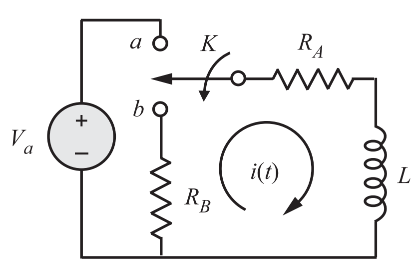


Figure E.P. 5.2

Ans: $\boldsymbol{i(t) = \frac{V_a}{R_A} e^{-\frac{R_A+R_B}{L}} u(t)}$

E.P 5.3

Find $i_1(t)$ and $i_2(t)$ for $t > 0$ for the circuit shown in Fig. E.P. 5.3 using Laplace transform.

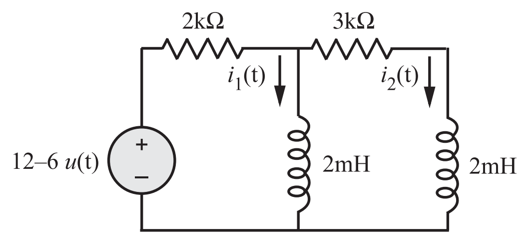


Figure E.P. 5.3

$$\textbf{Ans}: \quad i_1(t) = \left[2.4e^{-5\times10^5 t} + 0.6e^{-6\times5\times10^5 t} + 3\right] u(t)\,\text{mA}$$

$$i_2(t) = \left[1.2e^{-6\times5\times10^5 t} - 1.2e^{-5\times10^5 t}\right] u(t)\,\text{mA}$$

E.P 5.4

Using Laplace transform technique, find $i(t)$ when $i_1 = 0.1e^{-bt}$ A for the circuit shown in E.P. 5.4 when $b = 10^5$. Assume steady state conditions at $t = 0^-$.

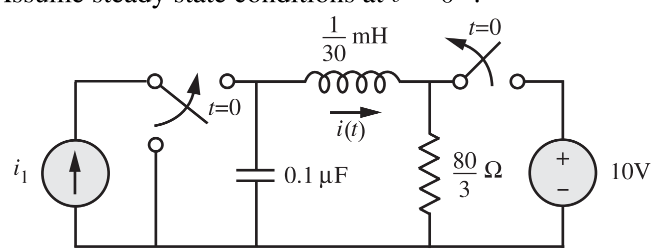


Figure E.P. 5.4

$$\textbf{Ans:} \quad i(t) = \left[\frac{1}{30}e^{-bt} + \frac{27}{40}e^{-6bt} - \frac{17}{24}e^{-10bt}\right] u(t)\,\text{A}$$

E.P 5.5

The current source shown in Fig. E.P. 5.5 is $i(t) = tu(t)\mu$ A. Find $v_o(t)$ when the initial value of v_o is zero.

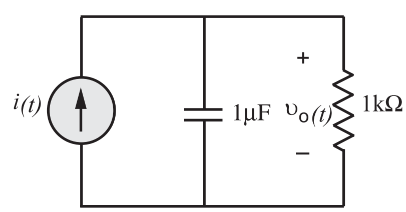


Figure E.P. 5.5

$$\textbf{Ans:} \quad v_o(t) = t - 10^{-3}\left(1 - e^{-10^3 t}\right)\text{mV}, \; t \geq 0$$

E.P 5.6

Find the inverse Laplace transform of the following $F(s)$:

(a) $F(s) = \dfrac{8s-3}{s^2+4s+13}$ (b) $F(s) = \dfrac{4s^2}{(s+3)^2}$

Ans: (a) $\boldsymbol{f(t) = 10.2e^{-2t}\cos(3t+38.3^\circ)\,u(t)}$

(b) $\boldsymbol{f(t) = \left[4e^{-3t} - 24te^{-3t} + 18t^2e^{-3t}\right]u(t)}$

E.P 5.7

Using convolution integral, find $f(t)$ if

$$F(s) = \frac{10}{s(s+5)}$$

Ans: $\boldsymbol{f(t) = 2\left[1-e^{-5t}\right]u(t)}$

E.P 5.8

Refer the network shown in Fig. E.P. 5.8. Assume the network is in steady state for $t < 0$. Determine the current $i(t)$ for $t > 0$.

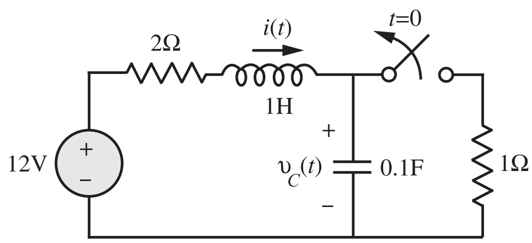


Figure E.P. 5.8

Ans: $\boldsymbol{i(t) = 4.22e^{-t}\cos(3t-18.4^\circ)\,u(t)}$ **A**

E.P 5.9

Find $v_o(t)$ in the circuit shown in Fig. E.P. 5.9.

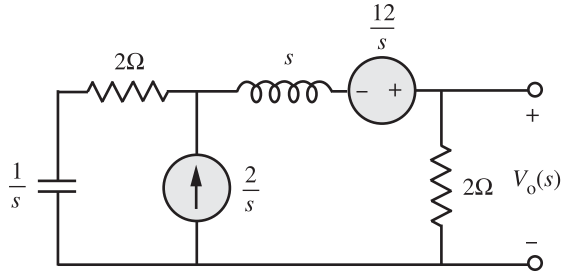


Figure E.P. 5.9

Ans: $\boldsymbol{v_o(t) = \left[4 - 8.93e^{-3.73t} + 4.93e^{-0.27t}\right]u(t)}$ **V**

E.P 5.10

Find $v_o(t)$ for $t > 0$. Refer the circuit shown in Fig. E.P. 5.10.

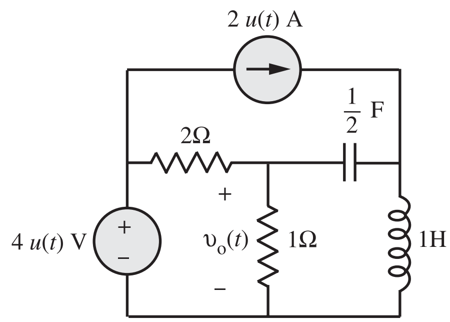


Figure E.P. 5.10

Ans: $$\boldsymbol{v_o(t) = \left[\frac{4}{3} + 2.55e^{\frac{-1}{3}t}\cos\left(\sqrt{17}t + 10.1°\right)\right]u(t)}$$

E.P 5.11

For the circuit shown in E.P. 5.11.

Find: (a) $H(s) = \dfrac{V_o(s)}{V_i(s)}$ (b) $h(t)$

(c) Step response (d) The response when $v_i(t) = 8\cos 2t$ V

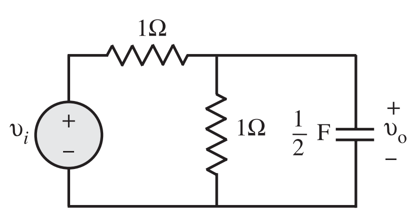


Figure E.P. 5.11

Ans: (a) $\boldsymbol{H(s) = \dfrac{2}{s+4}}$

(b) $\boldsymbol{h(t) = 2e^{-4t}u(t)}$

(c) $\boldsymbol{v_o(t) = 0.5\left(1 - e^{-4t}\right)u(t)}$ **V**

(d) $\boldsymbol{v_o(t) = 1.5\left[e^{-4t} + \cos 2t + 0.5\sin 2t\right]u(t)}$ **V**

E.P 5.12

Refer the circuit shown in Fig. E.P. 5.12. The switch is closed at $t = 0$

Find : (a) $i_1(t)$ and (b) $i_2(t)$

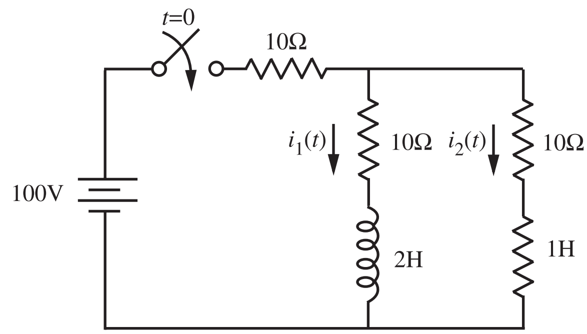


Figure E.P. 5.12

Ans: (a) $\boldsymbol{i_1(t) = \left[3.33 - 1.67e^{-6.34t} - 1.67e^{-23.66t}\right] u(t)}$

(b) $\boldsymbol{i_2(t) = \left[3.33 + 1.22e^{-6.34t} - 4.55e^{-23.66t}\right] u(t)}$

E.P 5.13

Find the Laplace transform of the waveform shown in Fig. E.P. 5.13.

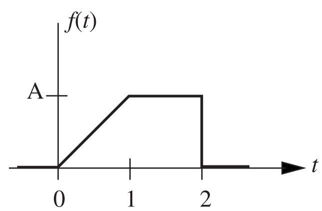


Figure E.P. 5.13

Ans: $\boldsymbol{F(s) = \frac{A\left(1 - e^{-s}\right)}{s^2} - \frac{A}{s}e^{-2s}}$

E.P 5.14

Find the Laplace transform of the periodic waveform shown in Fig. E.P. 5.14.

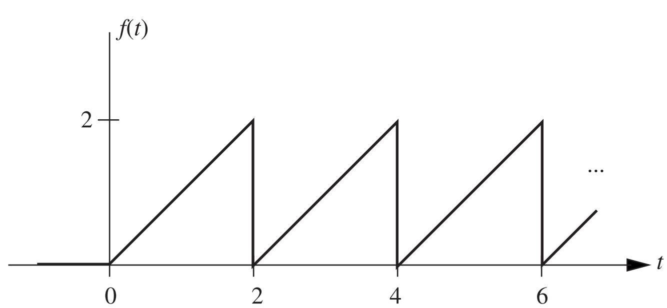


Figure E.P. 5.14

Ans: $\boldsymbol{F(s) = \left[\frac{1}{s^2} - \frac{1}{s^2}e^{-2s} - \frac{2}{s}e^{-2s}\right]\frac{1}{1 - e^{-2s}}}$

E.P 5.15

Find the Laplace transform of the waveform shown in Fig. E.P. 5.15

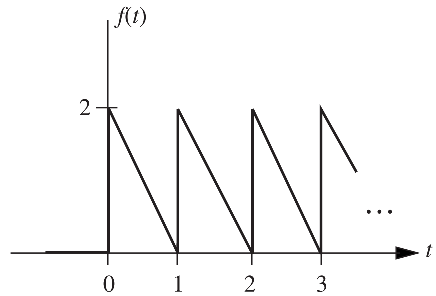


Figure E.P. 5.15

Ans: $\boldsymbol{F(s) = \left[-\frac{2}{s^2} + \frac{2}{s} + \frac{2}{s^2}e^{-s}\right]\frac{1}{1-e^{-s}}}$

E.P 5.16

Obtain the Laplace transform of the $f(t)$ shown in Fig. E.P. 5.16.

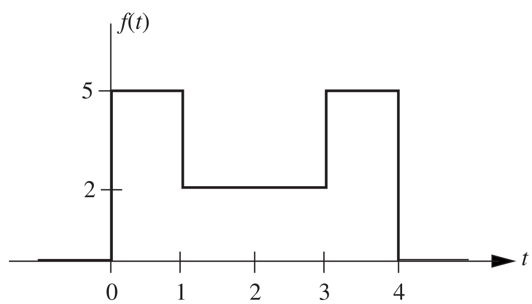


Figure E.P. 5.16

Ans: $\boldsymbol{F(s) = \frac{1}{s}\left[5 - 3e^{-s} + 3e^{-3s} - 5e^{-4s}\right]}$

E.P 5.17

Obtain the Laplace transform of the unit impulses shown in Fig. E.P. 5.17

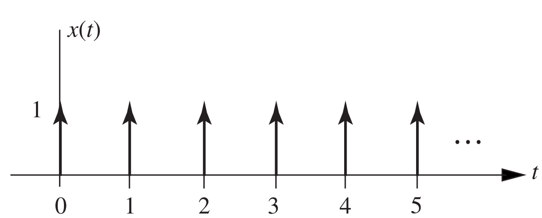


Figure E.P. 5.17

Ans: $\mathrm{X}(s) = \boldsymbol{\frac{1}{1-e^{-s}}}$

E.P 5.18

Refer the circuit shown in Fig. E.P. 5.18. Let $i(0) = 1\text{A}, v_o(0) = 2\text{V}$ and $v_s(t) = 4e^{-2t}u(t)\text{V}$. Find $v_o(t)$ for $t > 0$.

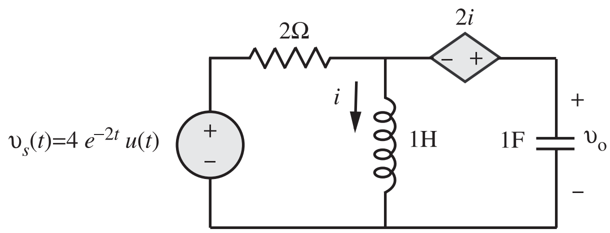


Figure E.P. 5.18

Ans: $\boldsymbol{v_o(t) = -\left[2 + 4.33e^{-0.5t} + 1.33e^{-2t}\right] u(t)}$ **volts**

E.P 5.19

Find $i(t)$ in the circuit shown in Fig. E.P. 5.19. Assume that the circuit is initially relaxed.

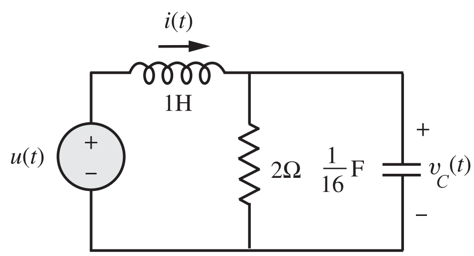


Figure E.P. 5.19

Ans: $\boldsymbol{i(t) = \left[0.5 - 0.5e^{-4t} - te^{-4t}\right] u(t)}$

E.P 5.20

Refer the circuit shown in Fig. E.P. 5.20. Assume zero initial conditions. Use convolution theorem to find $i(t)$.

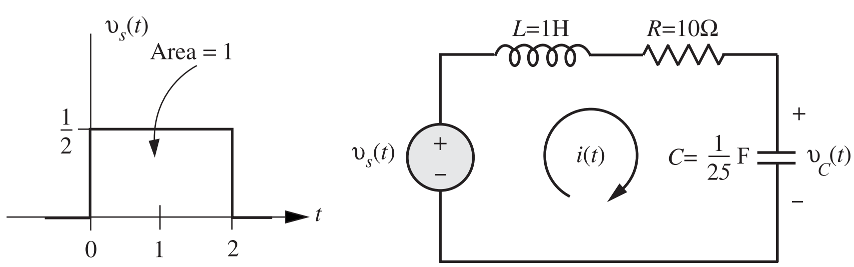


Figure E.P. 5.20

Ans: $\boldsymbol{i(t) = \frac{t}{2}e^{-5t} - \frac{(t-2)}{2}e^{-5(t-2)}}$

E.P 5.21

There is no energy stored in the circuit shown in Fig. E.P. 5.21 at the time when the switch is opened. Show that

$$V_2(s) = \frac{sI_g(s)}{C_1\left[s^2 + \left(\frac{R_1}{L_1}\right)s + \frac{1}{L_1C_1}\right]}$$

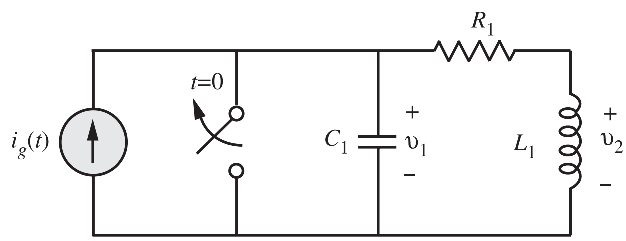


Figure E.P. 5.21

E.P 5.22

Refer the circuit shown in Fig. E.P. 5.22. If $i_s(t) = 6u(t)\text{mA}$, find $v_2(t)$.

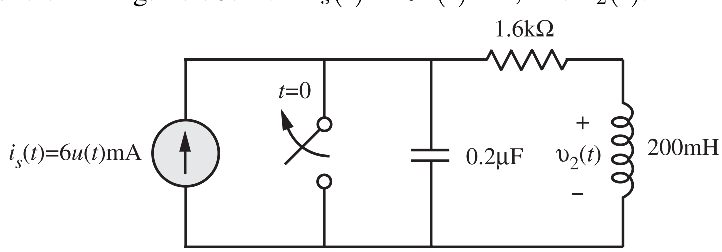


Figure E.P. 5.22

Ans: $\boldsymbol{v_2(t) = 10e^{-4000t}\cos(3000t - 90^\circ)\,u(t)\,\text{V}}$

E.P 5.23

Find $V_o(s)$ and $v_o(t)$ in the circuit shown in Fig. E.P. 5.23 if the initial energy is zero and the switch is closed at $t = 0$

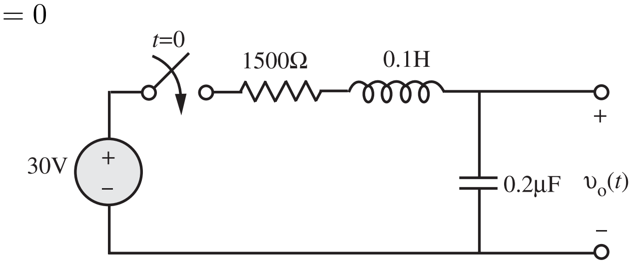


Figure E.P. 5.23

Ans: $\boldsymbol{v_o(t) = \left[30 - 60e^{-5000t} + 30e^{-10000t}\right]u(t)}$

E.P 5.24

The initial energy in the circuit in Fig. E.P. 5.24 is zero.

(a) Find $V_o(s)$.

(b) Use the initial and final value theorems to find $v_o(0^+)$ and $v_o(\infty)$.

(c) Do the values obtained in part (b) agree with known circuit behaviour? Explain.

(d) Find $v_o(t)$.

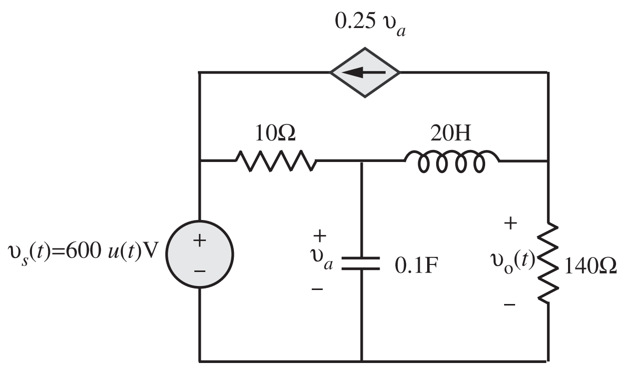


Figure E.P. 5.24

Ans: **(a)** $V_o(s) = \dfrac{-21 \times 10^3 s + 4200}{s(s^2 + 8s + 25)}$

(b) $v_o(0^+) = 0,\ v_o(\infty) = 168\text{V}$

(c) **YES**

(d) $v_o(t) = \left[168 + 7225.95e^{-4t}\cos(3t + 91.33°)\right] u(t)\ \text{V}$

E.P 5.25

Find the initial and final value of $H(s) = \dfrac{s^3 + 25 + 6}{s(s+1)^2(s+3)}$

Ans: 1, 2

E.P 5.26

Verify final value theorem and initial value theorem for the function,

$$f(t) = 2 + e^{-3t}\cos 2t$$

E.P 5.27

Using the convolution theorem, find the Laplace inverse of the following functions:

(i) $F(s) = \dfrac{1}{s(s+1)}$ (ii) $F(s) = \dfrac{1}{(s-a)^2}$ (iii) $F(s) = \dfrac{s}{(s+1)(s+2)}$

Ans: **(i)** $f(t) = 1 - e^{-t}$

(ii) $f(t) = te^{at}$

(iii) $f(t) = e^{-t} + 2e^{-2t} - 2e^{-t}$

E.P 5.28

In the circuit shown in Fig. E.P. 5.28, find the voltage across the resistance $v_R(t)$ using convolution integral. Given that $v_g(t) = e^{-2t}$ and $RC = 1$ second.

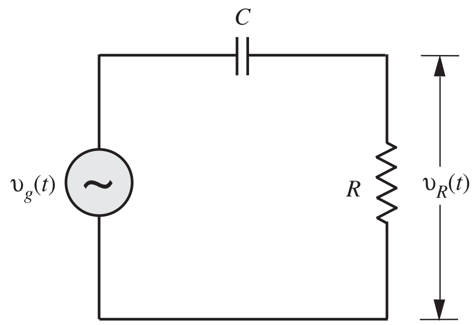


Figure E.P. 5.28

Ans: $\boldsymbol{v_R(t) = 2e^{-2t} - e^{-t}, \quad t \geq 0}$

E.P 5.29

Find the inverse Laplace transform of the following functions:

(i) $\dfrac{3s}{(s^2+1)(s^2+4)}$ (ii) $\dfrac{1}{(s+1)(s+2)^2}$ (iii) $\dfrac{s^2+3}{(s^2+2s+2)(s+2)}$

Ans: **(i)** $\boldsymbol{\cos t - \cos 2t}$

(ii) $\boldsymbol{e^{-t} - e^{-2t}(1+t)}$

(iii) $\boldsymbol{\frac{7}{2}e^{-2t} - 2.5e^{-t}\cos t + 0.5e^{-t}\sin t}$

E.P 5.30

In the circuit shown in Fig. E.P. 5.30, switch K is open for a long time so that steady state is reached and at $t = 0$, switch is closed. Determine the current $i(t)$ in 10 ohm resistor.

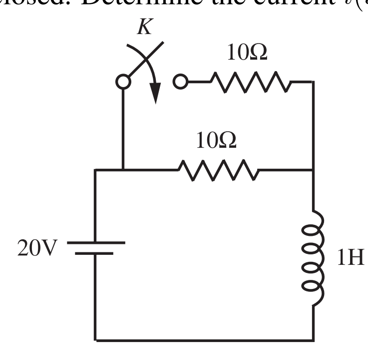


Figure E.P.5.30

Ans: **Current in each 10 Ω resistor =** $\boldsymbol{2u(t) - e^{-5t}}$

E.P 5.31

Synthesize the wave form shown in Fig. E.P. 5.31 using ramp function and obtain the Laplace transform of $f(t)$.

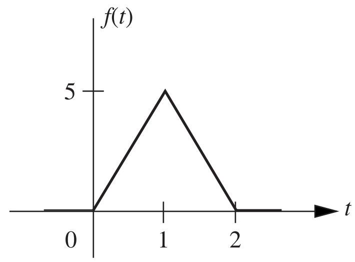


Figure E.P.5.31

Ans: $\boldsymbol{F(s) = \frac{5}{s^2}[1 - 2e^{-s} + e^{-2s}]}$

E.P 5.32

Find the Laplace transform of the voltage wave form as shown in Fig. E.P. 5.32.

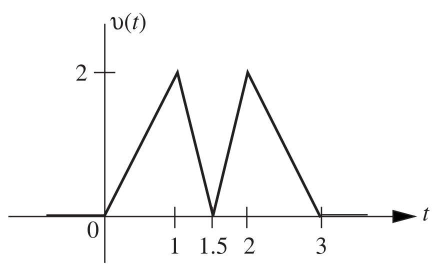


Figure E.P.5.32

Ans: $\boldsymbol{V(s) = \frac{2}{s^2}[1 - 3e^{-s} + 5e^{-1.5s} - 6e^{-2s} + 6e^{-3s}]}$

E.P 5.33

Find the Laplace transform of the perodic wave forms shown in Figs. E.P. 5.33(a) and (b).

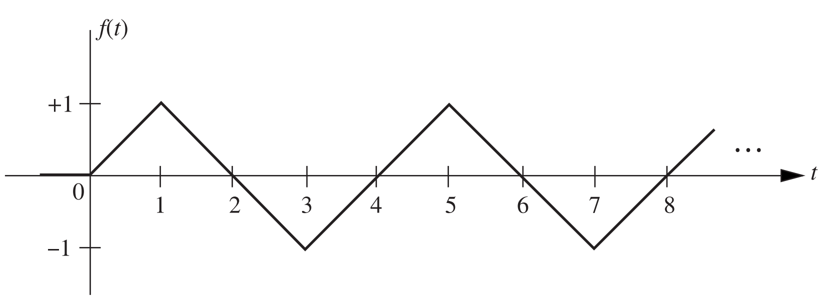


Figure E.P.5.33(a)

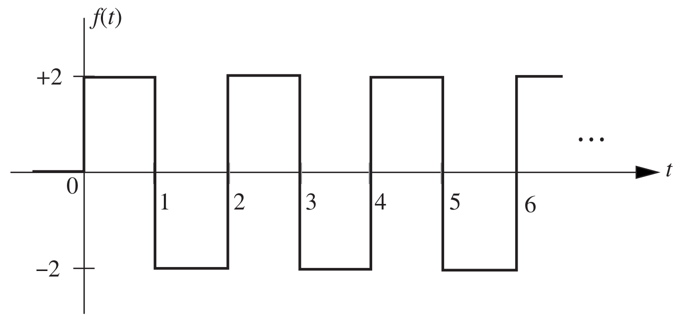


Figure E.P.5.33(b)

Ans: (i) $$F(s) = \frac{1}{1-e^{-4s}}\left[\frac{1}{s^2} - \frac{2}{s^2}e^{-s} + \frac{1}{s^2}e^{-2s} - \frac{1}{s}e^{-2s} + \frac{2}{s^2}e^{-3s} - \frac{1}{s^2}e^{-4s}\right]$$

(ii) $$F(s) = \frac{1}{1-e^{-2s}}\left[\frac{2}{s} - \frac{4e^{-s}}{s} + \frac{2}{s}e^{-2s}\right]$$

E.P 5.34

For the circuit shown in Fig. E.P. 5.34, find the current transients in both the loops using Laplace transformation method.

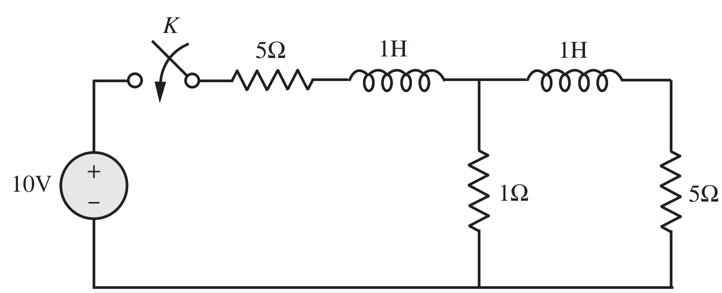


Figure E.P.5.34

Ans: $$i_1(t) = \frac{12}{7} - \frac{5}{7}e^{-2t} - e^{-5t} \text{ Ampere}, t \geq 0$$

$$i_2(t) = \frac{2}{7} + \frac{5}{7}e^{-7t} - e^{-5t} \text{ Ampere}, t \geq 0$$

E.P 5.35

Find the Laplace transform of the saw tooth wave as shown in Fig. E.P. 5.35.

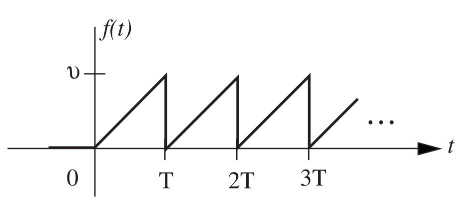


Figure E.P. 5.35

Ans: $F(s) = \dfrac{V(1-e^{-Ts} - Tse^{-Ts})}{Ts^2(1-e^{-Ts})}$

E.P 5.36

For the circuit shown in Fig. E.P. 5.36 switch K is closed at $t = 0$. Determine the current $i(t)$ for $t \geq 0$.

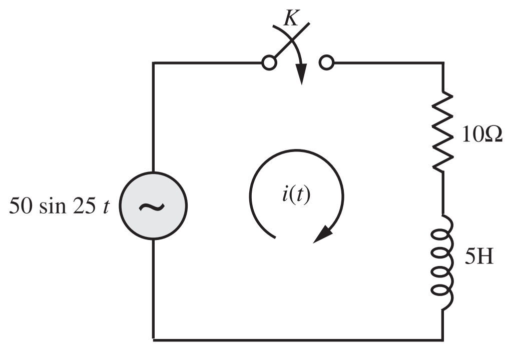


Figure E.P. 5.36

Ans: $i(t) = 0.357e^{-2t} - \dfrac{5}{25+j2}e^{-j25t} - \dfrac{5}{25-j2}e^{-j25t}$

E.P 5.37

For the circuit shown in Fig. E.P. 5.37, determine the source current when the switch K is closed at $t = 0$. Assume zero initial conditions.

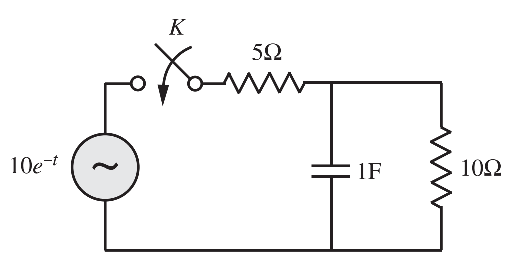


Figure E.P. 5.37

Ans: $i(t) = 2.57e^{-t} - 0.57e^{-0.3t}$ **Amperes**, $t \geq 0$

Chapter 7 Two Port Networks

7.1 Introduction

A pair of terminals through which a current may enter or leave a network is known as a port. A port is an access to the network and consists of a pair of terminals; the current entering one terminal leaves through the other terminal so that the net current entering the port equals zero. There are several reasons why we should study two-ports and the parameters that describe them. For example, most circuits have two ports. We may apply an input signal in one port and obtain an output signal from the other port. The parameters of a two-port network completely describes its behaviour in terms of the voltage and current at each port. Thus, knowing the parameters of a two port network permits us to describe its operation when it is connected into a larger network. Two-port networks are also important in modeling electronic devices and system components. For example, in electronics, two-port networks are employed to model transistors and Op-amps. Other examples of electrical components modeled by two-ports are transformers and transmission lines.

Four popular types of two-ports parameters are examined here: impedance, admittance, hybrid, and transmission. We show the usefulness of each set of parameters, demonstrate how they are related to each other.

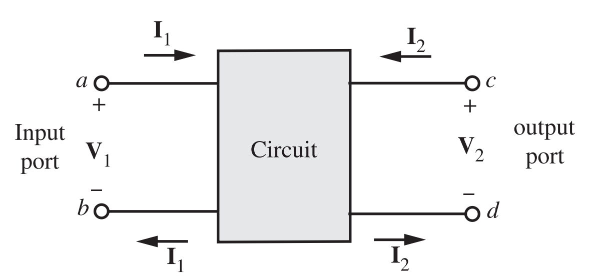


Figure 7.1 A two-port network

Fig. 7.1 represents a two-port network. A four terminal network is called a two-port network when the current entering one terminal of a pair exits the other terminal in the pair. For example, $\mathbf{I}_1$ enters terminal a and exits terminal b of the input terminal pair a-b.

We assume that there are no independent sources or nonzero initial conditions within the linear two-port network.

7.2 Admittance parameters

The network shown in Fig. 7.2 is assumed to be linear and contains no independent sources. Hence, principle of superposition can be applied to determine the current $\mathbf{I}_1$, which can be written as the sum of two components, one due to $\mathbf{V}_1$ and the other due to $\mathbf{V}_2$. Using this principle, we can write

$$\mathbf{I}_1 = \mathbf{y}_{11}\mathbf{V}_1 + \mathbf{y}_{12}\mathbf{V}_2$$

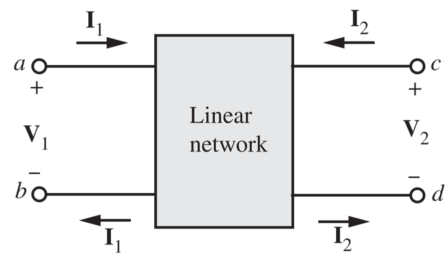


Figure 7.2 A linear two-port network

where $\mathbf{y}_{11}$ and $\mathbf{y}_{12}$ are the constants of proportionality with units of Siemens.

In a similar way, we can write

$$\mathbf{I}_2 = \mathbf{y}_{21}\mathbf{V}_1 + \mathbf{y}_{22}\mathbf{V}_2$$

Hence, the two equations that describe the two-port network are

$$\mathbf{I}_1 = \mathbf{y}_{11}\mathbf{V}_1 + \mathbf{y}_{12}\mathbf{V}_2 \tag{7.1}$$

$$\mathbf{I}_2 = \mathbf{y}_{21}\mathbf{V}_1 + \mathbf{y}_{22}\mathbf{V}_2 \tag{7.2}$$

Putting the above equations in matrix form, we get

$$\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{y}_{11} & \mathbf{y}_{12} \\ \mathbf{y}_{21} & \mathbf{y}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix}$$

Here the constants of proportionality $\mathbf{y}_{11}, \mathbf{y}_{12}, \mathbf{y}_{21}$ and $\mathbf{y}_{22}$ are called $\mathbf{y}$ parameters for a network. If these parameters $\mathbf{y}_{11}$, $\mathbf{y}_{12}$, $\mathbf{y}_{21}$ and $\mathbf{y}_{22}$ are known, then the input/output operation of the two-port is completely defined.

From equations (7.1) and (7.2), we can determine $\mathbf{y}$ parameters. We obtain $\mathbf{y}_{11}$ and $\mathbf{y}_{21}$ by connecting a current source $\mathbf{I}_1$ to port 1 and short-circuiting port 2 as shown in Fig. 7.3, finding $\mathbf{V}_1$ and $\mathbf{I}_2$, and then calculating,

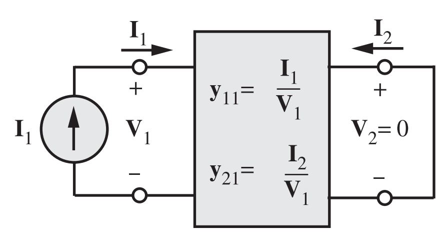


Figure 7.3 Determination of $\mathbf{y}_{11}$ and $\mathbf{y}_{12}$

$$\mathbf{y}_{11} = \left.\frac{\mathbf{I}_1}{\mathbf{V}_1}\right|_{\mathbf{V}_2=0} \qquad \mathbf{y}_{21} = \left.\frac{\mathbf{I}_2}{\mathbf{V}_1}\right|_{\mathbf{V}_2=0}$$

Since $\mathbf{y}_{11}$ is the admittance at the input measured in siemens with the output short-circuited, it is called short-circuit input admittance. Similarly, $\mathbf{y}_{21}$ is called the short-circuit transfer admittance.

Similarly, we obtain $\mathbf{y}_{12}$ and $\mathbf{y}_{22}$ by connecting a current source $\mathbf{I}_2$ to port 2 and short-circuiting port 1 as in Fig. 7.4, finding $\mathbf{I}_1$ and $\mathbf{V}_2$, and then calculating,

$$\mathbf{y}_{12} = \left.\frac{\mathbf{I}_1}{\mathbf{V}_2}\right|_{\mathbf{V}_1=0} \qquad \mathbf{y}_{22} = \left.\frac{\mathbf{I}_2}{\mathbf{V}_2}\right|_{\mathbf{V}_1=0}$$

$\mathbf{y}_{12}$ is called the *short-circuit transfer admittance* and $\mathbf{y}_{22}$ is called the *short-circuit output admittance*. Collectively the **y** parameters are referred to as short-circuit admittance parameters.

Please note that $\mathbf{y}_{12} = \mathbf{y}_{21}$ only when there are no dependent sources or Op-amps within the two-port network.

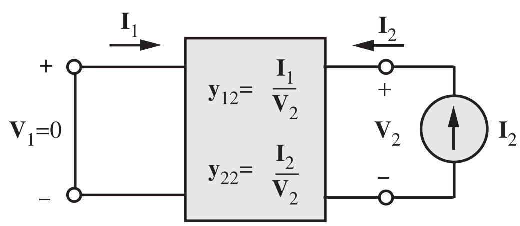


Figure 7.4 Determination of $\mathbf{y}_{12}$ and $\mathbf{y}_{22}$

EXAMPLE 7.1

Determine the admittance parameters of the T network shown in Fig. 7.5.

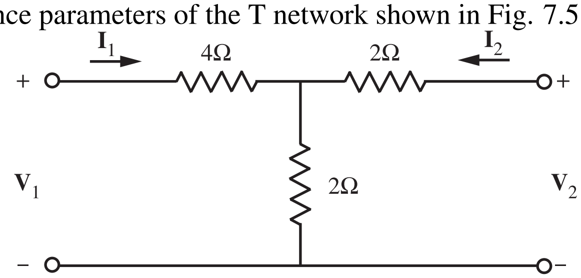


Figure 7.5

SOLUTION

To find $\mathbf{y}_{11}$ and $\mathbf{y}_{21}$, we have to short the output terminals and connect a current source $\mathbf{I}_1$ to the input terminals. The circuit so obtained is shown in Fig. 7.6(a).

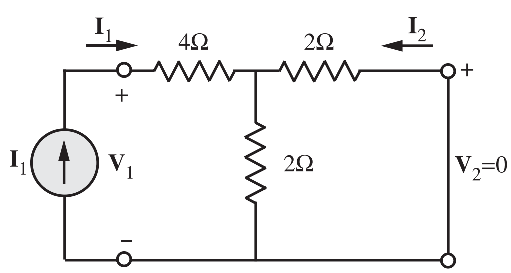


Figure 7.6(a)

$$\mathbf{I}_1 = \frac{\mathbf{V}_1}{4 + \dfrac{2\times 2}{2+2}} = \frac{\mathbf{V}_1}{5}$$

Hence, $\quad \mathbf{y}_{11} = \left.\dfrac{\mathbf{I}_1}{\mathbf{V}_1}\right|_{\mathbf{V}_2=0} = \dfrac{1}{5}\text{S}$

Using the principle of current division,

$$-\mathbf{I}_2 = \frac{\mathbf{I}_1 \times 2}{2+2} = \frac{\mathbf{I}_1}{2}$$

$$\Rightarrow \qquad -\mathbf{I}_2 = \frac{1}{2}\left[\frac{\mathbf{V}_1}{5}\right]$$

Hence, $\quad \mathbf{y}_{21} = \left.\dfrac{\mathbf{I}_2}{\mathbf{V}_1}\right|_{\mathbf{V}_2=0} = \dfrac{-1}{10}\text{S}$

To find $\mathbf{y}_{12}$ and $\mathbf{y}_{22}$, we have to short-circuit the input terminals and connect a current source $\mathbf{I}_2$ to the output terminals. The circuit so obtained is shown in Fig. 7.6(b).

$$\mathbf{I}_2 = \frac{\mathbf{V}_2}{2 + \dfrac{4 \times 2}{4+2}} = \frac{\mathbf{V}_2}{2 + \dfrac{4}{3}} = \frac{3\mathbf{V}_2}{10}$$

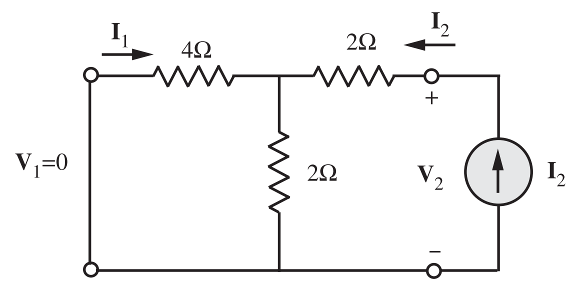


Figure 7.6(b)

Hence, $$\mathbf{y}_{22} = \left.\frac{\mathbf{I}_2}{\mathbf{V}_2}\right|_{\mathbf{V}_1=0} = \frac{3}{10}\mathrm{S}$$

Employing the principle of current division, we have

$$-\mathbf{I}_1 = \frac{\mathbf{I}_2 \times 2}{2+4}$$

$$\Rightarrow \quad -\mathbf{I}_1 = \frac{2\mathbf{I}_2}{6}$$

$$\Rightarrow \quad -\mathbf{I}_1 = \frac{1}{3}\left[\frac{3\mathbf{V}_2}{10}\right]$$

Hence, $$\mathbf{y}_{12} = \left.\frac{\mathbf{I}_1}{\mathbf{V}_2}\right|_{\mathbf{V}_1=0} = \frac{-1}{10}\mathrm{S}$$

It may be noted that, $\mathbf{y}_{12} = \mathbf{y}_{21}$. Thus, in matrix form we have

$$\mathbf{I} = \mathbf{YV}$$

$$\Rightarrow \quad \begin{bmatrix} \mathbf{I_1} \\ \mathbf{I_2} \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{-1}{10} \\ \frac{-1}{10} & \frac{3}{10} \end{bmatrix} \begin{bmatrix} \mathbf{V_1} \\ \mathbf{V_2} \end{bmatrix}$$

EXAMPLE 7.2

Find the $\mathbf{y}$ parameters of the two-port network shown in Fig. 7.7. Then determine the current in a 4Ω load, that is connected to the output port when a 2A source is applied at the input port.

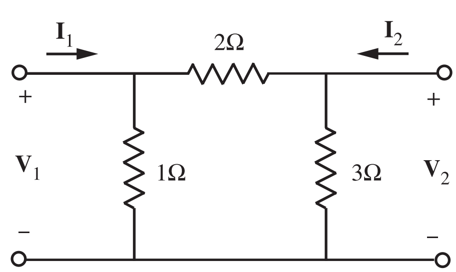


Figure 7.7

SOLUTION

To find $\mathbf{y}_{11}$ and $\mathbf{y}_{21}$, short-circuit the output terminals and connect a current source $\mathbf{I}_1$ to the input terminals. The resulting circuit diagram is shown in Fig. 7.8(a).

$$\mathbf{I}_1 = \frac{\mathbf{V}_1}{1\Omega||2\Omega} = \frac{\mathbf{V}_1}{\frac{1\times 2}{1+2}}$$

$$\Rightarrow \qquad \mathbf{I}_1 = \frac{3}{2}\mathbf{V}_1$$

Hence, $$\mathbf{y}_{11} = \left.\frac{\mathbf{I}_1}{\mathbf{V}_1}\right|_{\mathbf{V}_2=0} = \frac{3}{2}\text{S}$$

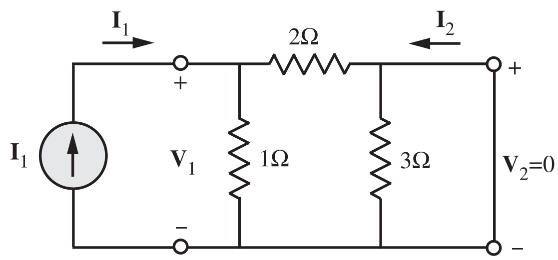


Figure 7.8(a)

Using the principle of current division,

$$-\mathbf{I}_2 = \frac{\mathbf{I}_1 \times 1}{1+2}$$

$$\Rightarrow \qquad -\mathbf{I}_2 = \frac{1}{3}\mathbf{I}_1$$

$$\Rightarrow \qquad -\mathbf{I}_2 = \frac{1}{3}\left[\frac{3}{2}\mathbf{V}_1\right]$$

Hence, $$\mathbf{y}_{21} = \frac{\mathbf{I}_2}{\mathbf{V}_1} = \frac{-1}{2}\text{S}$$

To find $\mathbf{y}_{12}$ and $\mathbf{y}_{22}$, short the input terminals and connect a current source $\mathbf{I}_2$ to the output terminals. The resulting circuit diagram is shown in Fig. 7.8(b).

$$\mathbf{I}_2 = \frac{\mathbf{V}_2}{2\Omega||3\Omega}$$

$$= \frac{\mathbf{V}_2}{\frac{2\times 3}{2+3}} = \frac{5\mathbf{V}_2}{6}$$

$$\mathbf{y}_{22} = \left.\frac{\mathbf{I}_2}{\mathbf{V}_2}\right|_{\mathbf{V}_1=0} = \frac{5}{6}\text{S}$$

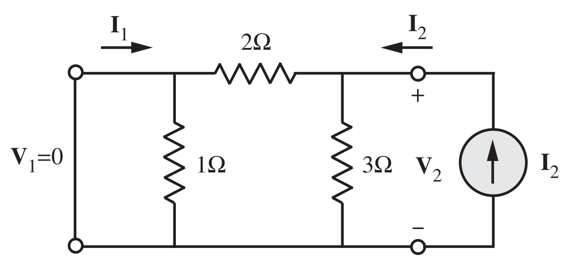


Figure 7.8(b)

Employing the current division principle,

$$-\mathbf{I}_1 = \frac{\mathbf{I}_2 \times 3}{2+3}$$

$$\Rightarrow \qquad -\mathbf{I}_1 = \frac{3}{5}\mathbf{I}_2$$

$$\Rightarrow \qquad -\mathbf{I}_1 = \frac{3}{5}\left[\frac{5\mathbf{V}_2}{6}\right]$$

$$\Rightarrow \qquad \mathbf{I}_1 = \frac{-1}{2}\mathbf{V}_2$$

Hence,

$$\mathbf{y}_{12} = \left.\frac{-\mathbf{I}_1}{\mathbf{V}_2}\right|_{\mathbf{V}_1=0} = \frac{-1}{2}\text{S}$$

Therefore, the equations that describe the two-port network are

$$\mathbf{I}_1 = \frac{3}{2}\mathbf{V}_1 - \frac{1}{2}\mathbf{V}_2 \tag{7.3}$$

$$\mathbf{I}_2 = -\frac{1}{2}\mathbf{V}_1 + \frac{5}{6}\mathbf{V}_2 \tag{7.4}$$

Putting the above equations (7.3) and (7.4) in matrix form, we get

$$\begin{bmatrix} \frac{3}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{5}{6} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix}$$

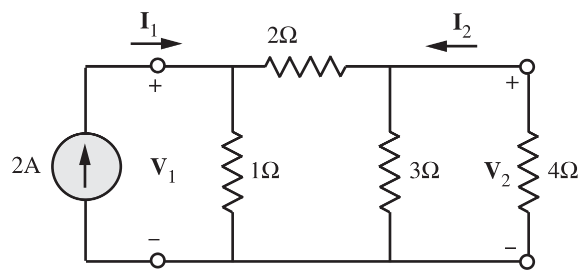


Figure 7.8(c)

Referring to Fig. 7.8(c), we find that $\mathbf{I}_1 = 2\text{A}$ and $\mathbf{V}_2 = -4\mathbf{I}_2$

Substituting $\mathbf{I}_1 = 2\text{A}$ in equation (7.3), we get

$$2 = \frac{3}{4}\mathbf{V}_1 - \frac{1}{2}\mathbf{V}_2 \tag{7.5}$$

Multiplying equation (7.4) by -4, we get

$$-4\mathbf{I}_2 = 2\mathbf{V}_1 - \frac{20}{6}\mathbf{V}_2$$

$$\Rightarrow \qquad \mathbf{V}_2 = 2\mathbf{V}_1 - \frac{20}{6}\mathbf{V}_2$$

$$\Rightarrow \qquad 0 = 2\mathbf{V}_1 - \left(\frac{20}{6} + 1\right)\mathbf{V}_2$$

$$\Rightarrow \qquad 0 = \frac{-1}{2}\mathbf{V}_1 + \frac{13}{12}\mathbf{V}_2 \tag{7.6}$$

Putting equations (7.5) and (7.6) in matrix form, we get

$$\begin{bmatrix} \frac{3}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{13}{12} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

It may be noted that the above equations are simply the nodal equations for the circuit shown in Fig. 7.8(c). Solving these equations, we get

$$\mathbf{V}_2 = \frac{3}{2}\text{V}$$

and hence,
$$\mathbf{I}_2 = \frac{-1}{4}\mathbf{V}_2 = \frac{-3}{8}\mathbf{A}$$

EXAMPLE 7.3

Refer the network shown in the Fig. 7.9 containing a current-controlled current source. For this network, find the **y** parameters.

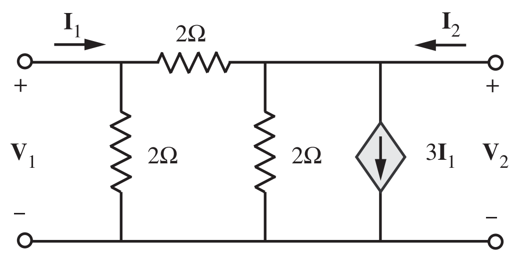


Figure 7.9

SOLUTION

To find $\mathbf{y}_{11}$ and $\mathbf{y}_{21}$ short the output terminals and connect a current source $\mathbf{I}_1$ to the input terminals. The resulting circuit diagram is as shown in Fig. 7.10(a) and it is further reduced to Fig. 7.10(b).

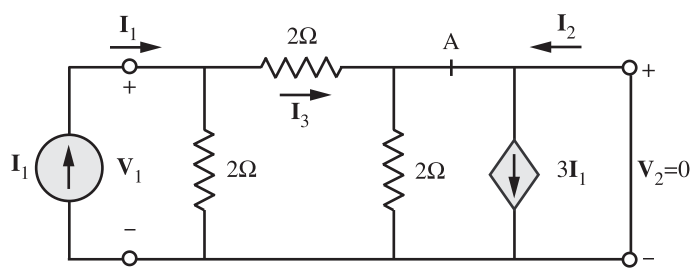


Figure 7.10(a)

$$\mathbf{I}_1 = \frac{\mathbf{V}_1}{\dfrac{2\times 2}{2+2}}$$

$$\Rightarrow \quad \mathbf{I}_1 = \mathbf{V}_1$$

Hence,
$$\mathbf{y}_{11} = \left.\frac{\mathbf{I}_1}{\mathbf{V}_1}\right|_{\mathbf{V}_2=0} = 1\text{S}$$

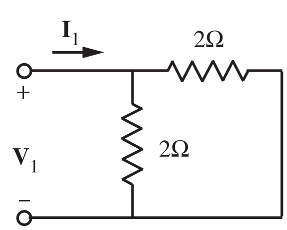


Figure 7.10(b)

Applying KCL at node A gives (Referring to Fig. 7.10(a)).

$$\mathbf{I}_3 + \mathbf{I}_2 = 3\mathbf{I}_1$$

$$\Rightarrow \qquad \frac{\mathbf{V}_1}{2} + \mathbf{I}_2 = 3\mathbf{I}_1$$

$$\Rightarrow \qquad \frac{\mathbf{V}_1}{2} + \mathbf{I}_2 = 3\mathbf{V}_1$$

$$\Rightarrow \qquad \frac{5\mathbf{V}_1}{2} = \mathbf{I}_2$$

Hence,
$$\mathbf{y}_{21} = \frac{\mathbf{I}_2}{\mathbf{V}_1} = \frac{5}{2}\text{S}$$

To find $\mathbf{y}_{22}$ and $\mathbf{y}_{12}$, short the input terminals and connect a current source $\mathbf{I}_2$ at the output terminals. The resulting circuit diagram is shown in Fig. 7.10(c) and further reduced to Fig. 7.10(d).

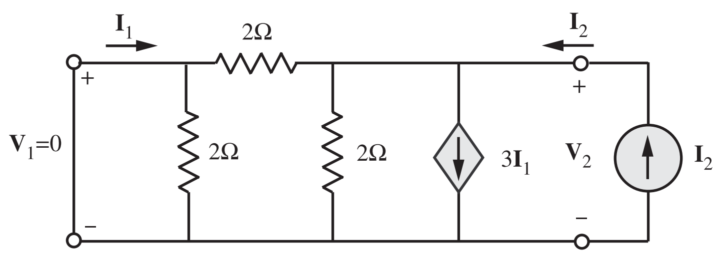


Figure 7.10(c)

$$\mathbf{I}_2 = -\mathbf{I}_1' = -\frac{\mathbf{V}_2}{2}$$

$$\Rightarrow \qquad -\mathbf{I}_1 = \frac{\mathbf{V}_2}{2}$$

Hence,
$$\mathbf{y}_{12} = \frac{\mathbf{I}_1}{\mathbf{V}_2} = \frac{-1}{2}\text{S}$$

Applying KCL at node B gives

$$\mathbf{I}_2 = \frac{\mathbf{V}_2}{2} + \frac{\mathbf{V}_2}{2} + 3\mathbf{I}_1$$

But
$$\mathbf{I}_1 = \frac{-\mathbf{V}_2}{2}$$

Hence,
$$\mathbf{I}_2 = \frac{\mathbf{V}_2}{2} + \frac{\mathbf{V}_2}{2} - 3\frac{\mathbf{V}_2}{2}$$

$$\Rightarrow \qquad \mathbf{y}_{22} = \left.\frac{\mathbf{I}_2}{\mathbf{V}_2}\right|_{\mathbf{V}_1=0} = -0.5\text{S}$$

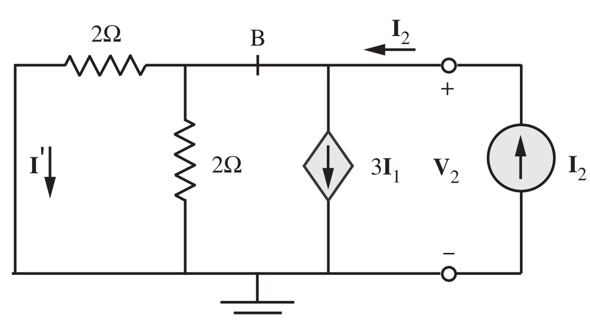


Figure 7.10(d)

Short-cut method:
Referring to Fig. 7.9, we have *KCL at node* $\mathbf{V}_1$:

$$\mathbf{I}_1 = \frac{\mathbf{V}_1}{2} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{2}$$
$$= \mathbf{V}_1 - 0.5\mathbf{V}_2$$

Comparing with

$$\mathbf{I}_1 = \mathbf{y}_{11}\mathbf{V}_1 + \mathbf{y}_{12}\mathbf{V}_2$$

we get

$$\mathbf{y}_{11} = 1\text{S and } \mathbf{y}_{12} = -0.5\text{S}$$

KCL at node $\mathbf{V}_2$:

$$\mathbf{I}_2 = 3\mathbf{I}_1 + \frac{\mathbf{V}_2}{2} + \frac{\mathbf{V}_2 - \mathbf{V}_1}{2}$$
$$= 3\left[\mathbf{V}_1 - 0.5\mathbf{V}_2\right] + \frac{\mathbf{V}_2}{2} + \frac{\mathbf{V}_2 - \mathbf{V}_1}{2}$$
$$\Rightarrow \quad \mathbf{I}_2 = \frac{5}{2}\mathbf{V}_1 - 0.5\mathbf{V}_2$$

Comparing with $\mathbf{I}_2 = \mathbf{y}_{21}\mathbf{V}_1 + \mathbf{y}_{22}\mathbf{V}_2$
we get

$$\mathbf{y}_{21} = 2.5\text{S and } \mathbf{y}_{22} = -0.5\text{S}$$

EXAMPLE 7.4

Find the **y** parameters for the two-port network shown in Fig. 7.11.

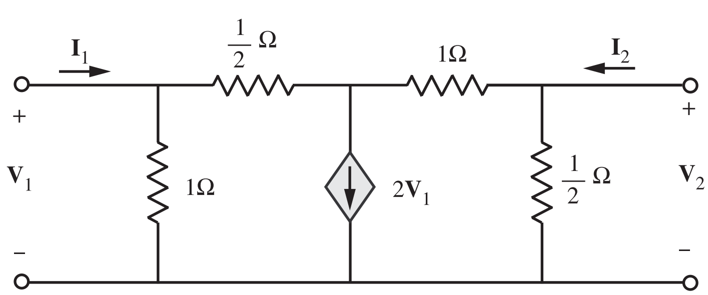


Figure 7.11

SOLUTION

To find $\mathbf{y}_{11}$ and $\mathbf{y}_{21}$ short-circuit the output terminals as shown in Fig. 7.12(a). Also connect a current source $\mathbf{I}_1$ to the input terminals.

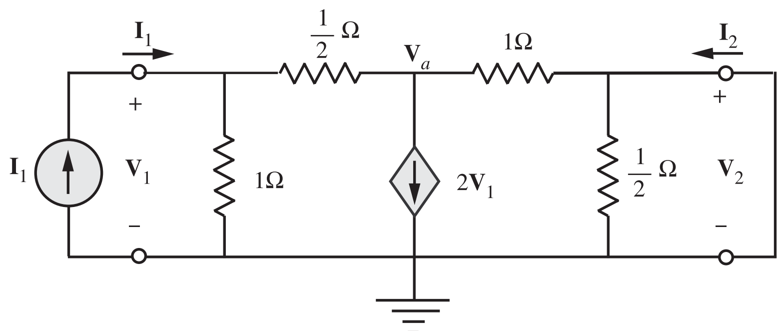


Figure 7.12(a)

KCL at node $\mathbf{V}_1$:

$$\mathbf{I}_1 = \frac{\mathbf{V}_1}{1} + \frac{\mathbf{V}_1 - \mathbf{V}_a}{\frac{1}{2}}$$

$$\Rightarrow \qquad 3\mathbf{V}_1 - 2\mathbf{V}_a = \mathbf{I}_1 \tag{7.7}$$

KCL at node $\mathbf{V}_a$:

$$\frac{\mathbf{V}_a - \mathbf{V}_1}{\frac{1}{2}} + \frac{\mathbf{V}_a - 0}{1} + 2\mathbf{V}_1 = 0$$

$$\Rightarrow \qquad 2\mathbf{V}_a - 2\mathbf{V}_1 + \mathbf{V}_a + 2\mathbf{V}_1 = 0$$

$$\Rightarrow \qquad \mathbf{V}_a = 0 \tag{7.8}$$

Making use of equation (7.8) in (7.7), we get

$$3\mathbf{V}_1 = \mathbf{I}_1$$

Hence,

$$\mathbf{y}_{11} = \left.\frac{\mathbf{I}_1}{\mathbf{V}_1}\right|_{\mathbf{V}_2=0} = 3\text{S}$$

Since $\mathbf{V}_a = 0$, $\mathbf{I}_2 = 0$,

$$\Rightarrow \qquad \mathbf{y}_{21} = \left.\frac{\mathbf{I}_2}{\mathbf{V}_1}\right|_{\mathbf{V}_2=0} = 0\text{S}$$

To find $\mathbf{y}_{22}$ and $\mathbf{y}_{12}$ short-circuit the input terminals and connect a current source $\mathbf{I}_2$ to the output terminals. The resulting circuit diagram is shown in Fig. 7.12(b).

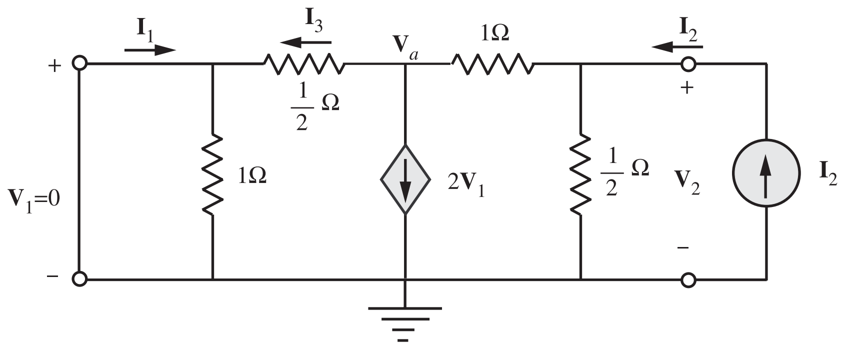


Figure 7.12(b)

KCL at node $\mathbf{V}_2$:

$$\frac{\mathbf{V}_2}{\frac{1}{2}} + \frac{\mathbf{V}_2 - \mathbf{V}_a}{1} = \mathbf{I}_2$$

$$\Rightarrow \qquad 3\mathbf{V}_2 - \mathbf{V}_a = \mathbf{I}_2 \tag{7.9}$$

KCL at node $\mathbf{V}_a$:

$$\frac{\mathbf{V}_a - \mathbf{V}_2}{1} + \frac{\mathbf{V}_a - 0}{\frac{1}{2}} + 0 = 0$$

$$\Rightarrow \qquad 3\mathbf{V}_a - \mathbf{V}_2 = 0$$

$$\text{or} \qquad \mathbf{V}_a = \frac{1}{3}\mathbf{V}_2 \tag{7.10}$$

Substituting equation (7.10) in (7.9), we get

$$3\mathbf{V}_2 - \frac{1}{3}\mathbf{V}_2 = \mathbf{I}_2$$

$$\Rightarrow \qquad \frac{8}{3}\mathbf{V}_2 = \mathbf{I}_2$$

Hence, $$\mathbf{y}_{22} = \frac{\mathbf{I}_2}{\mathbf{V}_2} = \frac{8}{3}\text{S}$$

We have, $$\mathbf{V}_a = \frac{1}{3}\mathbf{V}_2 \tag{7.11}$$

Also, $$\mathbf{I}_1 + \mathbf{I}_3 = 0$$

$$\Rightarrow \qquad \mathbf{I}_1 = -\mathbf{I}_3$$

$$= \frac{-\mathbf{V}_a}{\frac{1}{2}} = -2\mathbf{V}_a \tag{7.12}$$

Making use of equation (7.12) in (7.11), we get

$$-\frac{\mathbf{I}_1}{2} = \frac{1}{3}\mathbf{V}_2$$

Hence,
$$\mathbf{y}_{12} = \left.\frac{\mathbf{I}_1}{\mathbf{V}_2}\right|_{\mathbf{V}_1=0} = \frac{-2}{3}\mathrm{S}$$

EXAMPLE 7.5

Find the **y** parameters for the resistive network shown in Fig. 7.13.

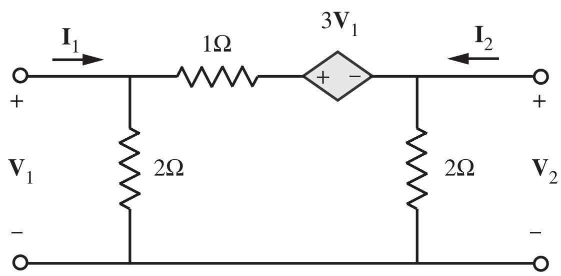


Figure 7.13

SOLUTION

Converting the voltage source into an equivalent current source, we get the circuit diagram shown in Fig. 7.14(a).

To find $\mathbf{y}_{11}$ and $\mathbf{y}_{21}$, the output terminals of Fig. 7.14(a) are shorted and connect a current source $\mathbf{I}_1$ to the input terminals. This results in a circuit diagram as shown in Fig. 7.14(b).

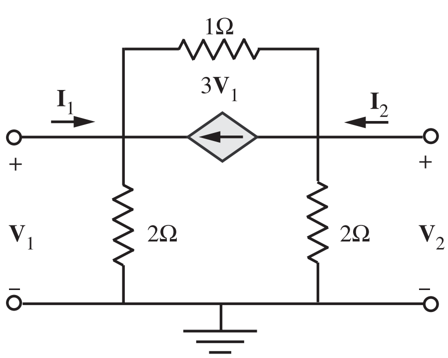


Figure 7.14(a)

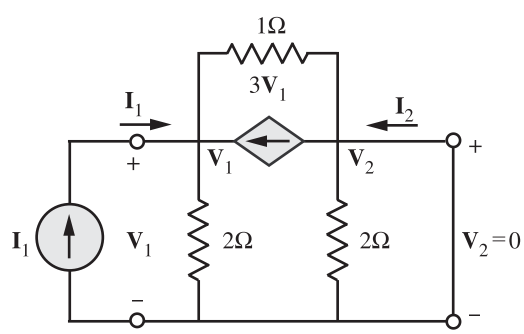


Figure 7.14(b)

KCL at node $\mathbf{V}_1$:

$$\frac{\mathbf{V}_1}{2} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{1} = \mathbf{I}_1 + 3\mathbf{V}_1$$

Since $\mathbf{V}_2 = 0$, we get

$$\frac{\mathbf{V}_1}{2} + \mathbf{V}_1 = \mathbf{I}_1 + 3\mathbf{V}_1$$

$$\Rightarrow \qquad \mathbf{I}_1 = \frac{-3}{2}\mathbf{V}_1$$

Hence,
$$\mathbf{y}_{11} = \left.\frac{\mathbf{I}_1}{\mathbf{V}_1}\right|_{\mathbf{V}_2=0} = \frac{-3}{2}\text{S}$$

KCL at node $\mathbf{V}_2$:

$$\frac{\mathbf{V}_2}{2} + 3\mathbf{V}_1 + \frac{\mathbf{V}_2 - \mathbf{V}_1}{1} = \mathbf{I}_2$$

Since $\mathbf{V}_2 = 0$, we get

$$0 + 3\mathbf{V}_1 - \mathbf{V}_1 = \mathbf{I}_2$$

$$\Rightarrow \qquad \mathbf{I}_2 = 2\mathbf{V}_1$$

Hence
$$\mathbf{y}_{21} = \frac{\mathbf{I}_1}{\mathbf{V}_2} = 2\text{S}$$

To find $\mathbf{y}_{21}$ and $\mathbf{y}_{22}$, the input terminals of Fig. 7.14(a) are shorted and connect a current source $\mathbf{I}_2$ to the output terminals. This results in a circuit diagram as shown in Fig. 7.14(c).

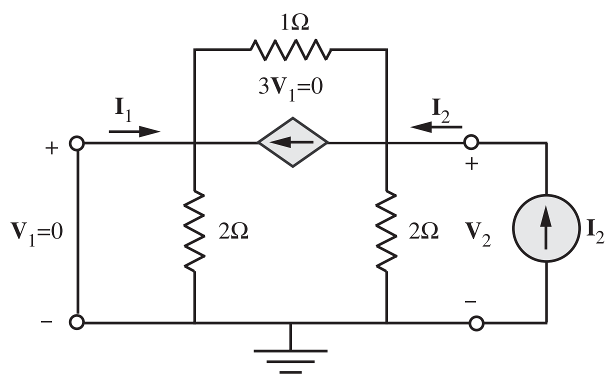


Figure 7.14(c)

$\Rightarrow$

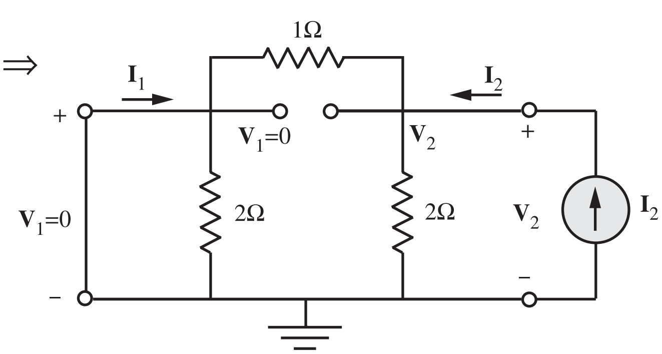


Figure 7.14(d)

KCL at node $\mathbf{V}_2$:

$$\frac{\mathbf{V}_2}{2} + \frac{\mathbf{V}_2 - 0}{1} = \mathbf{I}_2$$

$$\Rightarrow \qquad \frac{3}{2}\mathbf{V}_2 = \mathbf{I}_2$$

Hence,
$$\mathbf{y}_{22} = \frac{\mathbf{I}_2}{\mathbf{V}_2} = \frac{3}{2}\text{S}$$

KCL at node $\mathbf{V}_1$:

$$\mathbf{I}_1 = \frac{\mathbf{V}_1}{2} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{1} = 0$$

Since $\mathbf{V}_1 = 0$, we get

$$\mathbf{I}_1 = -\mathbf{V}_2$$

Hence,
$$\mathbf{y}_{12} = \frac{\mathbf{I}_1}{\mathbf{V}_2} = -1\text{S}$$

EXAMPLE 7.6

The network of Fig. 7.15 contains both a dependent current source and a dependent voltage source. Find the **y** parameters.

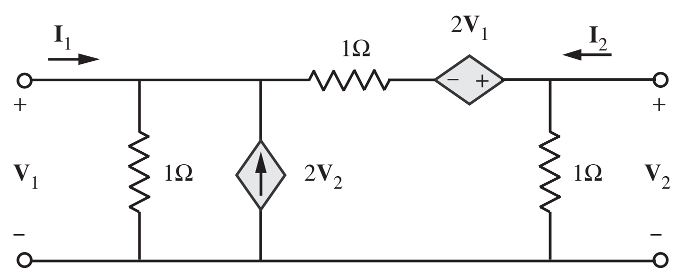


Figure 7.15

SOLUTION

While finding **y** parameters, we make use of *KCL* equations. Hence, it is preferable to have current sources rather than voltage sources. This prompts us to convert the dependent voltage source into an equivalent current source and results in a circuit diagram as shown in Fig. 7.16(a).

To find $\mathbf{y}_{11}$ and $\mathbf{y}_{21}$, refer the circuit diagram as shown in Fig. 7.16(b).

KCL at node $\mathbf{V}_1$:

$$\frac{\mathbf{V}_1}{1} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{1} + 2\mathbf{V}_1 = 2\mathbf{V}_2 + \mathbf{I}_1$$

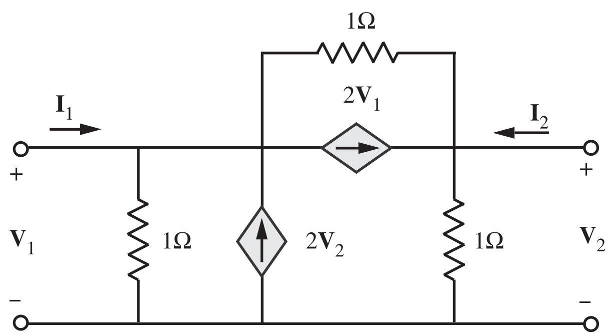


Figure 7.16(a)

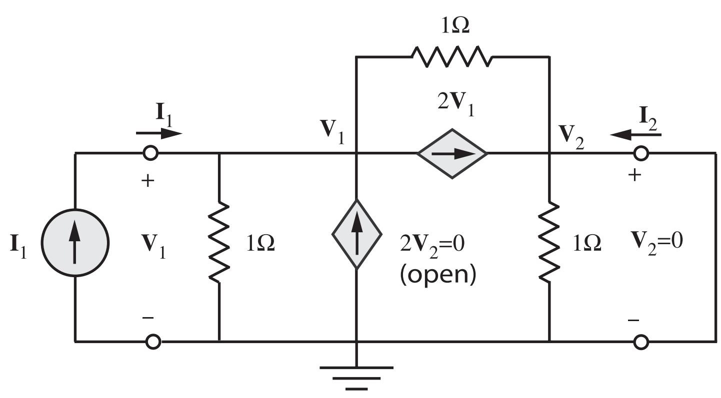


Figure 7.16(b)

Since $\mathbf{V}_2 = 0$, we get

$$\mathbf{V}_1 + \mathbf{V}_1 + 2\mathbf{V}_1 = \mathbf{I}_1$$

$$\Rightarrow \qquad 4\mathbf{V}_1 = \mathbf{I}_1$$

Hence,
$$\mathbf{y}_{11} = \frac{\mathbf{I}_1}{\mathbf{V}_1} = 4\text{S}$$

KCL at node $\mathbf{V}_2$:

$$\frac{\mathbf{V}_2}{1} + \frac{\mathbf{V}_2 - \mathbf{V}_1}{1} = 2\mathbf{V}_1 + \mathbf{I}_2$$

Since $\mathbf{V}_2 = 0$, we get

$$-\mathbf{V}_1 = 2\mathbf{V}_1 + \mathbf{I}_2$$

$$\Rightarrow \qquad -3\mathbf{V}_1 = \mathbf{I}_2$$

Hence,
$$\mathbf{y}_{21} = \left.\frac{\mathbf{I}_2}{\mathbf{V}_1}\right|_{\mathbf{V}_2=0} = -3\text{S}$$

To find $\mathbf{y}_{22}$ and $\mathbf{y}_{12}$, refer the circuit diagram shown in Fig. 7.16(c).

KCL at node $\mathbf{V}_1$:

$$\frac{\mathbf{V}_1}{1} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{1} + 2\mathbf{V}_1 = 2\mathbf{V}_2 + \mathbf{I}_1$$

Since $\mathbf{V}_1 = 0$, we get

$$-\mathbf{V}_2 = 2\mathbf{V}_2 + \mathbf{I}_1$$

$$\Rightarrow \qquad -3\mathbf{V}_2 = \mathbf{I}_1$$

Hence, $$\mathbf{y}_{12} = \left.\frac{\mathbf{I}_1}{\mathbf{V}_2}\right|_{\mathbf{V}_1=0} = -3\text{S}$$

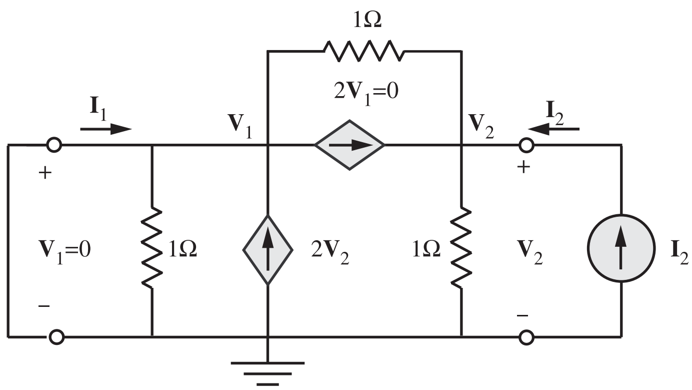


Figure 7.16(c)

KCL at node $\mathbf{V}_2$:

$$\frac{\mathbf{V}_2}{1} + \frac{\mathbf{V}_2 - \mathbf{V}_1}{1} = 2\mathbf{V}_1 + \mathbf{I}_2$$

Since $\mathbf{V}_1 = 0$, we get

$$\mathbf{V}_2 + \mathbf{V}_2 = 0 + \mathbf{I}_2$$

$$\Rightarrow \qquad -2\mathbf{V}_2 = \mathbf{I}_2$$

Hence, $$\mathbf{y}_{22} = \left.\frac{\mathbf{I}_2}{\mathbf{V}_2}\right|_{\mathbf{V}_1=0} = 2\text{S}$$

7.3 Impedance parameters

Let us assume the two port network shown in Fig. 7.17 is a linear network that contains no independent sources. Then using superposition theorem, we can write the input and output voltages as the sum of two components, one due to $\mathbf{I}_1$ and other due to $\mathbf{I}_2$:

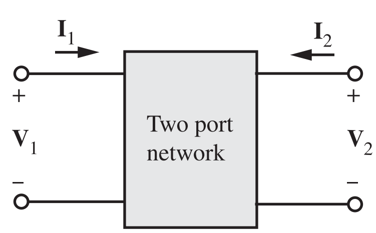


Figure 7.17

$$\mathbf{V}_1 = \mathbf{z}_{11}\mathbf{I}_1 + \mathbf{z}_{12}\mathbf{I}_2$$
$$\mathbf{V}_2 = \mathbf{z}_{21}\mathbf{I}_1 + \mathbf{z}_{22}\mathbf{I}_2$$

Putting the above equations in matrix from, we get

$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{z}_{11} & \mathbf{z}_{12} \\ \mathbf{z}_{21} & \mathbf{z}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix}$$

The **z** parameters are defined as follows:

$$\mathbf{z}_{11} = \left.\frac{\mathbf{V}_1}{\mathbf{I}_1}\right|_{\mathbf{I}_2=0} \qquad \mathbf{z}_{12} = \left.\frac{\mathbf{V}_1}{\mathbf{I}_2}\right|_{\mathbf{I}_1=0} \qquad \mathbf{z}_{21} = \left.\frac{\mathbf{V}_2}{\mathbf{I}_1}\right|_{\mathbf{I}_2=0} \qquad \mathbf{z}_{22} = \left.\frac{\mathbf{V}_2}{\mathbf{I}_2}\right|_{\mathbf{I}_1=0}$$

In the preceeding equations, letting $\mathbf{I}_1$ or $\mathbf{I}_2 = 0$ is equivalent to open-circuiting the input or output port. Hence, the **z** parameters are called *open-circuit impedance parameters.* $\mathbf{z}_{11}$ is defined as the *open-circuit input impedance,* $\mathbf{z}_{22}$ is called the *open-circuit output impedance,* and $\mathbf{z}_{12}$ and $\mathbf{z}_{21}$ are called the *open-circuit transfer impedances.*

If $\mathbf{z}_{12} = \mathbf{z}_{21}$, the network is said to be **reciprocal network**. Also, if all the **z**-parameter are identical, then it is called a **symmetrical network**.

EXAMPLE 7.7

Refer the circuit shown in Fig. 7.18. Find the **z** parameters of this circuit. Then compute the current in a 4Ω load if a $24\underline{/0^\circ}$ V source is connected at the input port.

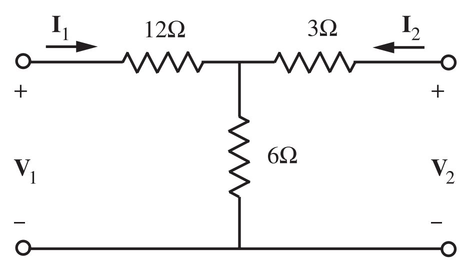


Figure 7.18

SOLUTION

To find $\mathbf{z}_{11}$ and $\mathbf{z}_{21}$, the output terminals are open circuited. Also connect a voltage source $\mathbf{V}_1$ to the input terminals. This gives a circuit diagram as shown in Fig. 7.19(a).

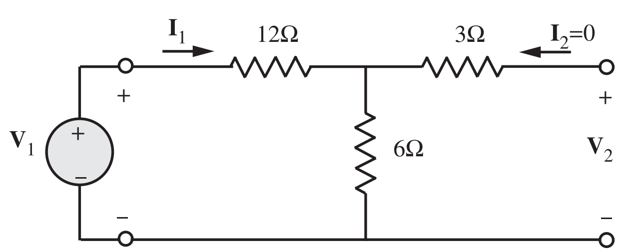


Figure 7.19(a)

Applying KVL to the left-mesh, we get

$$12\mathbf{I}_1 + 6\mathbf{I}_1 = \mathbf{V}_1$$

$$\Rightarrow \qquad \mathbf{V}_1 = 18\mathbf{I}_1$$

Hence,
$$\mathbf{z}_{11} = \left.\frac{\mathbf{V}_1}{\mathbf{I}_1}\right|_{\mathbf{I}_2=0} = 18\Omega$$

Applying KVL to the right-mesh, we get

$$-\mathbf{V}_2 + 3 \times 0 + 6\mathbf{I}_1 = 0$$

$$\Rightarrow \qquad \mathbf{V}_2 = 6\mathbf{I}_1$$

Hence,
$$\mathbf{z}_{21} = \frac{\mathbf{V}_2}{\mathbf{I}_1} = 6\Omega$$

To find $\mathbf{z}_{22}$ and $\mathbf{z}_{12}$, the input terminals are open circuited. Also connect a voltage source $\mathbf{V}_2$ to the output terminals. This results in a network as shown in Fig. 7.19(b).

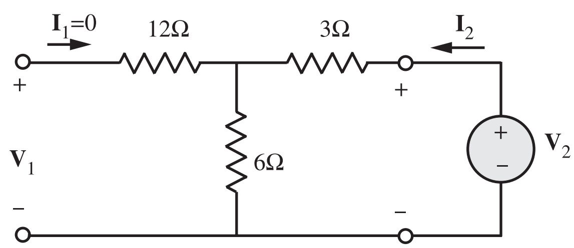


Figure 7.19(b)

Applying KVL to the left-mesh, we get

$$\mathbf{V}_1 = 12 \times 0 + 6\mathbf{I}_2$$

$$\Rightarrow \qquad \mathbf{V}_1 = 6\mathbf{I}_2$$

Hence,
$$\mathbf{z}_{12} = \left.\frac{\mathbf{V}_1}{\mathbf{I}_2}\right|_{\mathbf{I}_1=0} = 6\Omega$$

Applying KVL to the right-mesh, we get

$$-\mathbf{V}_2 + 3\mathbf{I}_2 + 6\mathbf{I}_2 = 0$$

$$\Rightarrow \qquad \mathbf{V}_2 = 9\mathbf{I}_2$$

Hence,
$$\mathbf{z}_{22} = \left.\frac{\mathbf{V}_2}{\mathbf{I}_2}\right|_{\mathbf{I}_1=0} = 9\Omega$$

The equations for the two-port network are, therefore

$$\mathbf{V}_1 = 18\mathbf{I}_1 + 6\mathbf{I}_2 \tag{7.13}$$

$$\mathbf{V}_2 = 6\mathbf{I}_1 + 9\mathbf{I}_2 \tag{7.14}$$

The terminal voltages for the network shown in Fig. 7.19(c) are

$$\mathbf{V}_1 = 24\,\underline{/0^\circ} \tag{7.15}$$

$$\mathbf{V}_2 = -4\mathbf{I}_2 \tag{7.16}$$

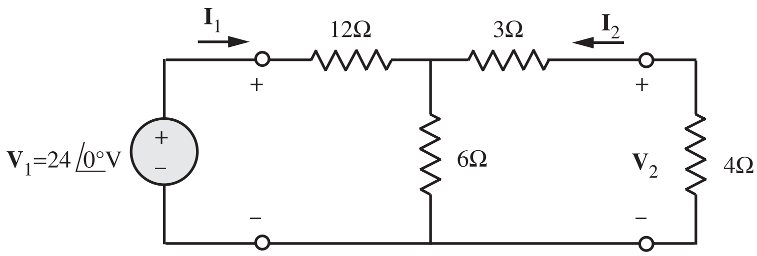


Figure 7.19(c)

Combining equations (7.15) and (7.16) with equations (7.13) and (7.14) yields

$$24\,\underline{/0^\circ} = 18\mathbf{I}_1 + 6\mathbf{I}_2$$

$$0 = 6\mathbf{I}_1 + 13\mathbf{I}_2$$

Solving, we get

$$\mathbf{I}_2 = -0.73\,\underline{/0^\circ}\text{ A}$$

EXAMPLE 7.8

Determine the **z** parameters for the two port network shown in Fig. 7.20.

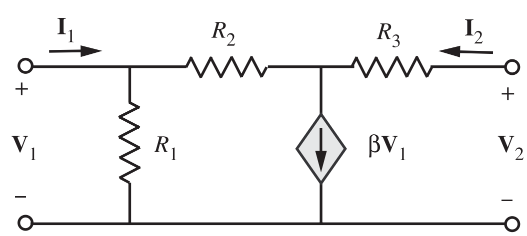


Figure 7.20

SOLUTION

To find $\mathbf{z}_{11}$ and $\mathbf{z}_{21}$, the output terminals are open-circuited and a voltage source is connected to the input terminals. The resulting circuit is shown in Fig. 7.21(a).

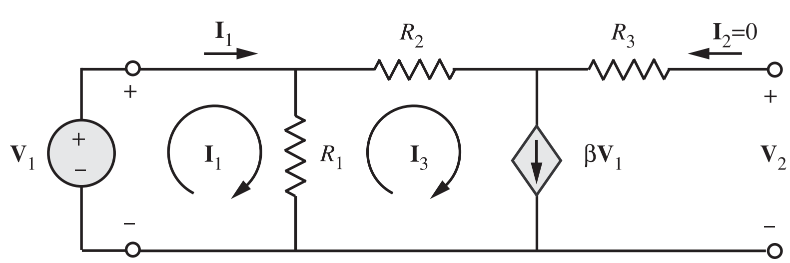


Figure 7.21(a)

By inspection, we find that $\mathbf{I}_3 = \beta \mathbf{V}_1$
Applying KVL to mesh 1, we get

$$R_1(\mathbf{I}_1 - \mathbf{I}_3) = \mathbf{V}_1$$
$$\Rightarrow \quad R_1\mathbf{I}_1 - R_1\mathbf{I}_3 = \mathbf{V}_1$$
$$\Rightarrow \quad R_1\mathbf{I}_1 - R_1\beta\mathbf{V}_1 = \mathbf{V}_1$$
$$\Rightarrow \quad (1 + R_1\beta)\,\mathbf{V}_1 = R_1\mathbf{I}_1$$

Hence,
$$\mathbf{z}_{11} = \left.\frac{\mathbf{V}_1}{\mathbf{I}_1}\right|_{\mathbf{I}_2=0} = \frac{R_1}{1+\beta R_1}$$

Applying KVL to the path $\mathbf{V}_1 \rightarrow R_2 \rightarrow R_3 \rightarrow \mathbf{V}_2$, we get

$$-\mathbf{V}_1 + R_2\mathbf{I}_3 - R_3\mathbf{I}_2 + \mathbf{V}_2 = 0$$

Since $\mathbf{I}_2 = 0$ and $\mathbf{I}_3 = \beta\mathbf{V}_1$, we get

$$-\mathbf{V}_1 + R_2\beta\mathbf{V}_1 - 0 + \mathbf{V}_2 = 0$$
$$\Rightarrow \quad \mathbf{V}_2 = \mathbf{V}_1(1 - \beta R_2)$$
$$= (1-\beta R_2)\frac{R_1\mathbf{I}_1}{1+\beta R_1}$$

Hence,
$$\mathbf{z}_{21} = \left.\frac{\mathbf{V}_2}{\mathbf{I}_1}\right|_{\mathbf{I}_2=0} = \frac{R_1(1-\beta R_2)}{1+\beta R_1}$$

The circuit used for finding $\mathbf{z}_{12}$ and $\mathbf{z}_{22}$ is shown in Fig. 7.21(b).

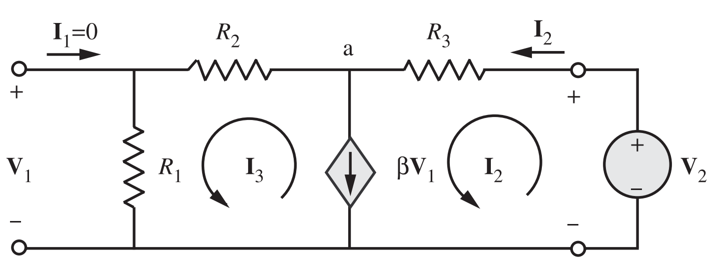


Figure 7.21(b)

By inspection, we find that

$$\mathbf{I}_2 - \mathbf{I}_3 = \beta\mathbf{V}_1 \text{ and } \mathbf{V}_1 = \mathbf{I}_3 R_1$$

$$\Rightarrow \quad \mathbf{I}_2 - \mathbf{I}_3 = \beta(\mathbf{I}_3 R_1)$$
$$\Rightarrow \quad \mathbf{I}_3(1+\beta R_1) = \mathbf{I}_2$$

Applying KVL to the path $R_3 \rightarrow R_2 \rightarrow R_1 \rightarrow \mathbf{V}_2$, we get

$$R_3\mathbf{I}_2 + (R_2 + R_1)\,\mathbf{I}_3 - \mathbf{V}_2 = 0$$

$$\Rightarrow \quad R_3\mathbf{I}_2 + (R_2 + R_1)\,\frac{\mathbf{I}_2}{1+\beta R_1} = \mathbf{V}_2$$

$$\Rightarrow \quad \mathbf{I}_2\left[R_3 + \frac{R_2 + R_1}{1+\beta R_1}\right] = \mathbf{V}_2$$

Hence,

$$\mathbf{z}_{22} = \left.\frac{\mathbf{V}_2}{\mathbf{I}_2}\right|_{\mathbf{I}_1=0} = R_3 + \frac{R_2 + R_1}{1+\beta R_1}\Omega$$

Applying KCL at node a, we get

$$\beta\mathbf{V}_1 + \mathbf{I}_3 = I_2$$

$$\Rightarrow \quad \beta\mathbf{V}_1 + \frac{\mathbf{V}_1}{R_1} = \mathbf{I}_2$$

$$\Rightarrow \quad \mathbf{V}_1\left[\beta + \frac{1}{R_1}\right] = \mathbf{I}_2$$

$$\Rightarrow \quad \mathbf{z}_{12} = \left.\frac{\mathbf{V}_1}{\mathbf{I}_2}\right|_{\mathbf{I}_1=0} = \frac{1}{\beta + \frac{1}{R_1}} = \frac{R_1}{1+\beta R_1}$$

EXAMPLE 7.9

Construct a circuit that realizes the following **z** parameters.

$$\mathbf{z} = \begin{bmatrix} 12 & 4 \\ 4 & 8 \end{bmatrix}$$

SOLUTION

Comparing **z** with = $\begin{bmatrix} \mathbf{z}_{11} & \mathbf{z}_{12} \\ \mathbf{z}_{21} & \mathbf{z}_{22} \end{bmatrix}$, we get

$$\mathbf{z}_{11} = 12\Omega, \quad \mathbf{z}_{12} = \mathbf{z}_{21} = 4\Omega, \quad \mathbf{z}_{22} = 8\Omega$$

Let us consider a T network as shown in Fig. 7.22(a). Our objective is to fit in the values of R_1, R_2 and R_3 for the given **z**.

Applying KVL to the input loop, we get

$$\begin{aligned}\mathbf{V}_1 &= R_1\mathbf{I}_1 + R_3\,(\mathbf{I}_1 + \mathbf{I}_2) \\ &= (R_1 + R_3)\,\mathbf{I}_1 + R_3\mathbf{I}_2\end{aligned}$$

Comparing the preceeding equation with

$$\mathbf{V}_1 = \mathbf{z}_{11}\mathbf{I}_1 + \mathbf{z}_{12}\mathbf{I}_2$$

we get

$$\begin{aligned} \mathbf{z}_{11} &= R_1 + R_3 = 12\Omega \\ \mathbf{z}_{12} &= R_3 = 4\Omega \\ \Rightarrow \quad R_1 &= 12 - R_3 = 8\Omega \end{aligned}$$

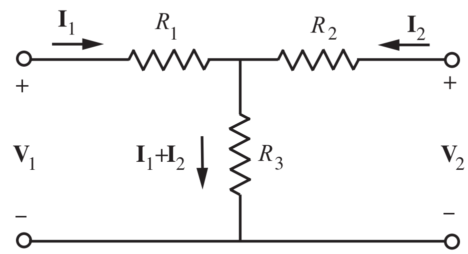


Figure 7.22(a)

Applying *KVL to the output loop*, we get

$$\mathbf{V}_2 = R_2\mathbf{I}_2 + R_3(\mathbf{I}_1 + \mathbf{I}_2)$$

$$\Rightarrow \quad \mathbf{V}_2 = R_3\mathbf{I}_1 + (R_2 + R_3)\mathbf{I}_2$$

Comparing the immediate preceeding equation with

$$\mathbf{V}_2 = \mathbf{z}_{21}\mathbf{I}_1 + \mathbf{z}_{22}\mathbf{I}_2$$

we get

$$\begin{aligned} \mathbf{z}_{21} &= R_3 = 4\Omega \\ \mathbf{z}_{22} &= R_2 + R_3 = 8\Omega \\ \Rightarrow \quad R_2 &= 8 - R_3 = 4\Omega \end{aligned}$$

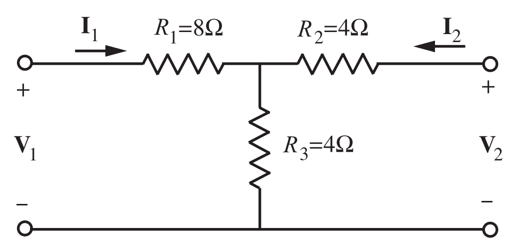


Figure 7.22(b)

Hence, the network to meet the given **z** parameter set is shown in Fig. 7.22(b).

EXAMPLE 7.10

If $\mathbf{z} = \begin{bmatrix} 40 & 10 \\ 20 & 30 \end{bmatrix}\Omega$ for the two-port network, calculate the average power delivered to 50Ω resistor.

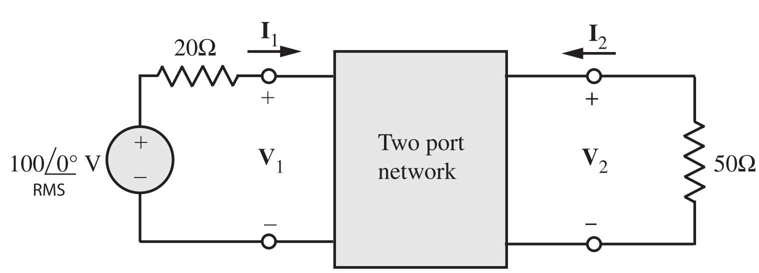


Figure 7.23

SOLUTION

We are given $\mathbf{z}_{11} = 40\Omega \quad \mathbf{z}_{12} = 10\Omega \quad \mathbf{z}_{21} = 20\Omega \quad \mathbf{z}_{22} = 30\Omega$

Since $\mathbf{z}_{12} \neq \mathbf{z}_{21}$, this is not a reciprocal network. Hence, it cannot be represented only by passive elements. We shall draw a network to satisfy the following two *KVL equations*:

$$\mathbf{V}_1 = 40\mathbf{I}_1 + 10\mathbf{I}_2$$
$$\mathbf{V}_2 = 20\mathbf{I}_1 + 30\mathbf{I}_2$$

One possible way of representing a network that is non-reciprocal is as shown in Fig. 7.24(a).

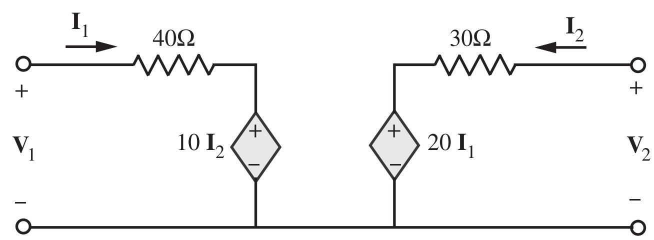


Figure 7.24(a)

Now connecting the source and the load to the two-port network, we get the network as shown in Fig. 7.24(b).

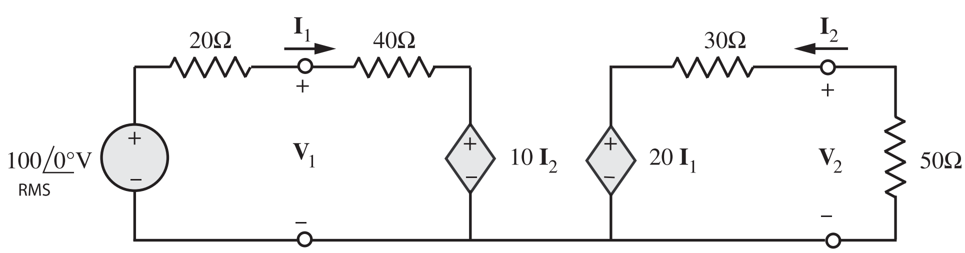


Figure 7.24(b)

KVL for mesh 1:

$$60\mathbf{I}_1 + 10\mathbf{I}_2 = 100$$
$$\Rightarrow \qquad 6\mathbf{I}_1 + \mathbf{I}_2 = 10$$

KVL for mesh 2:

$$80\mathbf{I}_2 + 20\mathbf{I}_1 = 0$$
$$\Rightarrow \qquad 4\mathbf{I}_2 + \mathbf{I}_1 = 0$$
$$\Rightarrow \qquad \mathbf{I}_1 = -4\mathbf{I}_2$$

Solving the above mesh equations, we get

$$-24\mathbf{I}_2 + \mathbf{I}_2 = 10$$

$$\Rightarrow \qquad -23\mathbf{I}_2 = 10$$

$$\Rightarrow \qquad \mathbf{I}_2 = \frac{-10}{23}$$

Power supplied to the load $= |\mathbf{I}_2|^2 R_L$

$$= \frac{100}{(23)^2} \times 50$$

$$= \mathbf{9.45\ W}$$

EXAMPLE 7.11

Refer the network shown in Fig. 7.25. Find the **z** parameters for the network. Take $\alpha = \dfrac{4}{3}$

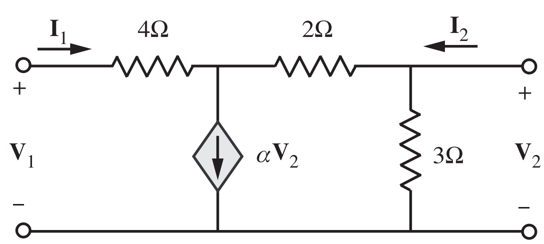


Figure 7.25

SOLUTION

To find $\mathbf{z}_{11}$ and $\mathbf{z}_{21}$, open-circuit the output terminals as shown in Fig. 7.26(a). Also connect a voltage source $\mathbf{V}_1$ to the input terminals.

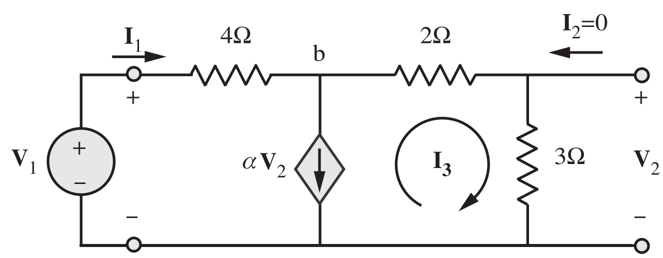


Figure 7.26(a)

Applying KVL around the path $\mathbf{V}_1 \to 4\Omega \to 2\Omega \to 3\Omega$, we get

$$4\mathbf{I}_1 + 5\mathbf{I}_3 = \mathbf{V}_1 \tag{7.17}$$

Also, $$\mathbf{V}_2 = 3\mathbf{I}_3, \text{ so } \mathbf{I}_3 = \frac{\mathbf{V}_2}{3} \tag{7.18}$$

KCL at node b leads to

$$\mathbf{I}_1 - \alpha\mathbf{V}_2 - \mathbf{I}_3 = 0 \tag{7.19}$$

Substituting equation (7.18) in (7.19), we get

$$\begin{aligned}\mathbf{I}_1 &= \alpha\mathbf{V}_2 + \frac{\mathbf{V}_2}{3} = \left[\alpha + \frac{1}{3}\right]\mathbf{V}_2 \\ &= \left[\frac{4}{3} + \frac{1}{3}\right]\mathbf{V}_2\end{aligned}$$

Hence,
$$\mathbf{z}_{21} = \left.\frac{\mathbf{V}_2}{\mathbf{I}_1}\right|_{\mathbf{I}_2=0} = \frac{3}{5}\Omega$$

Substituting equation (7.18) in (7.17), we get

$$\begin{aligned}\mathbf{V}_1 &= 4\mathbf{I}_1 + 5\frac{\mathbf{V_2}}{\mathbf{3}} \\ &= 4\mathbf{I_1} + \frac{5}{3}\left(\mathbf{I}_1 \times \frac{3}{5}\right) \qquad \left(\text{Since } \frac{\mathbf{V_2}}{\mathbf{I_1}} = \frac{3}{5}\right)\end{aligned}$$

Therefore,
$$\mathbf{z}_{11} = \frac{\mathbf{V}_1}{\mathbf{I}_1} = 5\Omega$$

To obtain $\mathbf{z}_{22}$ and $\mathbf{z}_{12}$, open-circuit the input terminals as shown in Fig. 7.26(b). Also, connect a voltage source $\mathbf{V}_2$ to the output terminals.

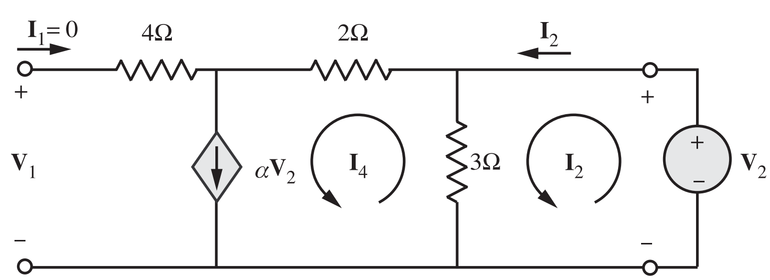


Figure 7.26(b)

KVL for the mesh on the left:

$$\mathbf{V}_1 + 5\mathbf{I}_4 - 3\mathbf{I}_2 = 0 \tag{7.20}$$

KVL for the mesh on the right:

$$\mathbf{V}_2 + 3\mathbf{I}_4 - 3\mathbf{I}_2 = 0 \tag{7.21}$$

Also,
$$\mathbf{I}_4 = \alpha\mathbf{V}_2 \tag{7.22}$$

Substituting equation (7.22) in (7.21), we get

$$\mathbf{V}_2 + 3\alpha\mathbf{V}_2 - 3\mathbf{I}_2 = 0$$

$$\Rightarrow \quad \mathbf{V}_2(1+3\alpha) = 3\mathbf{I}_2$$

Hence,

$$\mathbf{z}_{22} = \left.\frac{\mathbf{V}_2}{\mathbf{I}_2}\right|_{\mathbf{I}_1=0} = \frac{3}{1+3\alpha}$$

$$= \frac{3}{1+3\left(\frac{4}{3}\right)} = \frac{3}{5}\Omega$$

Substituting equation (7.22) in (7.20), we get

$$\mathbf{V}_1 + 5\alpha\mathbf{V}_2 = 3\mathbf{I}_2$$

Substituting $\mathbf{V}_2 = \frac{3}{5}\mathbf{I}_2$, we get

$$\mathbf{V}_1 + 5\alpha\left(\frac{3}{5}\times\mathbf{I}_2\right) = 3\mathbf{I}_2$$

Hence,

$$\mathbf{z}_{12} = \left.\frac{\mathbf{V}_1}{\mathbf{I}_2}\right|_{\mathbf{I}_1=0}$$

$$= 3 - 3\alpha$$

$$= 3 - 3\frac{4}{3} = -1\Omega$$

Finally, in the matrix form, we can write

$$\mathbf{z} = \begin{bmatrix} \mathbf{z}_{11} & \mathbf{z}_{12} \\ \mathbf{z}_{21} & \mathbf{z}_{22} \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ \frac{5}{3} & \frac{3}{5} \end{bmatrix}$$

Please note that $\mathbf{z}_{12} \neq \mathbf{z}_{21}$, since a dependent source is present in the circuit.

EXAMPLE 7.12

Find the Thevenin equivalent circuit with respect to port 2 of the circuit in Fig. 7.27 in terms of **z** parameters.

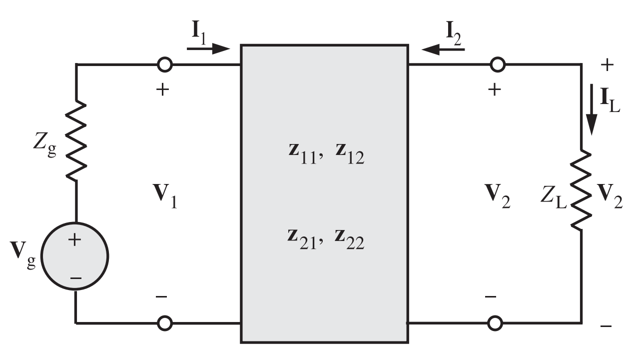


Figure 7.27

SOLUTION

The two port network is defined by

$$\mathbf{V}_1 = \mathbf{z}_{11}\mathbf{I}_1 + \mathbf{z}_{12}\mathbf{I}_2;$$

$$\mathbf{V}_2 = \mathbf{z}_{21}\mathbf{I}_1 + \mathbf{z}_{22}\mathbf{I}_2;$$

here, $\mathbf{V}_1 = \mathbf{V}_g - Z_g\mathbf{I}_1$
and $\mathbf{V}_2 = \mathbf{I}_L Z_L = -\mathbf{I}_2 Z_L$

To find Thevenin equivalent circuit as seen from the output terminals, we have to remove the load resistance R_L. The resulting circuit diagram is shown in Fig. 7.28(a).

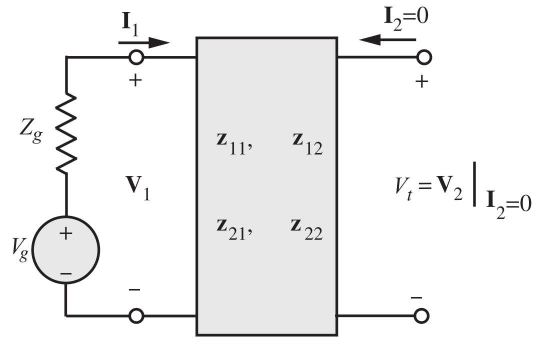


Figure 7.28(a)

$$V_t = \mathbf{V}_2|_{\mathbf{I}_2=0}$$
$$= \mathbf{z}_{21}\mathbf{I}_1 \tag{7.23}$$

With $\mathbf{I}_2 = 0$, we get

$$\mathbf{V}_1 = \mathbf{z}_{11}\mathbf{I}_1$$

$$\Rightarrow \qquad \mathbf{I}_1 = \frac{\mathbf{V}_1}{\mathbf{z}_{11}} = \frac{V_g - \mathbf{I}_1 Z_g}{\mathbf{z}_{11}}$$

Solving for $\mathbf{I}_1$, we get

$$\mathbf{I}_1 = \frac{V_g}{\mathbf{z}_{11} + Z_g} \tag{7.24}$$

Substituting equation (7.24) into equation (7.23), we get

$$V_t = \frac{\mathbf{z}_{21}V_g}{\mathbf{z}_{11} + Z_g}$$

To find Z_t, let us deactivate all the independent sources and then connect a voltage source $\mathbf{V}_2$ across the output terminals as shown in Fig. 7.28(b).

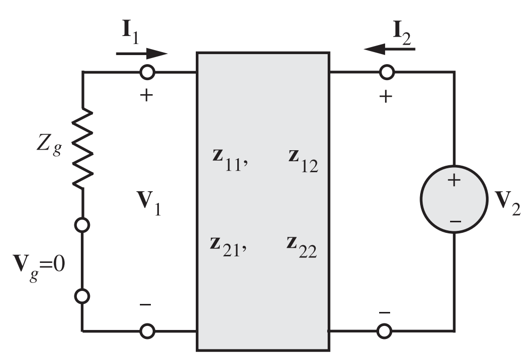


Figure 7.28(b)

$$Z_t = \left.\frac{\mathbf{V}_2}{\mathbf{I}_2}\right|_{\mathbf{V}_g=0}; \text{ where } \quad \mathbf{V}_2 = \mathbf{z}_{21}\mathbf{I}_1 + \mathbf{z}_{22}\mathbf{I}_2$$

We know that $\quad \mathbf{V}_1 = \mathbf{z}_{11}\mathbf{I}_1 + \mathbf{z}_{12}\mathbf{I}_2$

Substituting, $\mathbf{V}_1 = -\mathbf{I}_1 Z_g$ in the preceeding equation, we get

$$-\mathbf{I}_1 Z_g = \mathbf{z}_{11}\mathbf{I}_1 + \mathbf{z}_{12}\mathbf{I}_2$$

Solving, $$\mathbf{I}_1 = \frac{-\mathbf{z}_{12}\mathbf{I}_2}{\mathbf{z}_{11} + Z_g}$$

We know that,

$$\mathbf{V}_2 = \mathbf{z}_{21}\mathbf{I}_1 + \mathbf{Z}_{22}\mathbf{I}_2$$

$$= \mathbf{z}_{21}\left[\frac{-\mathbf{z}_{12}\mathbf{I}_2}{\mathbf{z}_{11} + Z_g}\right] + \mathbf{z}_{22}\mathbf{I}_2$$

Thus,

$$Z_t = \frac{\mathbf{V}_2}{\mathbf{I}_2} = \mathbf{z}_{22} - \frac{\mathbf{z}_{21}\mathbf{z}_{12}}{\mathbf{z}_{11} + Z_g}$$

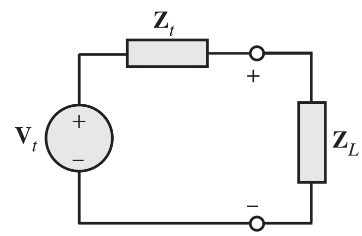


Figure 7.28(c)

The Thevenin equivalent circuit with respect to the output terminals along with load impedance Z_L is as shown in Fig. 7.28(c).

EXAMPLE 7.13

(a) Find the **z** parameters for the two-port network shown in Fig. 7.29.

(b) Find $\mathbf{V}_2(t)$ for $t > 0$ where $v_g(t) = 50u(t)\mathbf{V}$.

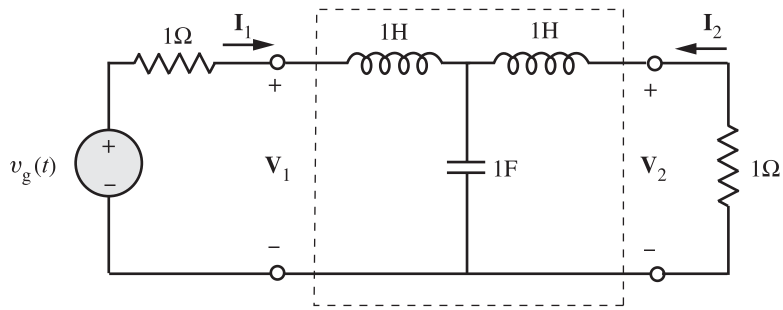


Figure 7.29

SOLUTION

The Laplace transformed network with all initial conditions set to zero is as shown in Fig. 7.30(a).

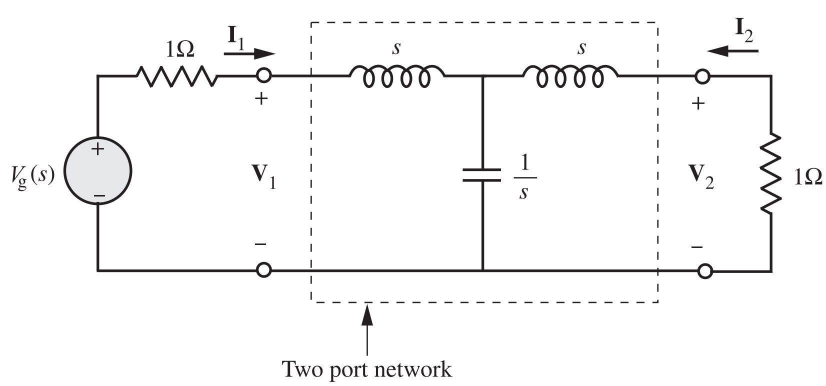


Figure 7.30(a)

(a) To find $\mathbf{z}_{11}$ and $\mathbf{z}_{21}$, open-circuit the output terminals and then connect a voltage source $\mathbf{V}_1$ across the input terminals as shown in Fig. 7.30(b).

Applying KVL to the left mesh, we get

$$\mathbf{V}_1 = \left(s + \frac{1}{s}\right)\mathbf{I}_1$$

Hence,
$$\mathbf{z}_{11} = \left.\frac{\mathbf{V}_1}{\mathbf{I}_1}\right|_{\mathbf{I}_2=0}$$
$$= s + \frac{1}{s} = \frac{s^2+1}{s}$$

Also,
$$\mathbf{V}_2 = \mathbf{I}_1\frac{1}{s}$$

Hence,
$$\mathbf{z}_{21} = \left.\frac{\mathbf{V}_2}{\mathbf{I}_1}\right|_{\mathbf{I}_2=0} = \frac{1}{s}$$

To find $\mathbf{z}_{21}$ and $\mathbf{z}_{22}$, open-circuit the input terminals and then connect a voltage source $\mathbf{V}_2$ across the output terminals as shown in Fig. 7.30(c).

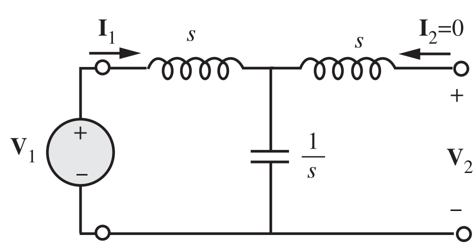


Figure 7.30(b)

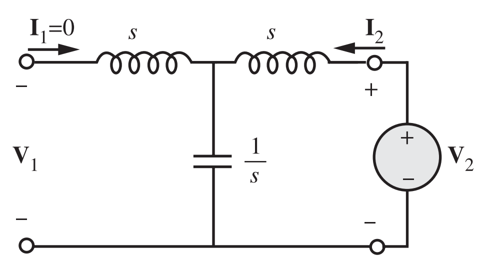


Figure 7.30(c)

Applying KVL to the right mesh, we get

$$\mathbf{V}_2 = \left[s + \frac{1}{s}\right]\mathbf{I}_2$$

$$\Rightarrow \qquad \mathbf{z}_{22} = \left.\frac{\mathbf{V}_2}{\mathbf{I}_2}\right|_{\mathbf{I}_1=0} = \frac{s^2+1}{s}$$

Also,
$$\mathbf{V}_1 = \frac{1}{s}\mathbf{I}_2$$

$$\Rightarrow \qquad \mathbf{z}_{12} = \left.\frac{\mathbf{V}_1}{\mathbf{I}_2}\right|_{\mathbf{I}_1=0} = \frac{1}{s}$$

Summarizing,

$$\mathbf{z} = \begin{bmatrix} \mathbf{z}_{11} & \mathbf{z}_{12} \\ \mathbf{z}_{21} & \mathbf{z}_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{s^2+1}{s} & \dfrac{1}{s} \\ \dfrac{1}{s} & \dfrac{s^2+1}{s} \end{bmatrix}$$

(b)

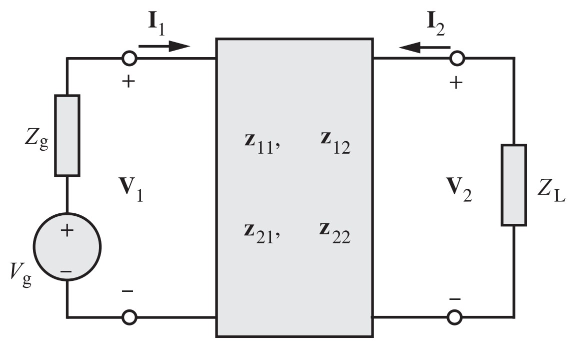


Figure 7.30(d)

Refer the two port network shown in Fig. 7.30(d).

$$\mathbf{V}_1 = V_g - \mathbf{I}_1 Z_g = \mathbf{z}_{11}\mathbf{I}_1 + \mathbf{z}_{12}\mathbf{I}_2$$

$$\Rightarrow \qquad V_g = (\mathbf{z}_{11} + Z_g)\,\mathbf{I}_1 + \mathbf{z}_{12}\mathbf{I}_2$$

$$\Rightarrow \qquad V_g = (\mathbf{z}_{11} + Z_g)\,\mathbf{I}_1 + \mathbf{z}_{12}\left[\frac{-\mathbf{V}_2}{Z_L}\right] \tag{7.25}$$

and

$$\mathbf{V}_2 = \mathbf{z}_{21}\mathbf{I}_1 + \mathbf{z}_{22}\mathbf{I}_2$$

$$\Rightarrow \qquad \mathbf{V}_2 = \mathbf{z}_{21}\mathbf{I}_1 - \mathbf{z}_{22}\frac{\mathbf{V}_2}{Z_L}$$

$$\Rightarrow \qquad \mathbf{I}_1 = \frac{1}{\mathbf{z}_{21}}\left[1 + \frac{\mathbf{z}_{22}}{Z_L}\right]\mathbf{V}_2 \tag{7.26}$$

Substituting equation (7.26) in equation (7.25) and simplifying, we get

$$\frac{\mathbf{V}_2}{V_g} = \frac{\mathbf{z}_{21}\mathbf{z}_L}{(Z_L + \mathbf{z}_{22})\,(\mathbf{z}_{11} + Z_g) - \mathbf{z}_{12}\mathbf{z}_{21}} \tag{7.27}$$

Substituting for Z_L, Z_g and **z**-parameters, we get

$$\frac{\mathbf{V}_2(s)}{V_g(s)} = \frac{\frac{1}{s}}{\left(\frac{s^2+1}{s} + 1\right)\left(\frac{s^2+1}{s} + 1\right) - \frac{1}{s^2}}$$

$$= \frac{s}{(s^2+s+1)^2 - 1}$$

$$\Rightarrow \qquad \frac{\mathbf{V}_2(s)}{V_g(s)} = \frac{1}{s^3 + 2s^2 + 3s + 2}$$

$$= \frac{1}{(s+1)\,(s^2+s+2)}$$

Hence,

$$\mathbf{V}_2(s) = \frac{\mathbf{V}_g(s)}{(s+1)\,(s^2+s+2)} \tag{7.28}$$

The equation $s^2 + s + 2 = 0$ gives

$$s_{1,2} = -\frac{1}{2} \pm j\frac{\sqrt{7}}{2}$$

This means that,

$$\mathbf{V}_2(s) = \frac{V_g(s)}{(s+1)\left(s+\frac{1}{2}-j\frac{\sqrt{7}}{2}\right)\left(s+\frac{1}{2}+j\frac{\sqrt{7}}{2}\right)}$$

Given $\quad v_g(t) = 50u(t)$

$$\Rightarrow \quad V_g(s) = \frac{50}{s}$$

Hence,

$$\mathbf{V}_2(s) = \frac{50}{s(s+1)\left(s+\frac{1}{2}-j\frac{\sqrt{7}}{2}\right)\left(s+\frac{1}{2}+j\frac{\sqrt{7}}{2}\right)}$$

$$= \frac{K_1}{s} + \frac{K_2}{s+1} + \frac{K_3}{s+\frac{1}{2}-j\frac{\sqrt{7}}{2}} + \frac{K_3^*}{s+\frac{1}{2}+j\frac{\sqrt{7}}{2}}$$

By performing partial fraction expansion, we get

$$K_1 = 25, \quad K_2 = -25, \quad K_3 = 9.45\,\underline{/90^\circ}$$

Hence,

$$\mathbf{V}_2(s) = \frac{25}{s} - \frac{25}{s+1} + \frac{9.45\,\underline{/90^\circ}}{s+\frac{1}{2}-j\frac{\sqrt{7}}{2}} + \frac{9.45\,\underline{/-90^\circ}}{s+\frac{1}{2}+j\frac{\sqrt{7}}{2}}$$

Taking inverse Laplace transform of the above equation, we get

$$\mathbf{V}_2(t) = \left[25 - 25e^{-t} + 18.9e^{-0.5t}\cos(1.32t + 90^\circ)\right] u(t)\text{V}$$

Verification:

$$\mathbf{V}_2(0) = 25 - 25 + 18.9\cos 90 = 0$$
$$\mathbf{V}_2(\infty) = 25 + 0 + 0 = 25\text{V}$$

Please note that at $t = \infty$, the circuit diagram of Fig. (7.29) looks as shown in Fig. 7.30(e).

$$\mathbf{I}(\infty) = \frac{50}{2} = 25\mathbf{A}$$

Hence, $\quad \mathbf{V}_2(\infty) = V_C(\infty) = 25\text{V}$

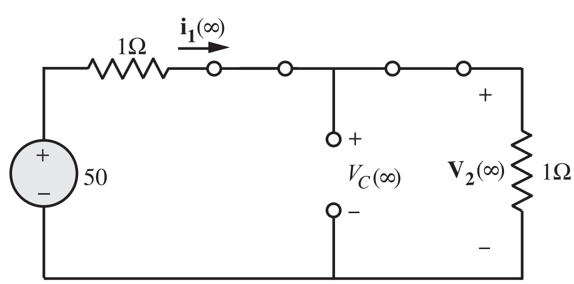


Figure 7.30(e)

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EXAMPLE 7.14

The following measurements were made on a resistive two-port network:

Measurement 1: With port 2 open and 100V applied to port 1, the port 1 current is 1.125A and port 2 voltage is 104V.

Measurement 2: With port 1 open and 50V applied to port 2, the port 2 current is 0.3A, and the port 1 voltage is 30 V.

Find the maximum power that can be delivered by this two-port network to a resistive load at port 2 when port 1 is driven by an ideal voltage source of 100 Vdc.

SOLUTION

$$\mathbf{z}_{11} = \left.\frac{\mathbf{V}_1}{\mathbf{I}_1}\right|_{\mathbf{I}_2=0} = \frac{100}{1.125} = 88.89\Omega$$

$$\mathbf{z}_{21} = \left.\frac{\mathbf{V}_2}{\mathbf{I}_1}\right|_{\mathbf{I}_2=0} = \frac{104}{1.125} = 92.44\Omega$$

$$\mathbf{z}_{12} = \left.\frac{\mathbf{V}_1}{\mathbf{I}_2}\right|_{\mathbf{I}_1=0} = \frac{30}{0.3} = 100\Omega$$

$$\mathbf{z}_{22} = \left.\frac{\mathbf{V}_2}{\mathbf{I}_2}\right|_{\mathbf{I}_1=0} = \frac{50}{0.3} = 166.67\Omega$$

We know from the previous example 7.12 that,

$$\begin{aligned} Z_t &= \mathbf{z}_{22} - \frac{\mathbf{z}_{12}\mathbf{z}_{21}}{\mathbf{z}_{11} + Z_g} \\ &= 166.67 - \frac{92.44 \times 100}{88.89 + 0} \\ &= 166.67 - 103.99 \\ &= 62.68\Omega \end{aligned}$$

For maximum power transfer,

$$\begin{aligned} Z_L &= Z_t \\ &= 62.68\Omega \text{ (For resistive load)} \end{aligned}$$

$$\begin{aligned} \mathbf{V}_t &= \frac{\mathbf{z}_{21}V_g}{\mathbf{z}_{11} + Z_g} \\ &= \frac{92.44 \times 100}{88.89 + 0} \\ &= 104\ \mathbf{V} \end{aligned}$$

The Thevenin equivalent circuit with respect to the output terminals with load resistance is as shown in Fig. 7.31.

$$P_{\max} = \mathbf{I}_t^2 R_L$$

$$= \left[\frac{104}{62.68 \times 2}\right]^2 \times 62.68$$

$$= \mathbf{43.14\ W}$$

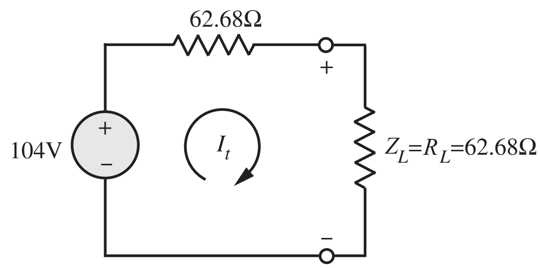


Figure 7.31

EXAMPLE 7.15

Refer the network shown in Fig. 7.32(a). Find the impedance parameters of the network.

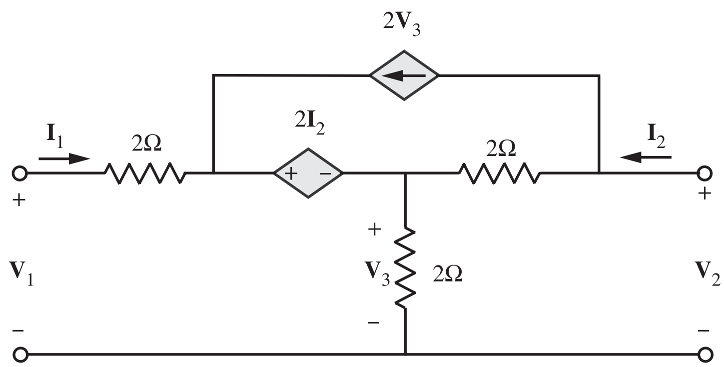


Figure 7.32(a)

SOLUTION

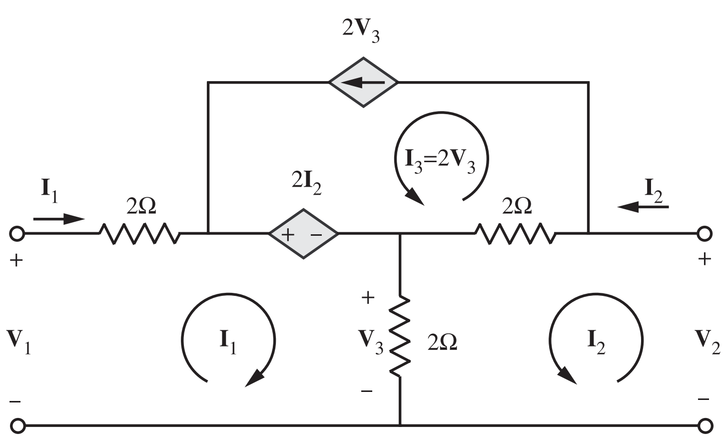


Figure 7.32(b)

Referring to Fig. 7.32(b), we can write

$$\mathbf{V}_3 = 2(\mathbf{I}_1 + \mathbf{I}_2)$$

KVL for mesh 1:

$$2\mathbf{I}_1 + 2\mathbf{I}_2 + 2(\mathbf{I}_1 + \mathbf{I}_2) = \mathbf{V}_1$$

$$\Rightarrow \quad 4\mathbf{I}_1 + 4\mathbf{I}_2 = \mathbf{V}_1$$

KVL for mesh 2:

$$2(\mathbf{I}_2 - 2\mathbf{V}_3) + 2(\mathbf{I}_1 + \mathbf{I}_2) = \mathbf{V}_2$$

$$\Rightarrow \quad 2\mathbf{I}_2 - 4 \times 2(\mathbf{I}_1 + \mathbf{I}_2) + 2(\mathbf{I}_1 + \mathbf{I}_2) = \mathbf{V}_2$$

$$\Rightarrow \quad 2\mathbf{I}_2 - 6(\mathbf{I}_1 + \mathbf{I}_2) = \mathbf{V}_2$$

$$\Rightarrow \quad -6\mathbf{I}_1 - 4\mathbf{I}_2 = \mathbf{V}_2$$

$$\mathbf{z}_{11} = \left.\frac{\mathbf{V}_1}{\mathbf{I}_1}\right|_{\mathbf{I}_2=0} = \left.\frac{4\mathbf{I}_1 + 4\mathbf{I}_2}{\mathbf{I}_1}\right|_{\mathbf{I}_2=0} = 4\Omega$$

$$\mathbf{z}_{21} = \left.\frac{\mathbf{V}_2}{\mathbf{I}_1}\right|_{\mathbf{I}_2=0} = \left.\frac{-6\mathbf{I}_1 - 4\mathbf{I}_2}{\mathbf{I}_1}\right|_{\mathbf{I}_2=0} = -6\Omega$$

$$\mathbf{z}_{12} = \left.\frac{\mathbf{V}_1}{\mathbf{I}_2}\right|_{\mathbf{I}_1=0} = \left.\frac{4\mathbf{I}_1 + 4\mathbf{I}_2}{\mathbf{I}_2}\right|_{\mathbf{I}_1=0} = 4\Omega$$

$$\mathbf{z}_{22} = \left.\frac{\mathbf{V}_2}{\mathbf{I}_2}\right|_{\mathbf{I}_1=0} = \left.\frac{-6\mathbf{I}_1 - 4\mathbf{I}_2}{\mathbf{I}_2}\right|_{\mathbf{I}_1=0} = -4\Omega$$

EXAMPLE 7.16

Is it possible to find **z** parameters for any two port network ? Explain.

SOLUTION

It should be noted that for some two-port networks, the **z** parameters do not exist because they cannot be described by the equations:

$$\left.\begin{aligned} \mathbf{V}_1 &= \mathbf{I}_1\mathbf{z}_{11} + \mathbf{I}_2\mathbf{z}_{12} \\ \mathbf{V}_2 &= \mathbf{I}_1\mathbf{z}_{21} + \mathbf{I}_2\mathbf{z}_{22} \end{aligned}\right\} \tag{7.29}$$

As an example, let us consider an ideal transformer as shown in Fig. 7.33.

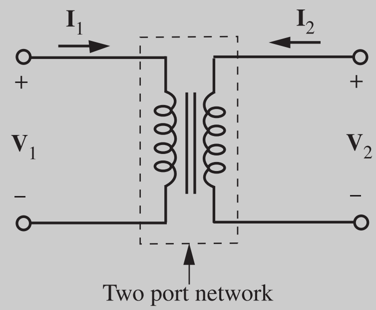


Figure 7.33

The defining equations for the two-port network shown in Fig. 7.33 are:

$$\mathbf{V}_1 = \frac{1}{n}\mathbf{V}_2 \quad \mathbf{I}_1 = -n\mathbf{I}_2$$

It is not possible to express the voltages in terms of the currents, and viceversa. Thus, the ideal transformer has no **z** parameters and no **y** parameters.

7.4 z and y parameters by matrix partitioning

For **z** parameters, the mesh equations are

$$\begin{aligned}
\mathbf{V}_1 &= \mathbf{z}_{11}\mathbf{I}_1 + \mathbf{z}_{12}\mathbf{I}_2 + \cdots + \mathbf{z}_{1n}\mathbf{I}_n \\
\mathbf{V}_2 &= \mathbf{z}_{21}\mathbf{I}_1 + \mathbf{z}_{22}\mathbf{I}_2 + \cdots + \mathbf{z}_{2n}\mathbf{I}_n \\
0 &= \cdots\cdots \\
0 &= \mathbf{z}_{n1}\mathbf{I}_1 + \mathbf{z}_{n2}\mathbf{I}_2 + \cdots + \mathbf{z}_{nn}\mathbf{I}_n
\end{aligned}$$

By matrix partitioning, the above equations can be written as

$$\left[\begin{array}{c} \mathbf{V}_1 \\ \mathbf{V}_2 \\ \hline 0 \\ - \\ 0 \end{array}\right] = \left[\begin{array}{cc|cc} \mathbf{z}_{11} & \mathbf{z}_{12} & - & \mathbf{z}_{1n} \\ \mathbf{z}_{21} & \mathbf{z}_{22} & - & \mathbf{z}_{2n} \\ \hline \mathbf{z}_{31} & \mathbf{z}_{32} & - & \mathbf{z}_{3n} \\ - & - & - & - \\ \mathbf{z}_{n1} & \mathbf{z}_{n2} & - & \mathbf{z}_{nn} \end{array}\right] \left[\begin{array}{c} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \hline \mathbf{I}_3 \\ - \\ \mathbf{I}_n \end{array}\right]$$

$$\Rightarrow \qquad \left[\begin{array}{c} \mathbf{V}_1 \\ \mathbf{V}_2 \\ \hline 0 \\ - \\ 0 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{M} & \mathbf{N} \\ \hline \mathbf{P} & \mathbf{Q} \end{array}\right] \left[\begin{array}{c} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \hline \mathbf{I}_3 \\ - \\ \mathbf{I}_n \end{array}\right]$$

The above equation can be simplified as (exact analysis not required)

$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{M} - \mathbf{N}\ \mathbf{Q}^{-1}\ \mathbf{P} \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix}$$

$\mathbf{M} - \mathbf{N}\mathbf{Q}^{-1}\mathbf{P}$ gives **z** parameters.
Similarly for **y** parameters,

$$\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \\ 0 \\ - \\ 0 \end{bmatrix} = \left[\begin{array}{cc:cc} \mathbf{y}_{11} & \mathbf{y}_{12} & - & \mathbf{y}_{1n} \\ \mathbf{y}_{21} & \mathbf{y}_{22} & - & \mathbf{y}_{2n} \\ \hdashline \mathbf{y}_{31} & \mathbf{y}_{32} & - & \mathbf{y}_{3n} \\ - & - & - & - \\ \mathbf{y}_{n1} & \mathbf{y}_{n2} & - & \mathbf{y}_{nn} \end{array}\right] \left[\begin{array}{c} \mathbf{V}_1 \\ \mathbf{V}_2 \\ \hdashline \mathbf{V}_3 \\ - \\ \mathbf{V}_n \end{array}\right]$$

$$\left[\begin{array}{c} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \hdashline 0 \\ - \\ 0 \end{array}\right] = \begin{bmatrix} \mathbf{M} & \mathbf{N} \\ \mathbf{P} & \mathbf{Q} \end{bmatrix} \left[\begin{array}{c} \mathbf{V}_1 \\ \mathbf{V}_2 \\ \hdashline \mathbf{V}_3 \\ - \\ \mathbf{V}_n \end{array}\right]$$

$$\Rightarrow \qquad \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{M} - \mathbf{N}\ \mathbf{Q}^{-1}\ \mathbf{P} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix}$$

$\mathbf{M} - \mathbf{N}\mathbf{Q}^{-1}\mathbf{P}$ gives **y** parameters.

EXAMPLE 7.17

Find **y** and **z** parameters for the resistive network shown in Fig. 7.34(a). Verify the result by using $\mathrm{Y} - \Delta$ transformation.

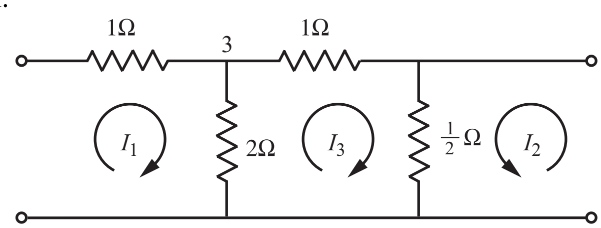


Figure 7.34(a)

SOLUTION

For the loops indicated, the equations in matrix form,

$$\left[\begin{array}{c} \mathbf{V}_2 \\ \mathbf{V}_2 \\ \hdashline 0 \end{array}\right] = \left[\begin{array}{cc:c} 3 & 0 & -2 \\ 0 & 0.5 & 0.5 \\ \hdashline -2 & 0.5 & 3.5 \end{array}\right] \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \mathbf{I}_3 \end{bmatrix}$$

Then,
$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \left\{ \begin{bmatrix} 3 & 0 \\ 0 & 0.5 \end{bmatrix} - \frac{1}{3.5} \begin{bmatrix} -2 \\ 0.5 \end{bmatrix} \begin{bmatrix} -2 & 0.5 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} \right\}$$
$$\Rightarrow \qquad \begin{bmatrix} 1.8571 & 0.2857 \\ 0.2857 & 0.4285 \end{bmatrix} = [\mathbf{z}]$$
$$\mathbf{y} = \mathbf{z}^{-1} = \begin{bmatrix} 0.6 & -0.4 \\ -0.4 & 2.5 \end{bmatrix}$$

Verification

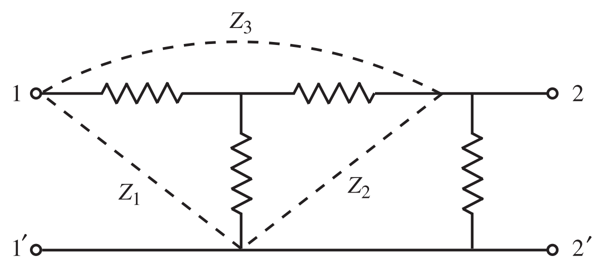


Figure 7.34(b)

Refer Fig 7.34(b), converting T of 1, 1', 2 into equation,

$$Z_1 = \frac{1 \times 1 + 1 \times 2 + 1 \times 2}{1} = 5$$
$$Z_2 = 5$$
$$Z_3 = \frac{5}{2}$$

Therefore,
$$Z_2' = \frac{5 \times \frac{1}{2}}{5.5} = \frac{5}{11}$$
$$\mathbf{y}_{11} = \frac{3}{5}; \quad \mathbf{y}_{12} = \mathbf{y}_{21} = -\frac{2}{5}; \quad \mathbf{y}_{22} = \frac{13}{5}$$

The values with transformed circuit is shown in Fig 7.34(c).

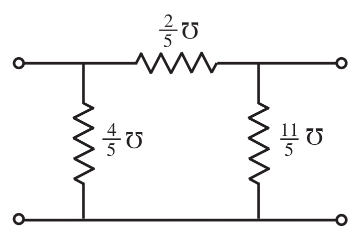


Figure 7.34(c)

EXAMPLE 7.18

Find **y** and **z** parameters for the network shown in Fig.7.35 which contains a current controlled source.

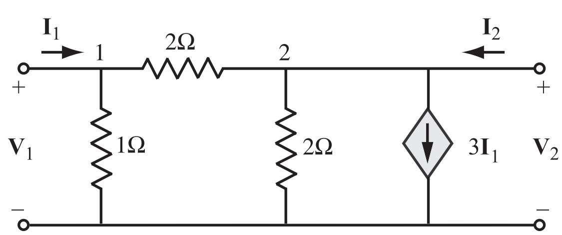


Figure 7.35

SOLUTION

At node 1,

$$1.5\mathbf{V}_1 - 0.5\mathbf{V}_2 = \mathbf{I}_1$$

At node 2,

$$-0.5\mathbf{V}_1 + \mathbf{V}_2 = \mathbf{I}_2 - 3\mathbf{I}_1$$

In matrix form,

$$\begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1 \end{bmatrix}\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix}$$

$$\Rightarrow \qquad \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix}$$

$$= \begin{bmatrix} -0.4 & 0.4 \\ -3.2 & 1.2 \end{bmatrix}\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix}$$

Therefore,

$$[\mathbf{z}] = \begin{bmatrix} -0.4 & 0.4 \\ -3.2 & 1.2 \end{bmatrix}$$

$$[\mathbf{y}] = [\mathbf{z}]^{-1} = \begin{bmatrix} 1.5 & -0.5 \\ 4 & -0.5 \end{bmatrix}$$

7.5 Hybrid parameters

The **z** and **y** parameters of a two-port network do not always exist. Hence, we define a third set of parameters known as hybrid parameters. In the pair of equations that define these parameters, $\mathbf{V}_1$ and $\mathbf{I}_2$ are the dependent variables. Hence, the two-port equations in terms of the hybrid parameters are

$$\mathbf{V}_1 = \mathbf{h}_{11}\mathbf{I}_1 + \mathbf{h}_{12}\mathbf{V}_2 \tag{7.30}$$

$$\mathbf{I}_2 = \mathbf{h}_{21}\mathbf{I}_1 + \mathbf{h}_{22}\mathbf{V}_2 \tag{7.31}$$

or in matrix form,

$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{I}_2 \end{bmatrix}\begin{bmatrix} \mathbf{h}_{11} & \mathbf{h}_{12} \\ \mathbf{h}_{21} & \mathbf{h}_{22} \end{bmatrix}\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{V}_2 \end{bmatrix}$$

These parameters are particularly important in transistor circuit analysis. These parameters are obtained via the following equations:

$$\mathbf{h}_{11} = \left.\frac{\mathbf{V}_1}{\mathbf{I}_1}\right|_{\mathbf{V}_2=0} \qquad \mathbf{h}_{12} = \left.\frac{\mathbf{V}_1}{\mathbf{V}_2}\right|_{\mathbf{I}_1=0} \qquad \mathbf{h}_{21} = \left.\frac{\mathbf{I}_2}{\mathbf{I}_1}\right|_{\mathbf{V}_2=0} \qquad \mathbf{h}_{22} = \left.\frac{\mathbf{I}_2}{\mathbf{V}_2}\right|_{\mathbf{I}_1=0}$$

The parameters $\mathbf{h}_{11}, \mathbf{h}_{12}, \mathbf{h}_{21}$ and $\mathbf{h}_{22}$ represent the *short-circuit input impedance*, the *open-circuit reverse voltage gain,* the *short-circuit forward current gain,* and the open-circuit *output admittance* respectively. Because of this mix of parameters, they are called **hybrid parameters.**

EXAMPLE 7.19

Refer the network shown in Fig. 7.36(a). For this network, determine the **h** parameters.

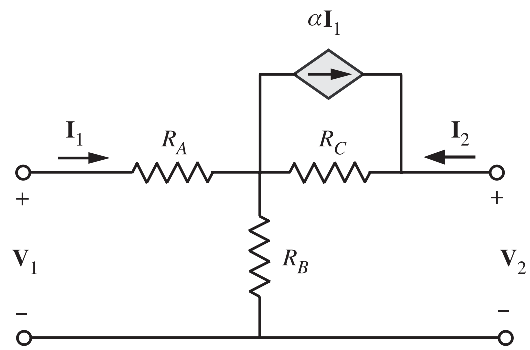


Figure 7.36(a)

SOLUTION

To find $\mathbf{h}_{11}$ and $\mathbf{h}_{21}$ short-circuit the output terminals so that $\mathbf{V}_2 = 0$. Also connect a current source $\mathbf{I}_1$ to the input port as in Fig. 7.36(b).

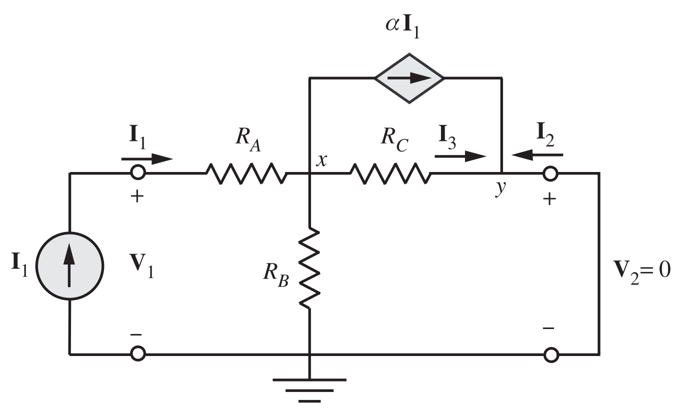


Figure 7.36(b)

Applying KCL at node x:

$$-\mathbf{I}_1 + \frac{V_x}{R_B} + \frac{V_x - 0}{R_C} + \alpha \mathbf{I}_1 = 0$$

$$\Rightarrow \qquad \mathbf{I}_1 [\alpha - 1] = -V_x \left[\frac{1}{R_B} + \frac{1}{R_C} \right]$$

$$\Rightarrow \qquad V_x = \frac{(1-\alpha)\mathbf{I}_1 R_B R_C}{R_B + R_C}$$

Hence,

$$\begin{aligned}
\mathbf{h}_{11} &= \left.\frac{\mathbf{V}_1}{\mathbf{I}_1}\right|_{\mathbf{V}_2=0} \\
&= \left.\frac{V_x + \mathbf{I}_1 R_A}{\mathbf{I}_1}\right|_{\mathbf{V}_2=0} \\
&= \frac{(1-\alpha)\mathbf{I}_1 R_B R_C}{(R_B+R_C)\,\mathbf{I}_1} + R_A \mathbf{I}_1 \\
&= \frac{(1-\alpha)R_B R_C}{R_B+R_C} + R_A
\end{aligned}$$

KCL at node y:

$$\alpha\mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 = 0$$

$$\Rightarrow \quad \alpha\mathbf{I}_1 + \mathbf{I}_2 + \frac{\mathbf{V}_x - 0}{R_C} = 0$$

$$\Rightarrow \quad \alpha\mathbf{I}_1 + \mathbf{I}_2 + \frac{1}{R_C}\left[\frac{(1-\alpha)\mathbf{I}_1 R_B R_C}{R_B+R_C}\right] = 0$$

Hence,

$$\begin{aligned}
\mathbf{h}_{21} &= \left.\frac{\mathbf{I}_2}{\mathbf{I}_1}\right|_{\mathbf{V}_2=0} \\
&= -\alpha - \frac{(1-\alpha)R_B}{R_B+R_C} \\
&= \frac{-(\alpha R_C + R_B)}{R_B+R_C}
\end{aligned}$$

To find $\mathbf{h}_{22}$ and $\mathbf{h}_{12}$ open-circuit the input port so that $\mathbf{I}_1 = 0$. Also, connect a voltage source $\mathbf{V}_2$ between the output terminals as shown in Fig. 7.36(c).

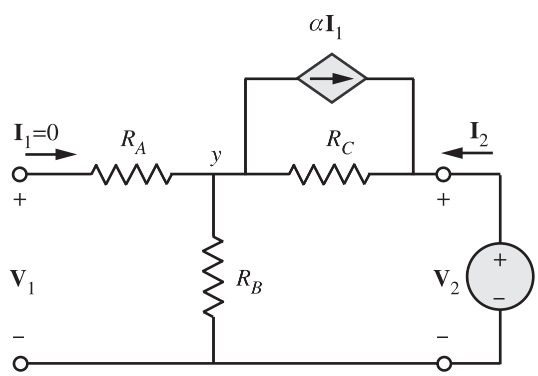


Figure 7.36(c)

KCL at node y:

$$\frac{\mathbf{V}_1}{R_B} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{R_C} + \alpha\mathbf{I}_1 = 0$$

Since $\mathbf{I}_1 = 0$, we get

$$\frac{\mathbf{V}_1}{R_B} + \frac{\mathbf{V}_1}{R_C} - \frac{\mathbf{V}_2}{R_C} = 0$$

$$\mathbf{V}_1\left[\frac{1}{R_B} + \frac{1}{R_C}\right] = \frac{\mathbf{V}_2}{R_C}$$

$$\Rightarrow \qquad \mathbf{h}_{12} = \left.\frac{\mathbf{V}_1}{\mathbf{V}_2}\right|_{\mathbf{I}_1=0} = \frac{R_B}{R_B + R_C}$$

Applying KVL to the output mesh, we get

$$-\mathbf{V}_2 + R_C\left(\alpha\mathbf{I}_1 + \mathbf{I}_2\right) + R_B\mathbf{I}_2 = 0$$

Since $\mathbf{I}_1 = 0$, we get

$$R_C\mathbf{I}_2 + R_B\mathbf{I}_2 = \mathbf{V}_2$$

$$\Rightarrow \qquad \mathbf{h}_{22} = \left.\frac{\mathbf{I}_2}{\mathbf{V}_2}\right|_{\mathbf{I}_1=0} = \frac{1}{R_C + R_B}$$

EXAMPLE 7.20

Find the hybrid parameters for the two-port network shown in Fig. 7.37(a).

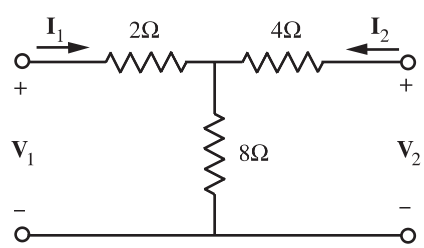


Figure 7.37(a)

SOLUTION

To find $\mathbf{h}_{11}$ and $\mathbf{h}_{21}$, short-circuit the output port and connect a current source $\mathbf{I}_1$ to the input port as shown in Fig. 7.37(b).

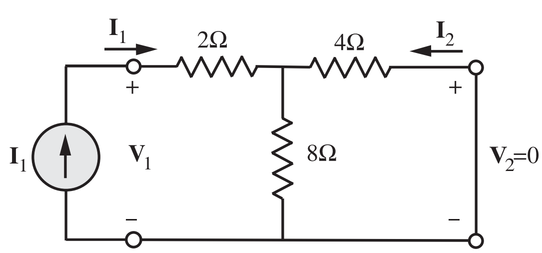


Figure 7.37(b)

Referring to Fig. 7.37(b), we find that

$$\mathbf{V}_1 = \mathbf{I}_1\left[2\Omega + (8\Omega||4\Omega)\right]$$
$$= \mathbf{I}_1 \times 4.67$$

Hence, $$\mathbf{h}_{11} = \left.\frac{\mathbf{V}_1}{\mathbf{I}_1}\right|_{\mathbf{V}_2=0} = 4.67\Omega$$

By using the principle of current division, we find that

$$-\mathbf{I}_2 = \frac{\mathbf{I}_1 \times 8}{8+4} = \frac{2}{3}\mathbf{I}_1$$

Hence, $$\mathbf{h}_{21} = \left.\frac{\mathbf{I}_2}{\mathbf{I}_1}\right|_{\mathbf{V}_2=0} = \frac{-2}{3}$$

To obtain $\mathbf{h}_{12}$ and $\mathbf{h}_{22}$, open-circuit the input port and connect a voltage source $\mathbf{V}_2$ to the output port as in Fig. 7.37(c).

Using the principle of voltage division,

$$\mathbf{V}_1 = \frac{8}{8+4}\mathbf{V}_2 = \frac{2}{3}\mathbf{V}_2$$

Hence, $$\mathbf{h}_{12} = \frac{\mathbf{V}_1}{\mathbf{V}_2} = \frac{2}{3}$$

Also, $$\mathbf{V}_2 = (8+4)\mathbf{I}_2$$
$$= 12\mathbf{I}_2$$

$$\Rightarrow \qquad \mathbf{h}_{22} = \left.\frac{\mathbf{I}_2}{\mathbf{V}_2}\right|_{\mathbf{I}_1=0} = \frac{1}{12}\text{S}$$

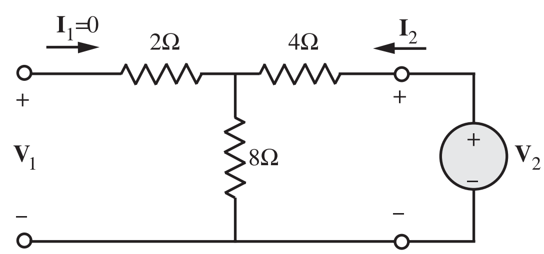


Figure 7.37(c)

EXAMPLE 7.21

Determine the **h** parameters of the circuit shown in Fig. 7.38(a).

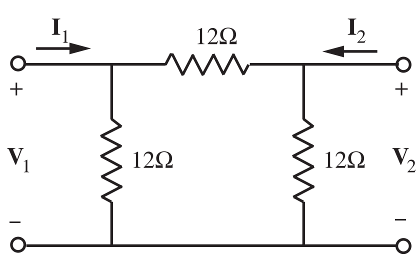


Figure 7.38(a)

SOLUTION

Performing Δ to Y transformation, the network shown in Fig. 7.38(a) takes the form as shown in Fig. 7.38(b). Please note that since all the resistors are of same value, $R_Y = \frac{1}{3}R_\Delta$.

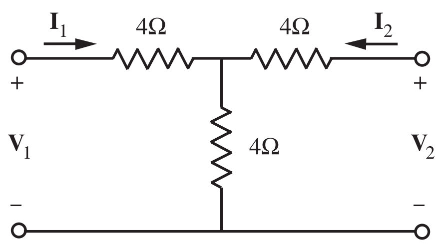


Figure 7.38(b)

To find $\mathbf{h}_{11}$ and $\mathbf{h}_{21}$, short-circuit the output port and connect a current source $\mathbf{I}_1$ to the input port as in Fig. 7.38(c).

$$\mathbf{V}_1 = \mathbf{I}_1\,[4\Omega + (4\Omega||4\Omega)]$$
$$= 6\mathbf{I}_1$$

Hence, $$\mathbf{h}_{11} = \left.\frac{\mathbf{V}_1}{\mathbf{I}_1}\right|_{\mathbf{V}_2=0} = 6\Omega$$

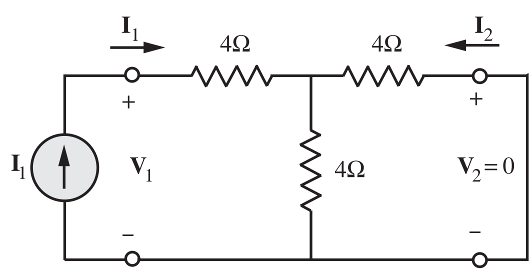


Figure 7.38(c)

Using the principle of current division,

$$-\mathbf{I}_2 = \frac{\mathbf{I}_1}{4+4} \times 4$$

$$\Rightarrow \qquad -\mathbf{I}_2 = \frac{\mathbf{I}_1}{2}$$

Hence, $$\mathbf{h}_{21} = \left.\frac{\mathbf{I}_2}{\mathbf{I}_1}\right|_{\mathbf{V}_2=0} = \frac{-1}{2}$$

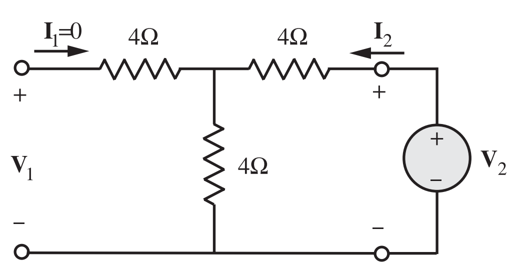


Figure 7.38(d)

To find $\mathbf{h}_{12}$ and $\mathbf{h}_{22}$, open-circuit the input port and connect a voltage source $\mathbf{V}_2$ to the output port as shown in Fig. 7.38(d).

Using the principle of voltage division, we get

$$\mathbf{V}_1 = \frac{\mathbf{V}_2}{4+4} \times 4$$

$$\Rightarrow \qquad \mathbf{h}_{12} = \left.\frac{\mathbf{V}_1}{\mathbf{V}_2}\right|_{\mathbf{I}_1=0} = \frac{1}{2}$$

Also, $$\mathbf{V}_2 = [4+4] \times \mathbf{I}_2 = 8\mathbf{I}_2$$

$$\Rightarrow \qquad \mathbf{h}_{22} = \left.\frac{\mathbf{I}_2}{\mathbf{V}_2}\right|_{\mathbf{I}_1=0} = \frac{1}{8}\text{ S}$$

EXAMPLE 7.22

Determine the Thevenin equivalent circuit at the output of the circuit in Fig. 7.39(a).

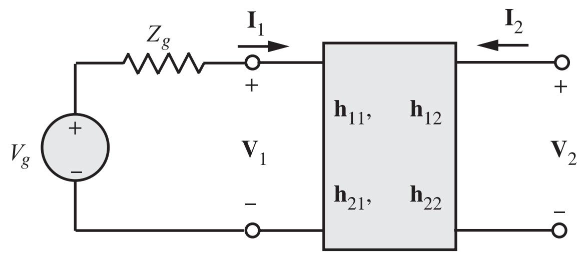


Figure 7.39(a)

SOLUTION

To find Z_t, deactivate the voltage source V_g and apply a 1 V voltage source at the output port, as shown in Fig. 7.39(b).

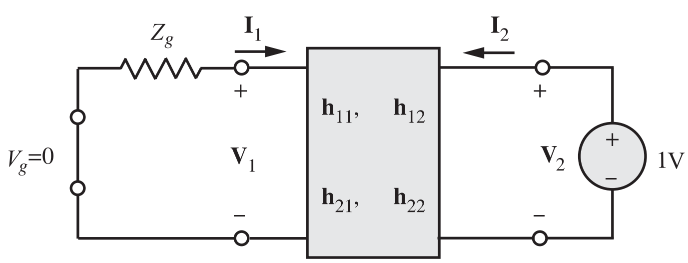


Figure 7.39(b)

The two-port circuit is described using **h** parameters by the following equations:

$$\mathbf{V}_1 = \mathbf{h}_{11}\mathbf{I}_1 + \mathbf{h}_{12}\mathbf{V}_2 \tag{7.32}$$

$$\mathbf{I}_2 = \mathbf{h}_{21}\mathbf{I}_1 + \mathbf{h}_{22}\mathbf{V}_2 \tag{7.33}$$

But $\mathbf{V}_2$ = 1 V and $\mathbf{V}_1 = -\mathbf{I}_1 Z_g$

Substituting these in equations (7.32) and (7.33), we get

$$-\mathbf{I}_1 Z_g = \mathbf{h}_{11}\mathbf{I}_1 + \mathbf{h}_{12}$$

$$\Rightarrow \qquad \mathbf{I}_1 = \frac{-\mathbf{h}_{12}}{Z_g + \mathbf{h}_{11}} \tag{7.34}$$

$$\mathbf{I}_2 = \mathbf{h}_{21}\mathbf{I}_1 + \mathbf{h}_{22} \tag{7.35}$$

Substituting equation (7.34) into equation (7.35), we get

$$\mathbf{I}_2 = \mathbf{h}_{22} - \frac{\mathbf{h}_{21}\mathbf{h}_{12}}{\mathbf{h}_{11} + Z_g}$$

$$= \frac{\mathbf{h}_{11}\mathbf{h}_{22} - \mathbf{h}_{21}\mathbf{h}_{12} + \mathbf{h}_{22}Z_g}{\mathbf{h}_{11} + Z_g}$$

Therefore,
$$Z_t = \frac{\mathbf{V}_2}{\mathbf{I}_2} = \frac{1}{\mathbf{I}_2} = \frac{\mathbf{h}_{11} + Z_g}{\mathbf{h}_{11}\mathbf{h}_{22} - \mathbf{h}_{12}\mathbf{h}_{21} + \mathbf{h}_{22}Z_g}$$

To get V_t, we find open circuit voltage $\mathbf{V}_2$ with $\mathbf{I}_2 = 0$. To find V_t, refer the Fig. 7.39(c).

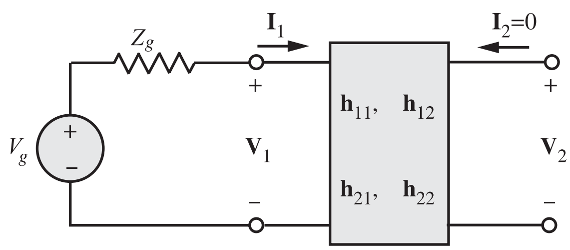


Figure 7.39(c)

At the input port, we can write

$$-V_g + \mathbf{I}_1 Z_g + \mathbf{V}_1 = 0$$

$$\Rightarrow \qquad \mathbf{V}_1 = V_g - \mathbf{I}_1 Z_g \tag{7.36}$$

Substituting equation (7.36) into equation (7.32), we get

$$V_g - \mathbf{I}_1 Z_g = \mathbf{h}_{11}\mathbf{I}_1 + \mathbf{h}_{12}\mathbf{V}_2$$

$$\Rightarrow \qquad V_g = (\mathbf{h}_{11} + Z_g)\,\mathbf{I}_1 + \mathbf{h}_{12}\mathbf{V}_2 \tag{7.37}$$

and substituting $\mathbf{I}_2 = 0$ in equation (7.33), we get

$$0 = \mathbf{h}_{21}\mathbf{I}_1 + \mathbf{h}_{22}\mathbf{V}_2$$

$$\Rightarrow \qquad \mathbf{I}_1 = \frac{-\mathbf{h}_{22}}{\mathbf{h}_{21}}\mathbf{V}_2 \tag{7.38}$$

Finally substituting equation (7.38) in (7.37), we get

$$V_g = (\mathbf{h}_{11} + Z_g)\left(\frac{-\mathbf{h}_{22}}{\mathbf{h}_{21}}\mathbf{V}_2\right) + \mathbf{h}_{12}\mathbf{V}_2$$

$$\Rightarrow \qquad \mathbf{V}_2 = V_t = \frac{V_g\mathbf{h}_{21}}{\mathbf{h}_{12}\mathbf{h}_{21} - \mathbf{h}_{11}\mathbf{h}_{22} - Z_g\mathbf{h}_{22}}$$

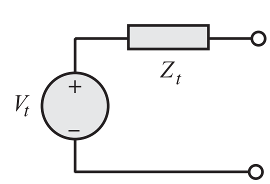


Figure 7.39(d)

Hence, the Thevenin equivalent circuit as seen from the output terminals is as shown in Fig. 7.39(d).

EXAMPLE 7.23

Find the input impedance of the network shown in Fig. 7.40.

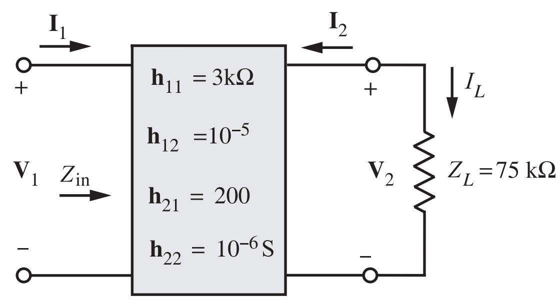


Figure 7.40

SOLUTION

For the two-port network, we can write

$$\mathbf{V}_1 = \mathbf{h}_{11}\mathbf{I}_1 + \mathbf{h}_{12}\mathbf{V}_2 \tag{7.39}$$

$$\mathbf{I}_2 = \mathbf{h}_{21}\mathbf{I}_1 + \mathbf{h}_{22}\mathbf{V}_2 \tag{7.40}$$

But

$$\mathbf{V}_2 = I_L Z_L = -\mathbf{I}_2 Z_L \tag{7.41}$$

where

$$Z_L = 75\ \text{k}\Omega$$

Substituting the value of $\mathbf{V}_2$ in equation (7.40), we get

$$\mathbf{I}_2 = \mathbf{h}_{21}\mathbf{I}_1 - \mathbf{h}_{22}\mathbf{I}_2 Z_L$$

$$\Rightarrow \qquad \mathbf{I}_2 = \frac{\mathbf{h}_{21}\mathbf{I}_1}{1 + Z_L\mathbf{h}_{22}} \tag{7.42}$$

Substituting equation (7.42) in equation (7.41), we get

$$\mathbf{V}_2 = \frac{-Z_L\mathbf{h}_{21}\mathbf{I}_1}{1 + Z_L\mathbf{h}_{22}} \tag{7.43}$$

Substituting equation (7.43) in equation (7.39), we get

$$\mathbf{V}_1 = \mathbf{h}_{11}\mathbf{I}_1 - \frac{\mathbf{h}_{12}Z_L\mathbf{h}_{21}\mathbf{I}_1}{1 + Z_L\mathbf{h}_{22}}$$

Hence

$$\begin{aligned} Z_{\text{in}} &= \frac{\mathbf{V}_1}{\mathbf{I}_1} \\ &= \mathbf{h}_{11} - \frac{Z_L\mathbf{h}_{12}\mathbf{h}_{21}}{1 + Z_L\mathbf{h}_{22}} \\ &= 3 \times 10^3 - \frac{75 \times 10^3 \times 10^{-5} \times 200}{1 + 75 \times 10^3 \times 10^{-6}} \\ &= \mathbf{2.86k\Omega} \end{aligned}$$

EXAMPLE 7.24

Find the voltage gain, $\dfrac{\mathbf{V}_2}{V_g}$ for the network shown in Fig. 7.41.

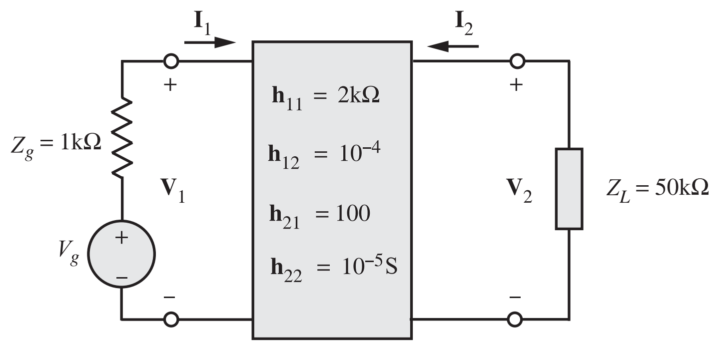


Figure 7.41

SOLUTION

For the two-port network we can write,

$$\mathbf{V}_1 = \mathbf{h}_{11}\mathbf{I}_1 + \mathbf{h}_{12}\mathbf{V}_2, \quad \text{here} \quad \mathbf{V}_1 = V_g - Z_g\mathbf{I}_1$$

$$\mathbf{I}_2 = \mathbf{h}_{21}\mathbf{I}_1 + \mathbf{h}_{22}\mathbf{V}_2, \quad \text{here} \quad \mathbf{V}_2 = -Z_L\mathbf{I}_2$$

Hence,

$$V_g - Z_g\mathrm{I}_1 = \mathbf{h}_{11}\mathbf{I}_1 + \mathbf{h}_{12}\mathbf{V}_2$$

$$\Rightarrow \quad V_g = (\mathbf{h}_{11} + Z_g)\,\mathbf{I}_1 + \mathbf{h}_{12}\mathbf{V}_2$$

$$\Rightarrow \quad \mathbf{I}_1 = \frac{V_g - \mathbf{h}_{12}\mathbf{V}_2}{\mathbf{h}_{11} + Z_g}$$

Also,

$$\mathbf{I}_2 = \frac{-\mathbf{V}_2}{Z_L} = \mathbf{h}_{21}\mathbf{I}_1 + \mathbf{h}_{22}\mathbf{V}_2$$

$$\Rightarrow \quad \frac{-\mathbf{V}_2}{Z_L} = \mathbf{h}_{21}\left[\frac{V_g - \mathbf{h}_{12}\mathbf{V}_2}{\mathbf{h}_{11} + Z_g}\right] + \mathbf{h}_{22}\mathbf{V}_2$$

From the above equation, we find that

$$\begin{aligned}
\frac{\mathbf{V}_2}{V_g} &= \frac{-\mathbf{h}_{21}Z_L}{(\mathbf{h}_{11}Z_g)\,(1 + \mathbf{h}_{22}Z_L) - \mathbf{h}_{12}\mathbf{h}_{21}Z_L} \\
&= \frac{-100 \times 50 \times 10^3}{(2 \times 10^3 + 1 \times 10^3)\,(1 + 10^{-5} \times 50 \times 10^3) - (10^{-4} \times 100 \times 50 \times 10^3)} \\
&= \mathbf{-1250}
\end{aligned}$$

EXAMPLE 7.25

The following dc measurements were done on the resistive network shown in Fig. 7.42(a).

Measurement 1	Measurement 2
$\mathbf{V}_1 = 20$ V	$\mathbf{V}_1 = 35$ V
$\mathbf{I}_1 = 0.8$ A	$\mathbf{I}_1 = 1$ A
$\mathbf{V}_2 = 0$ V	$\mathbf{V}_2 = 15$ V
$\mathbf{I}_2 = -0.4$ A	$\mathbf{I}_2 = 0$ A

Find the value of R_o for maximum power transfer.

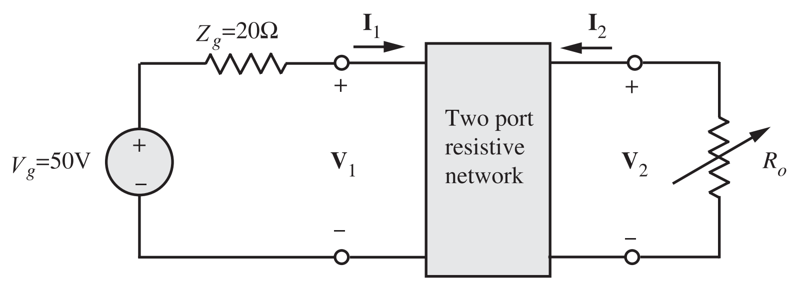


Figure 7.42(a)

SOLUTION

For the two-port network shown in Fig. 7.41, we can write:

$$\mathbf{V}_1 = \mathbf{h}_{11}\mathbf{I}_1 + \mathbf{h}_{12}\mathbf{V}_2$$
$$\mathbf{I}_2 = \mathbf{h}_{21}\mathbf{I}_1 + \mathbf{h}_{22}\mathbf{V}_2$$

From measurement 1:

$$\mathbf{h}_{11} = \left.\frac{\mathbf{V}_1}{\mathbf{I}_1}\right|_{\mathbf{V}_2=0} = \frac{20}{0.8} = 25\Omega$$
$$\mathbf{h}_{21} = \left.\frac{\mathbf{I}_2}{\mathbf{I}_1}\right|_{\mathbf{V}_2=0} = \frac{-0.4}{0.8} = -0.5$$

From measurement 2:

$$\mathbf{V}_1 = \mathbf{h}_{11}\mathbf{I}_1 + \mathbf{h}_{12}\mathbf{V}_2$$
$$\Rightarrow \quad 35 = 25 \times 1 + \mathbf{h}_{12} \times 15$$
$$\Rightarrow \quad \mathbf{h}_{12} = \frac{10}{15} = 0.67$$

Then,
$$\mathbf{I}_2 = \mathbf{h}_{21}\mathbf{I}_1 + \mathbf{h}_{22}\mathbf{V}_2$$
$$\Rightarrow \quad 0 = \mathbf{h}_{21} \times 1 + \mathbf{h}_{22} \times 15$$
$$\mathbf{h}_{22} = \frac{-\mathbf{h}_{21}}{15} = \frac{0.5}{15} = 0.033 \text{ S}$$

For example (7.22),

$$\mathbf{V}_t = \frac{V_g\mathbf{h}_{21}}{\mathbf{h}_{12}\mathbf{h}_{21} - \mathbf{h}_{11}\mathbf{h}_{22} - Z_g\mathbf{h}_{22}}$$
$$= \frac{50 \times (-0.5)}{0.67 \times (-0.5) - 25 \times 0.033 - 20 \times 0.033}$$
$$= \frac{-25}{-1.82} = 13.74 \text{ Volts}$$
$$Z_t = \frac{\mathbf{h}_{11} + Z_g}{\mathbf{h}_{11}\mathbf{h}_{22} - \mathbf{h}_{12}\mathbf{h}_{21} + \mathbf{h}_{22}Z_g}$$
$$= \frac{25 + 20}{25 \times 0.033 - 0.67 \times (-0.5) + 0.033 \times 20}$$
$$= \frac{45}{1.82} = 24.72\ \Omega$$

For maximum power transfer, $Z_L = Z_t = 24.72\ \Omega$ (Please note that, Z_L is purely resistive).

The Thevenin equivalent circuit as seen from the output terminals along with Z_L is shown in Fig. 7.42(b).

$$P_{\max} = I_t^2 \times 24.72$$
$$= \left[\frac{13.74}{24.72 + 24.72}\right]^2 \times 24.72$$
$$= \frac{(13.74)^2}{4 \times 24.72} = \mathbf{1.9\ Watts}$$

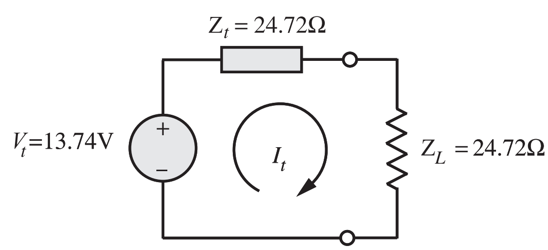


Figure 7.42(b)

EXAMPLE 7.26

Determine the hybird parameters for the network shown in Fig. 7.43.

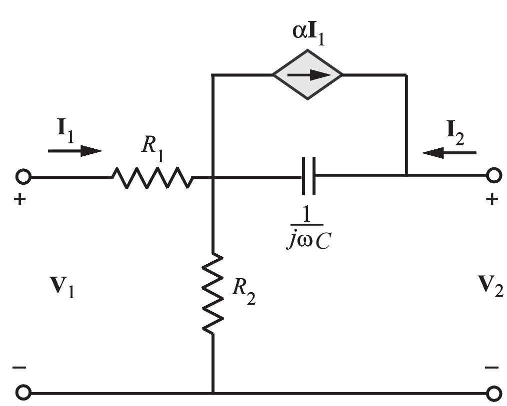


Figure 7.43

SOLUTION

To find $\mathbf{h}_{11}$ and $\mathbf{h}_{21}$, short-circuit the output terminals so that $\mathbf{V}_2 = 0$. Also connect a current source $\mathbf{I}_1$ to the input port as shown in Fig. 7.44(a).

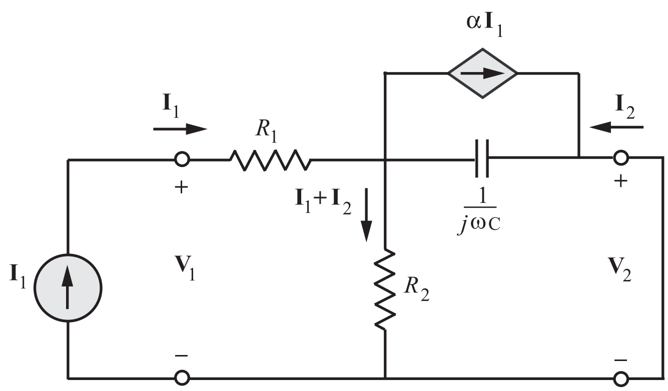


Figure 7.44(a)

Applying *KVL to the mesh on the right side*, we get

$$R_2[\mathbf{I}_1 + \mathbf{I}_2] + \frac{1}{j\omega C}[\alpha\mathbf{I}_1 + \mathbf{I}_2] = 0$$

$$\Rightarrow \qquad \left[R_2 + \frac{\alpha}{j\omega C}\right]\mathbf{I}_1 + \left[R_2 + \frac{1}{j\omega C}\right]\mathbf{I}_2 = 0$$

$$\Rightarrow \qquad [\alpha + j\omega R_2 C]\,\mathbf{I}_1 = -[1 + j\omega C R_2]\,\mathbf{I}_2$$

$$\Rightarrow \qquad \mathbf{I}_2 = \frac{-[\alpha + j\omega R_2 C]}{1 + j\omega R_2 C}\mathbf{I}_1$$

Hence,

$$\mathbf{h}_{21} = \left.\frac{\mathbf{I}_2}{\mathbf{I}_1}\right|_{\mathbf{V}_2=0} = -\left[\frac{\alpha + j\omega C R_2}{1 + j\omega R_2 C}\right]$$

Applying *KVL to the mesh on the left side*, we get

$$\begin{aligned}\mathbf{V}_1 &= R_1\mathbf{I}_1 + R_2[\mathbf{I}_1 + \mathbf{I}_2]\\ &= [R_1 + R_2]\,\mathbf{I}_1 + R_2\mathbf{I}_2\\ &= \left[R_1 + R_2 - \frac{R_2(\alpha + j\omega C R_2)}{1 + j\omega R_2 C}\right]\mathbf{I}_1\end{aligned}$$

Hence,

$$\mathbf{h}_{11} = \left.\frac{\mathbf{V}_1}{\mathbf{I}_1}\right|_{\mathbf{V}_2=0}$$

$$= R_1 + R_2 - \frac{R_2(\alpha + j\omega R_2 C)}{1 + j\omega R_2 C}$$
$$= \frac{R_1 + R_2(1-\alpha) + j\omega R_1 R_2 C}{1 + j\omega R_2 C}$$

To find $\mathbf{h}_{22}$ and $\mathbf{h}_{12}$, open-circuit the input terminals so that $\mathbf{I}_1 = 0$. Also connect a voltage source $\mathbf{V}_2$ to the output port as shown in Fig. 7.44(b). The dependent current source is open, because $\mathbf{I}_1 = 0$.

$$\mathbf{V}_1 = \mathbf{I}_2\mathbf{R}_2 = \frac{\mathbf{V}_2}{R_2 + \frac{1}{j\omega C}} R_2$$

Hence,
$$\mathbf{h}_{12} = \left.\frac{\mathbf{V}_1}{\mathbf{V}_2}\right|_{\mathbf{I}_1=0} = \frac{j\omega C R_2}{1 + j\omega C R_2}$$

$$\mathbf{I}_2 = \frac{\mathbf{V}_2}{R_2 + \frac{1}{j\omega C}} = \frac{j\omega C \mathbf{V}_2}{1 + j\omega C R_2}$$

$$\Rightarrow \quad \mathbf{h}_{22} = \left.\frac{\mathbf{I}_2}{\mathbf{V}_2}\right|_{\mathbf{I}_1=0} = \frac{j\omega C}{1 + j\omega C R_2}$$

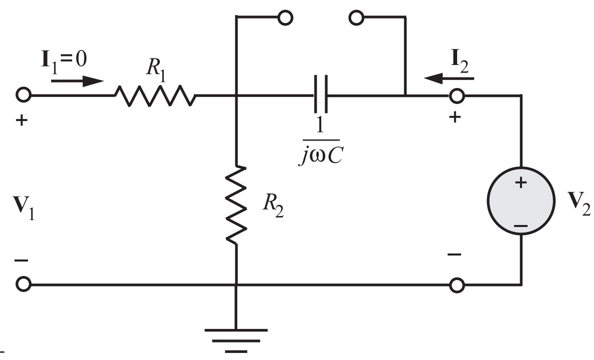


Figure 7.44(b)

7.6 Transmission parameters

The transmission parameters are defined by the equations:

$$\mathbf{V}_1 = \mathbf{A}\mathbf{V}_2 - \mathbf{B}\mathbf{I}_2$$
$$\mathbf{I}_1 = \mathbf{C}\mathbf{V}_2 - \mathbf{D}\mathbf{I}_2$$

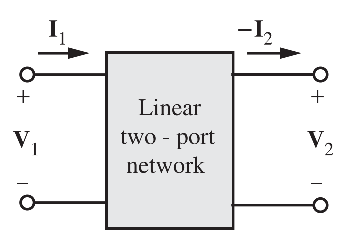


Figure 7.45 Terminal variables used to define the ABCD Parameters

Putting the above equations in matrix form we get

$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{I}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{V}_2 \\ -\mathbf{I}_2 \end{bmatrix}$$

Please note that in computing the transmission parameters, $-\mathbf{I}_2$ is used rather than $\mathbf{I}_2$, because the current is considered to be leaving the network as shown in Fig. 7.45.

These parameters are very useful in the analysis of circuits in cascade like transmission lines and cables. For this reason they are called Transmission Parameters. They are also known as **ABCD** parameters. The parameters are determined via the following equations:

$$\mathbf{A} = \left.\frac{\mathbf{V}_1}{\mathbf{V}_2}\right|_{\mathbf{I}_2=0} \qquad \mathbf{B} = \left.\frac{\mathbf{V}_1}{-\mathbf{I}_2}\right|_{\mathbf{V}_2=0} \qquad \mathbf{C} = \left.\frac{\mathbf{I}_1}{\mathbf{V}_2}\right|_{\mathbf{I}_2=0} \qquad \mathbf{D} = \left.\frac{\mathbf{I}_1}{-\mathbf{I}_2}\right|_{\mathbf{V}_2=0}$$

A, **B**, **C** and **D** represent the *open-circuit voltage ratio*, the *negative short-circuit transfer impedance*, the *open-circuit transfer admittance*, and the *negative short-circuit current ratio*, respectively. When the two-port network does not contain dependent sources, the following relation holds good.

$$\mathbf{AD} - \mathbf{BC} = 1$$

EXAMPLE 7.27

Determine the transmission parameters in the s domain for the network shown in Fig. 7.46.

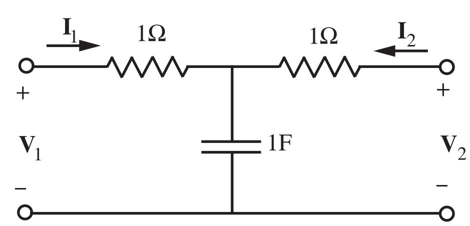


Figure 7.46

SOLUTION

The s domain equivalent circuit with the assumption that all the initial conditions are zero is shown in Fig. 7.47(a).

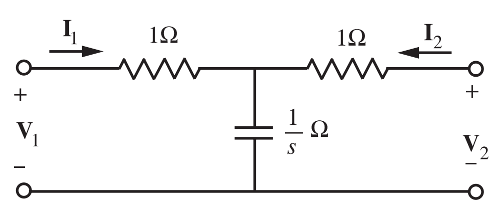


Figure 7.47(a)

To find the parameters **A** and **C**, open-circuit the output port and connect a voltage source $\mathbf{V}_1$ at the input port. The same is shown in Fig. 7.47(b).

$$\mathbf{I}_1 = \frac{\mathbf{V}_1}{1+\frac{1}{s}} = \frac{s\mathbf{V}_1}{s+1}$$

Then $$\mathbf{V}_2 = \frac{1}{s}\mathbf{I}_1$$

$$\Rightarrow \quad \mathbf{V}_2 = \frac{1}{s}\frac{s\mathbf{V}_1}{s+1} = \frac{\mathbf{V}_1}{s+1}$$

$$\Rightarrow \quad \mathbf{A} = \left.\frac{\mathbf{V}_1}{\mathbf{V}_2}\right|_{\mathbf{I}_2=0} = s+1$$

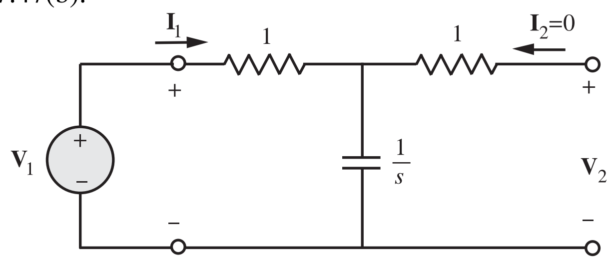


Figure 7.47(b)

Also, $$\mathbf{V}_2 = \frac{1}{s}\mathbf{I}_1$$

$$\Rightarrow \quad \mathbf{C} = \left.\frac{\mathbf{I}_1}{\mathbf{V}_2}\right|_{\mathbf{I}_2=0} = s$$

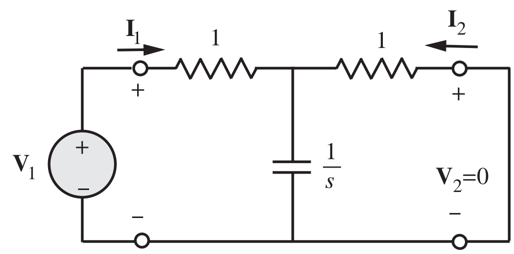


Figure 7.47(c)

To find the parameters **B** and **D**, short-circuit the output port and connect a voltage source $\mathbf{V}_1$ to the input port as shown in Fig. 7.47(c).

The total impedance as seen by the source $\mathbf{V}_1$ is

$$\mathbf{Z} = 1 + \frac{\frac{1}{s}\times 1}{\frac{1}{s}+1}$$

$$= 1 + \frac{1}{s+1} = \frac{s+2}{s+1}$$

$$\mathbf{I}_1 = \frac{\mathbf{V}_1}{\mathbf{Z}} = \frac{\mathbf{V}_1(s+1)}{(s+2)} \tag{7.44}$$

Using the principle of current division, we have

$$-\mathbf{I}_2 = \frac{\mathbf{I}_1\left(\frac{1}{s}\right)}{\frac{1}{s}+1} = \frac{\mathbf{I}_1}{s+1} \tag{7.45}$$

Hence, $$\mathbf{D} = \left.\frac{\mathbf{I}_1}{-\mathbf{I}_2}\right|_{\mathbf{V}_2=0} = s+1$$

From equation (7.44) and (7.45), we can write

$$-\mathbf{I}_2(s+1) = \frac{\mathbf{V}_1(s+1)}{(s+2)}$$

Hence, $$\mathbf{B} = \left.\frac{-\mathbf{V}_1}{\mathbf{I}_2}\right|_{\mathbf{V}_2=0} = s+2$$

Verification

We know that for a two port network without any dependent sources,

$$\mathbf{AD} - \mathbf{BC} = 1$$
$$(s+1)(s+1) - s(s+2) = 1$$

EXAMPLE 7.28

Determine the **ABCD** parameters for the two port network shown in Fig. 7.48.

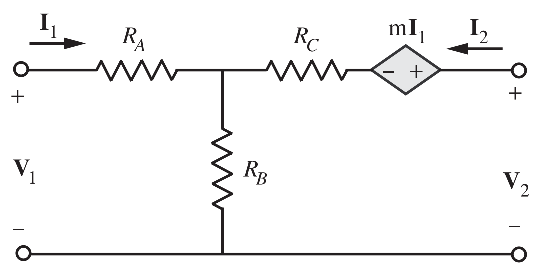


Figure 7.48

SOLUTION

To find the parameters **A** and **C**, open-circuit the output port as shown in Fig. 7.49(a) and connect a voltage source $\mathbf{V}_1$ to the input port.

Applying KVL to the output mesh, we get

$$-\mathbf{V}_2 + \mathrm{m}\mathbf{I}_1 + 0 \times R_C + \mathbf{I}_1 R_A = 0$$

$$\Rightarrow \quad \mathbf{V}_2 = \mathbf{I}_1 (\mathrm{m} + R_A)$$

Hence, $\mathbf{C} = \left.\dfrac{\mathbf{I}_1}{\mathbf{V}_2}\right|_{\mathbf{I}_2=0} = \dfrac{1}{\mathrm{m} + R_A}$

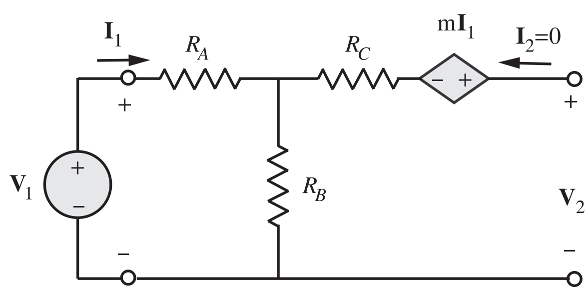


Figure 7.49(a)

Applying KVL to the input mesh, we get

$$\mathbf{V}_1 = \mathbf{I}_1 (R_A + R_B)$$

Hence, $$\mathbf{A} = \left.\frac{\mathbf{V}_1}{\mathbf{V}_2}\right|_{\mathbf{I}_2=0} = \frac{R_A + R_B}{\mathrm{m} + R_A}$$

To find the parameters **B** and **D**, short-ciruit the output port and connect a voltage source $\mathbf{V}_1$ to the input port as shown in Fig. 7.49(b).

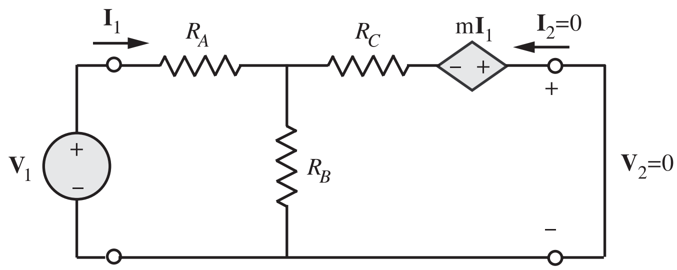


Figure 7.49(b)

Applying KVL to the right-mesh, we get

$$m\mathbf{I}_1 + R_C\mathbf{I}_2 + R_B(\mathbf{I}_1 + \mathbf{I}_2) = 0$$

$$\Rightarrow \quad (m + R_B)\,\mathbf{I}_1 = -(R_C + R_B)\,\mathbf{I}_2$$

$$\Rightarrow \quad \mathbf{I}_1 = \frac{-(R_C + R_B)}{(m + R_B)}\mathbf{I}_2$$

Hence,
$$\mathbf{D} = \left.\frac{\mathbf{I}_1}{-\mathbf{I}_2}\right|_{\mathbf{V}_2=0} = \frac{(R_C + R_B)}{(m + R_B)}$$

Applying KVL to the left-mesh, we get

$$-\mathbf{V}_1 + R_A\mathbf{I}_1 + R_B(\mathbf{I}_1 + \mathbf{I}_2) = 0$$

$$\Rightarrow \quad \mathbf{V}_1 = (R_A + R_B)\,\mathbf{I}_1 + R_B\mathbf{I}_2$$

$$= (R_A + R_B)\left[\frac{-(R_C + R_B)}{(m + R_B)}\mathbf{I}_2\right] + R_B\mathbf{I}_2$$

$$= -\left[\frac{R_CR_A + R_CR_B + R_BR_A - mR_B}{m + R_B}\right]\mathbf{I}_2$$

Hence,
$$\mathbf{B} = \left.\frac{\mathbf{V}_1}{-\mathbf{I}_2}\right|_{\mathbf{V}_2=0}$$

$$= \frac{R_CR_A + R_CR_B + R_BR_A - mR_B}{m + R_B}$$

EXAMPLE 7.29

The following direct-current measurements were done on a two port network:

Port 1 open	**Port 1 Short-circuited**
$\mathbf{V}_1 = 1$ mV	$\mathbf{I}_1 = -0.5\ \mu$A
$\mathbf{V}_2 = 10$ V	$\mathbf{I}_2 = 80\ \mu$A
$\mathbf{I}_2 = 200\ \mu$A	$\mathbf{V}_2 = 5$ V

Calculate the transmission parameters for the two port network.

SOLUTION

For the two port network, we can write

$$\mathbf{V}_1 = \mathbf{AV}_2 - \mathbf{BI}_2$$
$$\mathbf{I}_1 = \mathbf{CV}_2 - \mathbf{DI}_2$$

From $\mathbf{I}_1 = 0$ (port 1 open): $\quad 1 \times 10^{-3} = \mathbf{A} \times 10 - \mathbf{B} \times 200 \times 10^{-6}$

From $\mathbf{V}_1 = 0$ (Port 1 short): $\quad 0 = \mathbf{A} \times 5 - \mathbf{B} \times 80 \times 10^{-6}$

Solving simultaneously yields,

$$\mathbf{A} = -4 \times 10^{-4}, \mathbf{B} = -25\Omega$$

From $\mathbf{I}_1 = 0$: $\quad 0 = \mathbf{C} \times 10 - \mathbf{D} \times \left(200 \times 10^{-6}\right)$

From $\mathbf{V}_1 = 0$: $\quad -0.5 \times 10^{-6} = \mathbf{C} \times 5 - \mathbf{D} \times 80 \times 10^{-6}$

Solving simultaneously yields,

$$\mathbf{C} = -5 \times 10^{-7}\text{S}, \quad \mathbf{D} = -0.025$$

In summary,

$$\mathbf{A} = -4 \times 10^{-4}$$
$$\mathbf{B} = -25\Omega$$
$$\mathbf{C} = -5 \times 10^{-7}\ \text{S}$$
$$\mathbf{D} = -0.025$$

EXAMPLE 7.30

Find the transmission parameters for the network shown in Fig. 7.50.

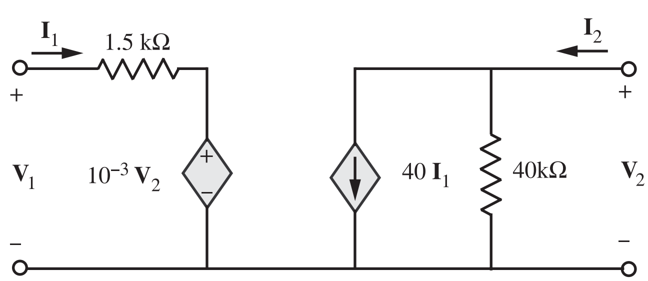


Figure 7.50

SOLUTION

To find the parameters **A** and **C**, open the output port and connect a voltage source $\mathbf{V}_1$ to the input port as shown in Fig. 7.51(a).

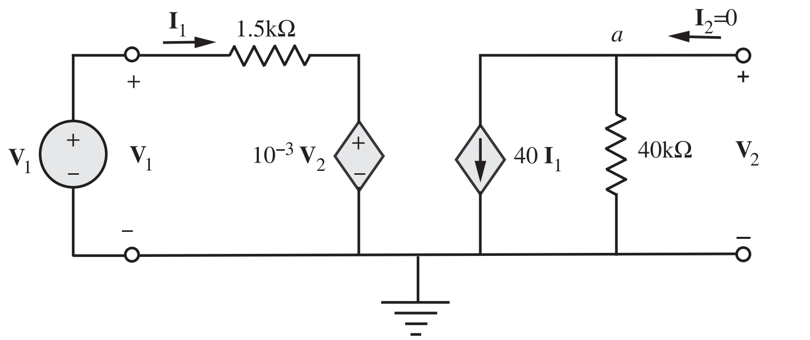


Figure 7.51(a)

Applying KVL to the input loop, we get

$$\mathbf{V}_1 = 1.5 \times 10^3 \mathbf{I}_1 + 10^{-3}\mathbf{V}_2$$

Also *KCL at node a* gives

$$40\mathbf{I}_1 + \frac{\mathbf{V}_2}{40 \times 10^3} = 0$$

$$\Rightarrow \qquad \mathbf{I}_1 = \frac{-\mathbf{V}_2}{160 \times 10^3} = -6.25 \times 10^{-6}\mathbf{V}_2$$

Substituting the value of $\mathbf{I}_1$ in the preceeding loop equation, we get

$$\mathbf{V}_1 = 1.5 \times 10^3 \left(-6.25 \times 10^{-6}\mathbf{V}_2\right) + 10^{-3}\mathbf{V}_2$$

$$\Rightarrow \qquad \mathbf{V}_1 = -9.375 \times 10^{-3}\mathbf{V}_2 + 10^{-3}\mathbf{V}_2$$
$$= -8.375 \times 10^{-3}\mathbf{V}_2$$

Hence, $$\mathbf{A} = \left.\frac{\mathbf{V}_1}{\mathbf{V}_2}\right|_{\mathbf{I}_2=0} = \mathbf{-8.375 \times 10^{-3}}$$

Also, $$\mathbf{C} = \left.\frac{\mathbf{I}_1}{\mathbf{V}_2}\right|_{\mathbf{I}_2=0} = \mathbf{-6.25 \times 10^{-6}}$$

To find the parameters **B** and **D**, refer the circuit shown in Fig. 7.51(b).

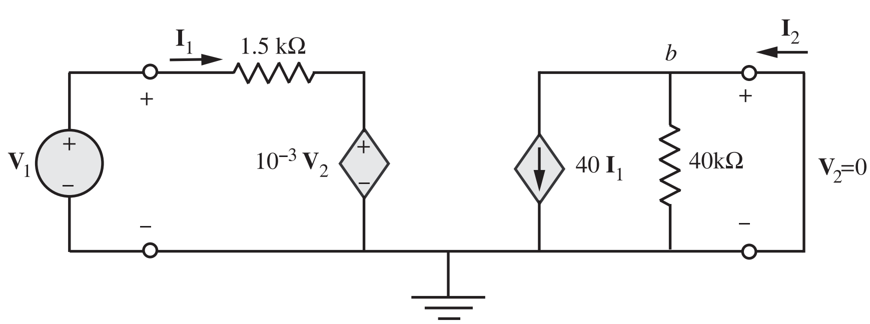


Figure 7.51(b)

Applying KCL at node b, we find

$$40\mathbf{I}_1 + 0 = \mathbf{I}_2$$

$$\Rightarrow \qquad \mathbf{I}_2 = 40\mathbf{I}_1$$

Hence,

$$\mathbf{D} = \left.\frac{-\mathbf{I}_1}{\mathbf{I}_2}\right|_{\mathbf{V}_2=0} = \frac{\mathbf{-1}}{\mathbf{40}}$$

Applying KVL to the input loop, we get

$$\mathbf{V}_1 = 1.5 \times 10^3 \mathbf{I}_1$$

$$\Rightarrow \qquad \mathbf{V}_1 = 1.5 \times 10^3 \times \frac{\mathbf{I}_2}{40}$$

Hence,

$$\mathbf{B} = \left.\frac{\mathbf{V}_1}{-\mathbf{I}_2}\right|_{\mathbf{V}_2=0}$$

$$= \frac{-1.5}{40} \times 10^3$$

$$= \mathbf{-37.5\Omega}$$

EXAMPLE 7.31

Find the Thevenin equivalent circuit as seen from the output port using the transmission parameters for the network shown in Fig. 7.52.

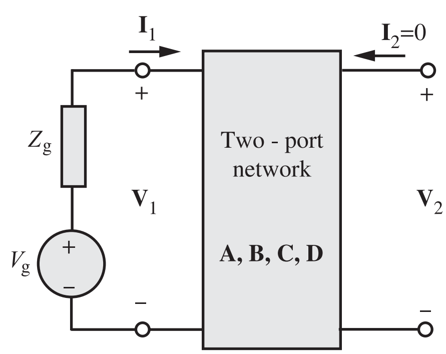


Figure 7.52

SOLUTION

For the two-port network, we can write

$$\mathbf{V}_1 = \mathbf{A}\mathbf{V}_2 - \mathbf{B}\mathbf{I}_2 \tag{7.46}$$

$$\mathbf{I}_1 = \mathbf{C}\mathbf{V}_2 - \mathbf{D}\mathbf{I}_2 \tag{7.47}$$

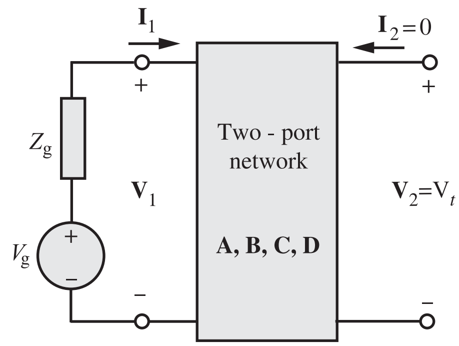


Figure 7.53(a)

Refer the network shown in Fig. 7.53(a) to find V_t.
At the input port,

$$V_g - \mathbf{I}_1 Z_g = \mathbf{V}_1 \tag{7.48}$$

Also,

$$\mathbf{I}_2 = 0 \tag{7.49}$$

Making use of equations (7.48) and (7.49) in equations (7.46) and (7.47) we get,

$$V_g - \mathbf{I}_1 Z_g = \mathbf{A}\mathbf{V}_2 \tag{7.50}$$

and

$$\mathbf{I}_1 = \mathbf{C}\mathbf{V}_2 \tag{7.51}$$

Making use of equation (7.51) in (7.50), we get

$$V_g - \mathbf{C}\mathbf{V}_2 Z_g = \mathbf{A}\mathbf{V}_2$$

$$\Rightarrow \qquad \mathbf{V}_2 = V_t = \frac{V_g}{\mathrm{A} + \mathbf{C}Z_g}$$

To find R_t, deactivate the voltage source V_g and then connect a voltage source $\mathbf{V}_2 = 1$ V at the output port. The resulting circuit diagram is shown in Fig. 7.53(b).

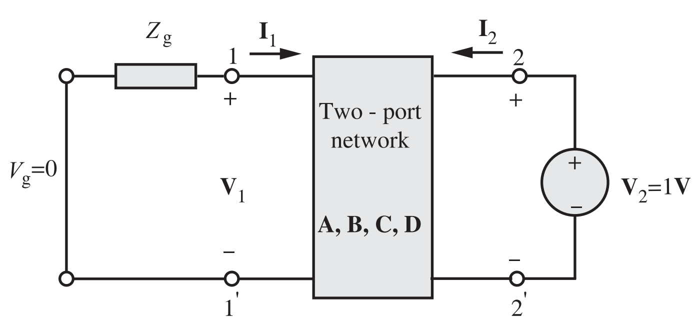


Figure 7.53(b)

Referring Fig. 7.53(b), we can write

$$\mathbf{V}_1 = -\mathbf{I}_1 Z_g$$

Substituting the value of $\mathbf{V}_1$ in equation (7.46), we get

$$-\mathbf{I}_1 Z_g = \mathbf{A}\mathbf{V}_2 - \mathbf{B}\mathbf{I}_2$$

$$\Rightarrow \qquad \mathbf{I}_1 = \frac{-\mathbf{A}}{Z_g}\mathbf{V}_2 + \frac{\mathbf{B}}{Z_g}\mathbf{I}_2 \tag{7.52}$$

Equating equations (7.47) and (7.52) results

$$\mathbf{C}\mathbf{V}_2 - \mathbf{D}\mathbf{I}_2 = \frac{-\mathbf{A}}{Z_g}\mathbf{V}_2 + \frac{\mathbf{B}}{Z_g}\mathbf{I}_2$$

$$\mathbf{V}_2\left[\mathbf{C} + \frac{-\mathbf{A}}{Z_g}\right] = \left[\mathbf{D} + \frac{\mathbf{B}}{Z_g}\right]\mathbf{I}_2$$

Hence

$$Z_t = \frac{\mathbf{V}_2}{\mathbf{I}_2} = \frac{\mathbf{D} + \dfrac{\mathbf{B}}{Z_g}}{\mathbf{C} + \dfrac{\mathbf{A}}{Z_g}}$$

$$= \frac{\mathbf{B} + \mathbf{D}Z_g}{\mathbf{A} + \mathbf{C}Z_g}$$

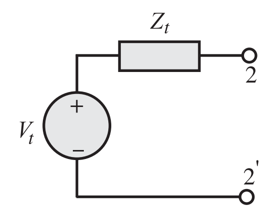


Figure 7.54

Hence, the Thevenin equivalent circuit as seen from the output port is as shown in Fig. 7.54.

EXAMPLE 7.32

For the network shown in Fig. 7.55(a), find R_L for maximum power transfer and the maximum power transferred.

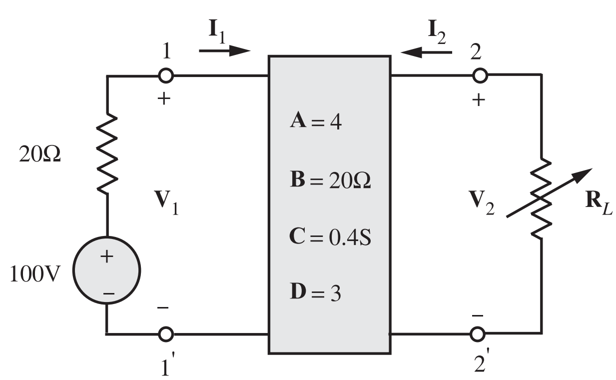


Figure 7.55(a)

SOLUTION

From the previous example 7.31,

$$Z_t = \frac{\mathbf{B} + \mathbf{D}Z_g}{\mathbf{A} + \mathbf{C}Z_g} = \frac{20 + 3 \times 20}{4 + 0.4 \times 20} = \frac{20 + 60}{4 + 8} = \mathbf{6.67\Omega}$$
$$V_t = \frac{V_g}{\mathbf{A} + \mathbf{C}Z_g} = \frac{100}{4 + 0.4 \times 20} = \frac{100}{12} = \mathbf{8.33V}$$

For maximum power transfer,
$R_L = Z_t = 6.67\Omega$ (purely resistive)
Hence, the Thevenin equivalent circuit as seen from output terminals along with R_L is as shown in Fig. 7.55(b).

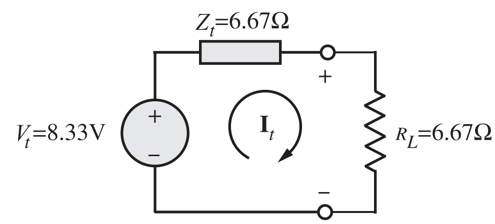


Figure 7.55(b)

$$I_t = \frac{8.33}{6.67 + 6.67} = 0.624\text{A}$$
$$\begin{aligned}(P_L)_{\max} &= I_t^2 \times 6.67 \\ &= (0.624)^2 \times 6.67 \\ &= \mathbf{2.6 Watts}\end{aligned}$$

EXAMPLE 7.33

Refer the bridge circuit shown in Fig. 7.56. Find the transmission parameters.

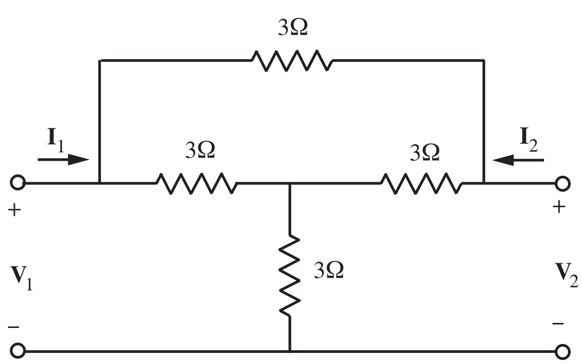


Figure 7.56

SOLUTION

Performing $\mathbf{\Delta}$ to Y transformation, as shown in Fig. 7.57(a) the network reduces to the form as shown in Fig. 7.57(b). Please note that, when all resistors are of equal value,

$$R_{\text{Y}} = \frac{1}{3}R_{\Delta}$$

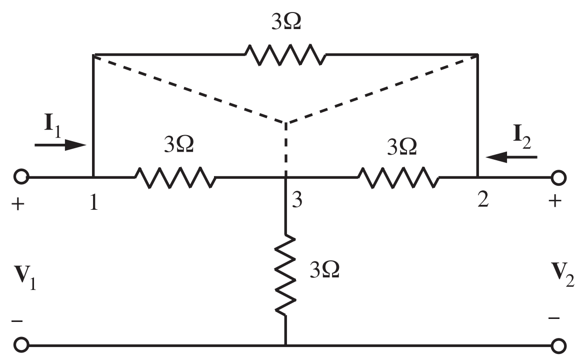


Figure 7.57(a)

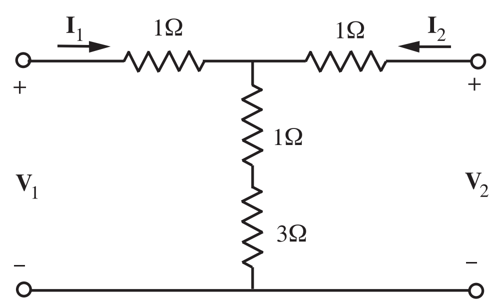


Figure 7.57(b)

To find the parameters **A** and **D**, open the output port and connect a voltage source $\mathbf{V}_1$ at the input port as shown is Fig. 7.57(c).

Applying KVL to the input loop we get

$\mathbf{I}_1 + 4\mathbf{I}_1 = \mathbf{V}_1$

$$\Rightarrow \quad \mathbf{V}_1 = 5\mathbf{I}_1$$

Also, $\quad \mathbf{I}_1 = \dfrac{\mathbf{V}_2}{4} \Rightarrow \mathbf{C} = \left.\dfrac{\mathbf{I}_1}{\mathbf{V}_2}\right|_{\mathbf{I}_2=0} = \dfrac{1}{4}\text{S}$

Also, $\quad \Rightarrow \mathbf{V}_1 = 5\mathbf{I}_1 = \dfrac{5}{4}\mathbf{V}_2$

Hence, $\quad \mathbf{A} = \left.\dfrac{\mathbf{V}_1}{\mathbf{V}_2}\right|_{\mathbf{I}_2=0} = \dfrac{5}{4}$

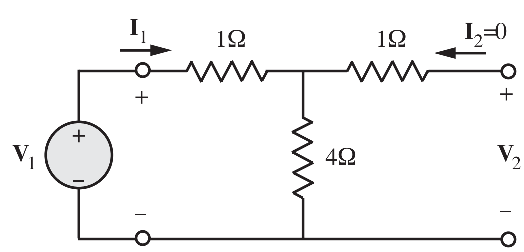


Figure 7.57(c)

To find the parameters **B** and **D**, refer the circuit shown in Fig. 7.57(d).

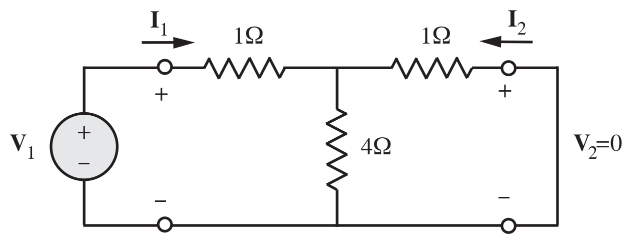


Figure 7.57(d)

$$-\mathbf{I}_2 = \frac{\mathbf{I}_1 \times 4}{4+1} = \frac{4}{5}\mathbf{I}_1$$

Hence $$\mathbf{D} = -\left.\frac{\mathbf{I}_1}{\mathbf{I}_2}\right|_{\mathbf{v}_2=0} = \frac{5}{4}$$

Applying KVL to the input loop, we get

$$-\mathbf{V}_1 + 1 \times \mathbf{I}_1 + 4 \times (\mathbf{I}_1 + \mathbf{I}_2) = 0$$

Substituting $\mathbf{I}_1 = -\frac{5}{4}\mathbf{I}_2$ in the preceeding equation, we get

$$-\mathbf{V}_1 - \frac{5}{4}\mathbf{I}_2 + 4\left(-\frac{5}{4}\mathbf{I}_2 + \mathbf{I}_2\right) = 0$$

$$\Rightarrow \quad -\mathbf{V}_1 - \frac{5}{4}\mathbf{I}_2 - 5\mathbf{I}_2 + 4\mathbf{I}_2 = 0$$

$$\Rightarrow \quad 4\mathbf{V}_1 = -9\mathbf{I}_2$$

Hence, $$\mathbf{B} = \left.\frac{\mathbf{V}_1}{-\mathbf{I}_2}\right|_{\mathbf{v}_2=0} = \frac{9}{4}\Omega$$

Verification:
For a two port network which does not contain any dependent sources, we have

$$\mathbf{AD} - \mathbf{BC} = 1$$

$$\frac{5}{4} \times \frac{5}{4} - \frac{1}{4} \times \frac{9}{4} = \frac{25}{16} - \frac{9}{16} = 1$$

7.7 Relations between two-port parameters

If all the two-port parameters for a network exist, it is possible to relate one set of parameters to another, since these parameters interrelate the variables $\mathbf{V}_1, \mathbf{I}_1, \mathbf{V}_2$ and $\mathbf{I}_2$. To begin with let us first derive the relation between the **z** parameters and **y** parameters.

The matrix equation for the **z** parameters is

$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{z}_{11} & \mathbf{z}_{12} \\ \mathbf{z}_{21} & \mathbf{z}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix}$$

$$\Rightarrow \quad \mathbf{V} = \mathbf{zI} \qquad (7.53)$$

Similarly, the equation for **y** parameters is

$$\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{y}_{11} & \mathbf{y}_{12} \\ \mathbf{y}_{21} & \mathbf{y}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix}$$

$$\Rightarrow \quad \mathbf{I} = \mathbf{yV} \qquad (7.54)$$

Substituting equation (7.54) into equation (7.53), we get

$$\mathbf{V} = \mathbf{zyV}$$

Hence, $$\mathbf{z} = \mathbf{y}^{-1} = \frac{\mathrm{adj}(\mathbf{y})}{\boldsymbol{\Delta}\mathbf{y}}$$

where $$\boldsymbol{\Delta}\mathbf{y} = \mathbf{y}_{11}\mathbf{y}_{22} - \mathbf{y}_{21}\mathbf{y}_{12}$$

This means that we can obtain **z** matrix by inverting **y** matrix. It is quite possible that a two-port network has a **y** matrix or a **z** matrix, but not both.

Next let us proceed to find **z** parameters in terms of **ABCD** parameters.

The **ABCD** parameters of a two-port network are defined by

$$\mathbf{V}_1 = \mathbf{AV}_2 - \mathbf{BI}_2$$

$$\mathbf{I}_1 = \mathbf{CV}_2 - \mathbf{DI}_2$$

$$\Rightarrow \qquad \mathbf{V}_2 = \frac{1}{\mathbf{C}}(\mathbf{I_1}{+}\mathbf{DI}_2)$$

$$\Rightarrow \qquad \mathbf{V}_2 = \frac{1}{\mathbf{C}}\mathbf{I}_1 + \frac{\mathbf{D}}{\mathbf{C}}\mathbf{I}_2 \tag{7.55}$$

$$\mathbf{V}_1 = \mathbf{A}\left(\frac{\mathbf{I}_1}{\mathbf{C}} + \frac{\mathbf{DI}_2}{\mathbf{C}}\right) - \mathbf{BI}_2$$

$$= \frac{\mathbf{AI}_1}{\mathbf{C}} + \left(\frac{\mathbf{AD}}{\mathbf{C}} - \mathbf{B}\right)\mathbf{I}_2 \tag{7.56}$$

Comparing equations (7.56) and (7.55) with

$$\mathbf{V}_1 = \mathbf{z}_{11}\mathbf{I}_1 + \mathbf{z}_{12}\mathbf{I}_2$$

$$\mathbf{V}_2 = \mathbf{z}_{21}\mathbf{I}_1 + \mathbf{z}_{22}\mathbf{I}_2$$

respectively, we find that

$$\mathbf{z}_{11} = \frac{\mathbf{A}}{\mathbf{C}} \qquad \mathbf{z}_{12} = \frac{\mathbf{AD} - \mathbf{BC}}{\mathbf{C}} \qquad \mathbf{z}_{21} = \frac{1}{\mathbf{C}} \qquad \mathbf{z}_{22} = \frac{\mathbf{D}}{\mathbf{C}}$$

Next, let us derive the relation between hybrid parameters and **z** parameters.

$$\mathbf{V}_1 = \mathbf{z}_{11}\mathbf{I}_1 + \mathbf{z}_{12}\mathbf{I}_2 \tag{7.57}$$

$$\mathbf{V}_2 = \mathbf{z}_{21}\mathbf{I}_1 + \mathbf{z}_{22}\mathbf{I}_2 \tag{7.58}$$

From equation (7.58), we can write

$$\mathbf{I}_2 = \frac{-\mathbf{z}_{21}}{\mathbf{z}_{22}}\mathbf{I}_1 + \frac{\mathbf{V}_2}{\mathbf{z}_{22}} \tag{7.59a}$$

Substituting this value of $\mathbf{I}_2$ in equation (7.57), we get

$$\mathbf{V}_1 = \mathbf{z}_{11}\mathbf{I}_1 + \mathbf{z}_{12}\left[\frac{-\mathbf{z}_{21}\mathbf{I}_1}{\mathbf{z}_{22}} + \frac{\mathbf{V}_2}{\mathbf{z}_{22}}\right]$$

$$= \left[\frac{\mathbf{z}_{11}\mathbf{z}_{22} - \mathbf{z}_{12}\mathbf{z}_{21}}{\mathbf{z}_{22}}\right]\mathbf{I}_1 + \frac{\mathbf{z}_{12}\mathbf{V}_2}{\mathbf{z}_{22}} \tag{7.59b}$$

Comparing equations (7.59b) and (7.59a) with

$$\mathbf{V}_1 = \mathbf{h}_{11}\mathbf{I}_1 + \mathbf{h}_{12}\mathbf{V}_2$$
$$\mathbf{I}_2 = \mathbf{h}_{21}\mathbf{I}_1 + \mathbf{h}_{22}\mathbf{V}_2$$

we get,
$$\mathbf{h}_{11} = \frac{\mathbf{z}_{11}\mathbf{z}_{22} - \mathbf{z}_{12}\mathbf{z}_{21}}{\mathbf{z}_{22}} = \frac{\Delta\mathbf{z}}{\mathbf{Z}_{22}}$$
$$\mathbf{h}_{12} = \frac{\mathbf{z}_{12}}{\mathbf{z}_{22}} \qquad \mathbf{h}_{21} = \frac{-\mathbf{z}_{21}}{\mathbf{z}_{22}} \qquad \mathbf{h}_{22} = \frac{1}{\mathbf{z}_{22}}$$

where
$$\Delta\mathbf{z} = \mathbf{z}_{11}\mathbf{z}_{22} - \mathbf{z}_{12}\mathbf{z}_{21}$$

Finally, let us derive the relationship between **y** parameters and **ABCD** parameters.

$$\mathbf{I}_1 = \mathbf{y}_{11}\mathbf{V}_1 + \mathbf{y}_{12}\mathbf{V}_2 \tag{7.60}$$
$$\mathbf{I}_2 = \mathbf{y}_{21}\mathbf{V}_1 + \mathbf{y}_{22}\mathbf{V}_2 \tag{7.61}$$

From equation (7.61), we can write

$$\mathbf{V}_1 = \frac{\mathbf{I}_2}{\mathbf{y}_{21}} - \frac{\mathbf{y}_{22}}{\mathbf{y}_{21}}\mathbf{V}_2$$
$$= \frac{-\mathbf{y}_{22}}{\mathbf{y}_{21}}\mathbf{V}_2 + \frac{1}{\mathbf{y}_{21}}\mathbf{I}_2 \tag{7.62}$$

Substituting equation (7.62) in equation (7.60), we get

$$\mathbf{I}_1 = \frac{-\mathbf{y}_{11}\mathbf{y}_{22}}{\mathbf{y}_{21}}\mathbf{V}_2 + \mathbf{y}_{12}\mathbf{V}_2 + \frac{\mathbf{y}_{11}}{\mathbf{y}_{21}}\mathbf{I}_2$$
$$= \frac{-\Delta\mathbf{y}}{\mathbf{y}_{21}}\mathbf{V}_2 + \frac{\mathbf{y}_{11}}{\mathbf{y}_{21}}\mathbf{I}_2 \tag{7.63}$$

Comparing equations (7.62) and (7.63) with the following equations,

$$\mathbf{V}_1 = \mathbf{A}\mathbf{V}_2 - \mathbf{B}\mathbf{I}_2$$
$$\mathbf{I}_1 = \mathbf{C}\mathbf{V}_2 - \mathbf{D}\mathbf{I}_2$$

we get
$$\mathbf{A} = \frac{-\mathbf{y}_{22}}{\mathbf{y}_{21}} \qquad \mathbf{B} = \frac{-1}{\mathbf{y}_{21}} \qquad \mathbf{C} = \frac{-\Delta\mathbf{y}}{\mathbf{y}_{21}} \qquad \mathbf{D} = \frac{-\mathbf{y}_{11}}{\mathbf{y}_{21}}$$

where
$$\Delta\mathbf{y} = \mathbf{y}_{11}\mathbf{y}_{22} - \mathbf{y}_{12}\mathbf{y}_{21}$$

Table 7.1 lists all the conversion formulae that relate one set of two-port parameters to another. Please note that $\Delta\mathbf{z}$, $\Delta\mathbf{y}$, $\Delta\mathbf{h}$, and $\Delta\mathbf{T}$, refer to the determinants of the matrices for **z**, **y**, **hybrid**, and **ABCD** parameters respectively.

Table 7.1 Parameter relationships

	$\mathbf{z}$	$\mathbf{y}$	$\mathbf{T}$	$\mathbf{h}$
$\mathbf{z}$	$\begin{bmatrix} \mathbf{z}_{11} & \mathbf{z}_{12} \\ \mathbf{z}_{21} & \mathbf{z}_{22} \end{bmatrix}$	$\begin{bmatrix} \frac{\mathbf{y}_{22}}{\Delta\mathbf{y}} & \frac{-\mathbf{y}_{12}}{\Delta\mathbf{y}} \\ \frac{-\mathbf{y}_{21}}{\Delta\mathbf{y}} & \frac{\mathbf{y}_{11}}{\Delta\mathbf{y}} \end{bmatrix}$	$\begin{bmatrix} \frac{\mathbf{A}}{\mathbf{C}} & \frac{\Delta\mathbf{T}}{\mathbf{C}} \\ \frac{1}{\mathbf{C}} & \frac{\mathbf{D}}{\mathbf{C}} \end{bmatrix}$	$\begin{bmatrix} \frac{\Delta\mathbf{h}}{\mathbf{h}_{22}} & \frac{\mathbf{h}_{12}}{\mathbf{h}_{22}} \\ \frac{-\mathbf{h}_{21}}{\mathbf{h}_{22}} & \frac{1}{\mathbf{h}_{22}} \end{bmatrix}$
$\mathbf{y}$	$\begin{bmatrix} \frac{\mathbf{z}_{22}}{\Delta\mathbf{z}} & \frac{-\mathbf{z}_{12}}{\Delta\mathbf{z}} \\ \frac{-\mathbf{z}_{21}}{\Delta\mathbf{z}} & \frac{\mathbf{z}_{11}}{\Delta\mathbf{z}} \end{bmatrix}$	$\begin{bmatrix} \mathbf{y}_{11} & \mathbf{y}_{12} \\ \mathbf{y}_{21} & \mathbf{y}_{22} \end{bmatrix}$	$\begin{bmatrix} \frac{\mathbf{D}}{\mathbf{B}} & \frac{-\Delta\mathbf{T}}{\mathbf{B}} \\ \frac{-1}{\mathbf{B}} & \frac{\mathbf{A}}{\mathbf{B}} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{\mathbf{h}_{11}} & \frac{-\mathbf{h}_{22}}{\mathbf{h}_{11}} \\ \frac{\mathbf{h}_{21}}{\mathbf{h}_{11}} & \frac{\Delta\mathbf{h}}{\mathbf{h}_{11}} \end{bmatrix}$
$\mathbf{T}$	$\begin{bmatrix} \frac{\mathbf{z}_{11}}{\mathbf{z}_{21}} & \frac{\Delta\mathbf{z}}{\mathbf{z}_{21}} \\ \frac{1}{\mathbf{z}_{21}} & \frac{\mathbf{z}_{22}}{\mathbf{z}_{21}} \end{bmatrix}$	$\begin{bmatrix} \frac{-\mathbf{y}_{22}}{\mathbf{y}_{21}} & \frac{-1}{\mathbf{y}_{21}} \\ \frac{-\Delta\mathbf{y}}{\mathbf{y}_{21}} & \frac{-\mathbf{y}_{11}}{\mathbf{y}_{21}} \end{bmatrix}$	$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$	$\begin{bmatrix} \frac{-\Delta\mathbf{h}}{\mathbf{h}_{21}} & \frac{-\mathbf{h}_{11}}{\mathbf{h}_{21}} \\ \frac{-\mathbf{h}_{22}}{\mathbf{h}_{21}} & \frac{-1}{\mathbf{h}_{21}} \end{bmatrix}$
$\mathbf{h}$	$\begin{bmatrix} \frac{\Delta\mathbf{z}}{\mathbf{z}_{22}} & \frac{\mathbf{z}_{12}}{\mathbf{z}_{22}} \\ \frac{-\mathbf{z}_{21}}{\mathbf{z}_{22}} & \frac{1}{\mathbf{z}_{22}} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{\mathbf{y}_{11}} & \frac{-\mathbf{y}_{12}}{\mathbf{y}_{11}} \\ \frac{\mathbf{y}_{21}}{\mathbf{y}_{11}} & \frac{\Delta\mathbf{y}}{\mathbf{y}_{11}} \end{bmatrix}$	$\begin{bmatrix} \frac{\mathbf{B}}{\mathbf{D}} & \frac{\Delta\mathbf{T}}{\mathbf{D}} \\ -\frac{1}{\mathbf{D}} & \frac{\mathbf{C}}{\mathbf{D}} \end{bmatrix}$	$\begin{bmatrix} \mathbf{h}_{11} & \mathbf{h}_{12} \\ \mathbf{h}_{21} & \mathbf{h}_{22} \end{bmatrix}$

$$\Delta\mathbf{z} = \mathbf{z}_{11}\mathbf{z}_{22} - \mathbf{z}_{12}\mathbf{z}_{21}, \Delta\mathbf{y} = \mathbf{y}_{11}\mathbf{y}_{22} - \mathbf{y}_{12}\mathbf{y}_{21}, \Delta\mathbf{h} = \mathbf{h}_{11}\mathbf{h}_{22} - \mathbf{h}_{12}\mathbf{h}_{21}, \Delta\mathbf{T} = \mathbf{AD} - \mathbf{BC}$$

EXAMPLE 7.34

Determine the **y** parameters for a two-port network if the **z** parameters are

$$\mathbf{z} = \begin{bmatrix} 10 & 5 \\ 5 & 9 \end{bmatrix}$$

SOLUTION

$$\Delta\mathbf{z} = 10 \times 9 - 5 \times 5 = 65$$

$$\mathbf{y}_{11} = \frac{\mathbf{z}_{22}}{\Delta\mathbf{z}} = \frac{9}{65}\mathrm{S}$$

$$\mathbf{y}_{12} = \frac{-\mathbf{z}_{12}}{\Delta\mathbf{z}} = \frac{-5}{65}\mathrm{S}$$

$$\mathbf{y}_{21} = \frac{-\mathbf{z}_{21}}{\Delta\mathbf{z}} = \frac{-5}{65}\mathrm{S}$$

$$\mathbf{y}_{22} = \frac{\mathbf{z}_{11}}{\Delta\mathbf{z}} = \frac{10}{65}\mathrm{S}$$

EXAMPLE 7.35

Following are the hybrid parameters for a network:

$$\begin{bmatrix} \mathbf{h}_{11} & \mathbf{h}_{12} \\ \mathbf{h}_{21} & \mathbf{h}_{22} \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 3 & 6 \end{bmatrix}$$

Determine the **y** parameters for the network.

SOLUTION

$$\Delta\mathbf{h} = 5 \times 6 - 3 \times 2 = 24$$

$$\mathbf{y}_{11} = \frac{1}{\mathbf{h}_{11}} = \frac{1}{5}\text{S}$$

$$\mathbf{y}_{12} = \frac{-\mathbf{h}_{22}}{\mathbf{h}_{11}} = \frac{-6}{5}\text{S}$$

$$\mathbf{y}_{21} = \frac{\mathbf{h}_{21}}{\mathbf{h}_{11}} = \frac{3}{5}\text{S}$$

$$\mathbf{y}_{22} = \frac{\Delta\mathbf{h}}{\mathbf{h}_{11}} = \frac{24}{5}\text{S}$$

Reinforcement problems

R.P 7.1

The network of Fig. R.P. 7.1 contains both a dependent current source and a dependent voltage source. Determine **y** and **z** parameters.

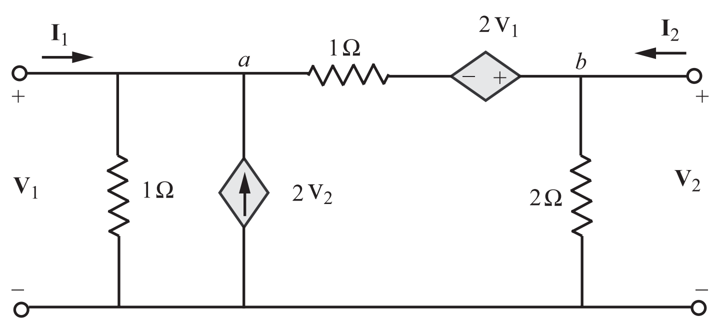


Figure R.P. 7.1

SOLUTION

From the figure, the node equations are

$$\mathbf{I}_{ab} = -\left(\mathbf{I}_2 - \frac{\mathbf{V}_2}{2}\right)$$

At node a,

$$\mathbf{I}_1 = \mathbf{V}_1 - 2\mathbf{V}_2 - \left(\mathbf{I}_2 - \frac{\mathbf{V}_2}{2}\right)$$

At node b,

$$\mathbf{V}_1 = \mathbf{V}_2 - 2\mathbf{V}_1 - \left(\mathbf{I}_2 - \frac{\mathbf{V}_2}{2}\right)$$

Simplifying, the nodal equations, we get

$$\mathbf{I}_1 + \mathbf{I}_2 = \mathbf{V}_1 - \frac{3}{2}\mathbf{V}_2$$

$$\mathbf{I}_2 = -3\mathbf{V}_1 + \frac{3}{2}\mathbf{V}_2$$

In matrix form,

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{3}{2} \\ -3 & \frac{3}{2} \end{bmatrix}\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix}$$

$$\Rightarrow \qquad \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 & -\frac{3}{2} \\ -3 & \frac{3}{2} \end{bmatrix}\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix}$$

Therefore,

$$\mathbf{y} = \begin{bmatrix} 4 & -3 \\ -3 & \frac{3}{2} \end{bmatrix}$$

and

$$\mathbf{Z} = -\frac{1}{3}\begin{bmatrix} \frac{3}{2} & 3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -0.5 & 1 \\ 1 & -\frac{4}{3} \end{bmatrix}$$

R.P 7.2

Compute $\mathbf{y}$ parameters for the network shown in Fig. R.P. 7.2.

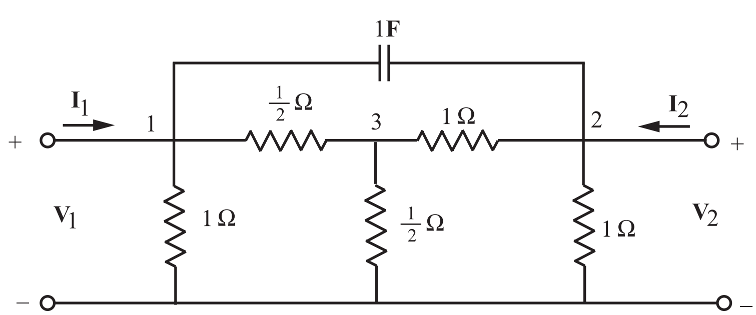


Figure R.P. 7.2

SOLUTION

The circuit shall be transformed into s-domain and then we shall use the matrix partitioning method to solve the problem. From Fig 7.2, Node equations in matrix form,

$$\left[\begin{array}{c}\mathbf{I}_1\\ \mathbf{I}_2\\ \hline 0\end{array}\right]\left[\begin{array}{cc|c}s+3 & -s & -2\\ -s & s+2 & -1\\ \hline -2 & -1 & 5\end{array}\right]\left[\begin{array}{c}\mathbf{V}_1\\ \mathbf{V}_2\\ \hline \mathbf{V}_3\end{array}\right] = \left[\begin{array}{c|c}\mathbf{P} & \mathbf{Q}\\ - & -\\ \hline \mathbf{M} & \mathbf{N}\end{array}\right]\left[\begin{array}{c}\mathbf{V}_1\\ \mathbf{V}_2\\ \hline \mathbf{V}_3\end{array}\right]$$

$$\begin{bmatrix}\mathbf{I}_1\\ \mathbf{I}_2\end{bmatrix} = \begin{bmatrix}\mathbf{P} - \mathbf{Q}\ \mathbf{N}^{-1}\ \mathbf{M}\end{bmatrix}\begin{bmatrix}\mathbf{V}_1\\ \mathbf{V}_2\end{bmatrix}$$

$$= \left\{\begin{bmatrix}s+3 & -s\\ -s & s+2\end{bmatrix} - \frac{1}{5}\begin{bmatrix}2\\ 1\end{bmatrix}\begin{bmatrix}2 & 1\end{bmatrix}\right\}\begin{bmatrix}\mathbf{V}_1\\ \mathbf{V}_2\end{bmatrix}$$

$$= \left\{\begin{bmatrix}s+3 & -s\\ -s & s+2\end{bmatrix} - \begin{bmatrix}\frac{4}{5} & \frac{2}{5}\\ \frac{2}{5} & \frac{1}{5}\end{bmatrix}\right\}\begin{bmatrix}\mathbf{V}_1\\ \mathbf{V}_2\end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix}s+2.2 & -(s+0.4)\\ -(s+0.4) & s+1.8\end{bmatrix}$$

R.P 7.3

Determine for the circuit shown in Fig. R.P. 7.3(a): (a) Y_1, Y_2, Y_3 and g_m in terms of **y** parameters. (b) Repeat the problem if the current source is connected across Y_3 with the arrow pointing to the left.

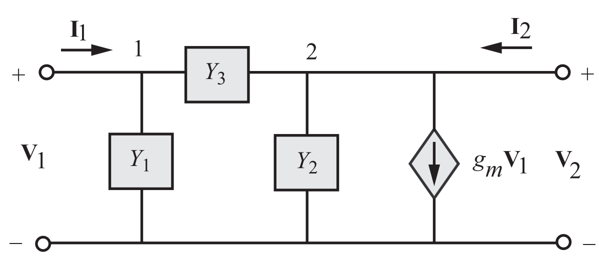


Figure R.P. 7.3(a)

SOLUTION

(a) Refering Fig. R.P. 7.3(a), the node equations are:

At node 1

$$\begin{aligned}\mathbf{I}_1 &= Y_1\mathbf{V}_1 + (\mathbf{V}_1 - \mathbf{V}_2)Y_3\\ &= \mathbf{V}_1(Y_1 + Y_3) - Y_3Y_2 \qquad (7.64)\end{aligned}$$

At node 2

$$\begin{aligned}\mathbf{I}_2 &= g_m\mathbf{V}_1 + \mathbf{V}_2Y_2 + (\mathbf{V}_2 - \mathbf{V}_1)Y_3\\ &= (g_m - Y_3)\mathbf{V}_1 + (Y_2 + Y_3)\mathbf{V}_2 \qquad (7.65)\end{aligned}$$

Then from equations (7.64) and (7.65),

$$\mathbf{y}_{11} = Y_1 + Y_3; \quad \mathbf{y}_{12} = -Y_3$$
$$\mathbf{y}_{21} = g_m - Y_3; \quad \mathbf{y}_{22} = Y_2 + Y_3$$

Solving,

$$Y_3 = -\mathbf{y}_{12}; \quad Y_1 = \mathbf{y}_{11} + \mathbf{y}_{12}$$
$$Y_2 = \mathbf{y}_{22} + \mathbf{y}_{12}; \quad g_m = \mathbf{y}_{21} - \mathbf{y}_{12}$$

(b) Making the changes as given in the problem, we get the circuit shown in Fig R.P. 7.3(b).

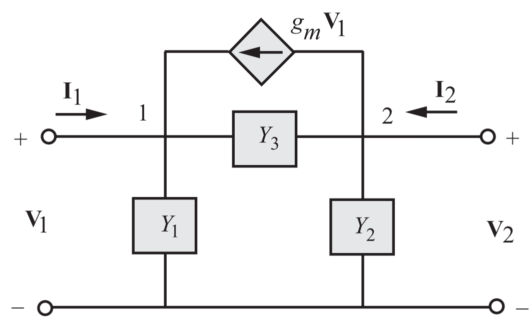


Figure R.P. 7.3(b)

Node equations : At node 1

$$\begin{aligned}\mathbf{I}_1 &= Y_1\mathbf{V}_1 + (\mathbf{V}_1 - \mathbf{V}_2)Y_3 - g_m\mathbf{V}_1 \\ &= (Y_1 + Y_3 - g_m)\mathbf{V}_1 - Y_3\mathbf{V}_2 \end{aligned} \tag{7.66}$$

At node 2,

$$\begin{aligned}\mathbf{I}_2 &= \mathbf{V}_2Y_2 + (\mathbf{V}_2 - \mathbf{V}_1)Y_3 + g_m\mathbf{V}_1 \\ &= (g_m - Y_3)\mathbf{V}_1 + \mathbf{V}_2(Y_2 + Y_3) \end{aligned} \tag{7.67}$$

From equations (7.66) and (7.67),

$$\mathbf{y}_{11} = Y_1 + Y_3 - g_m; \quad \mathbf{y}_{12} = -Y_3$$
$$\mathbf{y}_{21} = g_m - Y_3; \quad \mathbf{y}_{22} = Y_2 + Y_3$$

Solving,

$$Y_3 = -\mathbf{y}_{12}; \quad Y_2 = \mathbf{y}_{22} - \mathbf{y}_{12}$$
$$g_m = \mathbf{y}_{21} - \mathbf{y}_{12}$$
$$\begin{aligned} Y_1 &= \mathbf{y}_{11} - Y_3 + g_m \\ &= \mathbf{y}_{11} + \mathbf{y}_{12} + \mathbf{y}_{21} - \mathbf{y}_{12} = \mathbf{y}_{11} - \mathbf{y}_{21}\end{aligned}$$

R.P 7.4

Complete the table given as part of Fig. R.P. 7.4. Also find the values for **y** parameters.

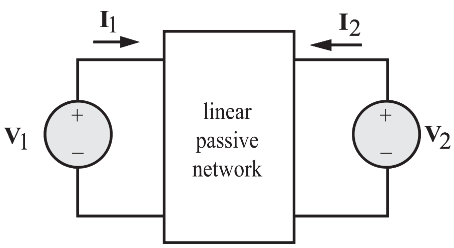


Figure R.P. 7.4

Table

Sl.no	$\mathbf{V}_1$	$\mathbf{V}_2$	$\mathbf{I}_1$	$\mathbf{I}_2$
1	50	100	-1	27
2	100	50	7	24
3	200	0	–	–
4	–	–	20	0
5	–	–	10	30

SOLUTION

From article 7.2,

$$\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{y}_{11} & \mathbf{y}_{12} \\ \mathbf{y}_{21} & \mathbf{y}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix}$$

Substituting the values from rows 1 and 2,

$$\begin{bmatrix} -1 & 7 \\ 27 & 24 \end{bmatrix} = \begin{bmatrix} \mathbf{y}_{11} & \mathbf{y}_{12} \\ \mathbf{y}_{21} & \mathbf{y}_{22} \end{bmatrix} \begin{bmatrix} 50 & 100 \\ 100 & 50 \end{bmatrix}$$

Post multiplying by $[\mathbf{V}]^{-1}$,

$$\begin{bmatrix} \mathbf{y}_{11} & \mathbf{y}_{12} \\ \mathbf{y}_{21} & \mathbf{y}_{22} \end{bmatrix} = \begin{bmatrix} -1 & 7 \\ 27 & 24 \end{bmatrix} \begin{bmatrix} 50 & 100 \\ 100 & 50 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 0.1 & -0.06 \\ 0.14 & 0.2 \end{bmatrix}$$

For row 3:

$$\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 0.1 & -0.06 \\ 0.14 & 0.2 \end{bmatrix} \begin{bmatrix} 200 \\ 0 \end{bmatrix} = \begin{bmatrix} 20 \\ 28 \end{bmatrix}$$

For row 4:

$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} 0.1 & -0.06 \\ 0.14 & 0.2 \end{bmatrix}^{-1} \begin{bmatrix} 20 \\ 0 \end{bmatrix} = \begin{bmatrix} 140.84 \\ -98.59 \end{bmatrix}$$

For row 5:

$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} 0.1 & -0.06 \\ 0.14 & 0.2 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 30 \end{bmatrix} = \begin{bmatrix} 133.8 \\ 56.338 \end{bmatrix}$$

R.P 7.5

Find the condition on a and b for reciprocity for the network shown in Fig. R.P. 7.5.

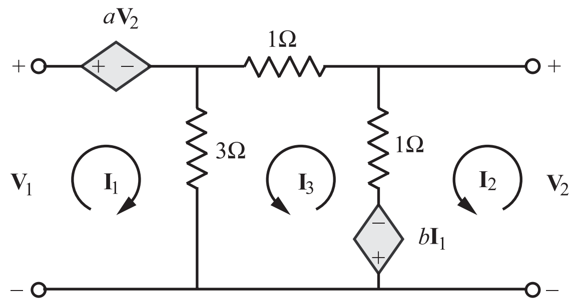


Figure R.P. 7.5

SOLUTION

The loop equations are

$$\mathbf{V}_1 - a\mathbf{V}_2 = 3(\mathbf{I}_1 + \mathbf{I}_3) \tag{7.68}$$

$$\mathbf{V}_2 = (\mathbf{I}_2 - \mathbf{I}_3) - b\mathbf{I}_1 \tag{7.69}$$

$$\begin{aligned} \mathbf{I}_3 &= \mathbf{V}_2 - (\mathbf{V}_1 - a\mathbf{V}_2) \\ &= (1+a)\mathbf{V}_2 - \mathbf{V}_1 \end{aligned} \tag{7.70}$$

Substituting equation (7.70) in equations (7.68) and (7.69),

$$\mathbf{V}_1 - a\mathbf{V}_2 = 3\mathbf{I}_1 + 3(1+a)\mathbf{V}_2 - 3\mathbf{V}_1$$

$$\Rightarrow \quad 4\mathbf{V}_1 = (3+4a)\mathbf{V}_2 + 3\mathbf{I}_1 \tag{7.71}$$

$$\mathbf{V}_2 = \mathbf{I}_2 - [(1+a)\,\mathbf{V}_2 - \mathbf{V}_1] - b\mathbf{I}_1$$

$$\Rightarrow \quad -\mathbf{V}_1 + (2+a)\,\mathbf{V}_2 = -b\mathbf{I}_1 + \mathbf{I}_2 \tag{7.72}$$

Putting equations (7.71) and (7.72) in matrix form and solving

$$\begin{aligned} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} &= \begin{bmatrix} 4 & -(3+4a) \\ -1 & 2+a \end{bmatrix}^{-1} \begin{bmatrix} 3 & 0 \\ -b & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} \\ &= \frac{1}{\Delta} \begin{bmatrix} 2+a & 3+4a \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ -b & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} \\ &= \frac{1}{\Delta} \begin{bmatrix} M & 3+4a \\ 3-4b & N \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} \end{aligned}$$

For reciprocity,

$$3 + 4a = 3 - 4b$$

Therefore, $\qquad a = -b$

R.P 7.6

For what value of a is the circuit reciprocal? Also find **h** parameters.

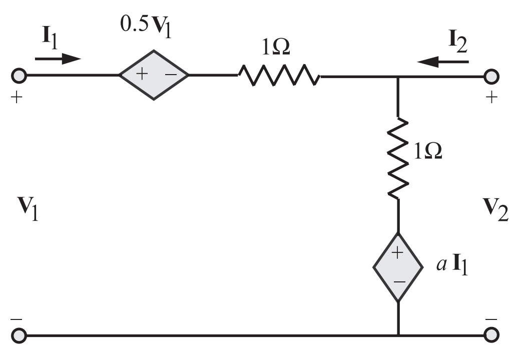


Figure R.P. 7.6

SOLUTION

The node equations are

$$\mathbf{V}_1 - 0.5\mathbf{V}_1 - \mathbf{I}_1 = \mathbf{V}_2$$
$$\mathbf{V}_2 = (\mathbf{I}_1 + \mathbf{I}_2)2 + a\mathbf{I}_1$$
$$\mathbf{h} = \begin{bmatrix} 0.5 & 0 \\ 0 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 2+a & -1 \end{bmatrix}$$
$$= \frac{1}{\Delta} \begin{bmatrix} -2 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2+a & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 2 \\ \dfrac{2+a}{2} & 0.5 \end{bmatrix} \qquad (\Delta = -1)$$

For reciprocity,

$$\mathbf{h}_{12} = -\mathbf{h}_{21}$$
$$\Rightarrow \qquad 2 = \frac{2+a}{2}$$
$$\Rightarrow \qquad -4 = 2 + a; \quad a = 2$$

Therefore $$\mathbf{h} = \begin{bmatrix} 2 & 2 \\ 2 & 0.5 \end{bmatrix}$$

R.P 7.7

Find $\mathbf{y}_{12}$ and $\mathbf{y}_{21}$ for the network shown in Fig. R.P. 7.7 for n = 10. What is the value of n for the network to be reciprocal?

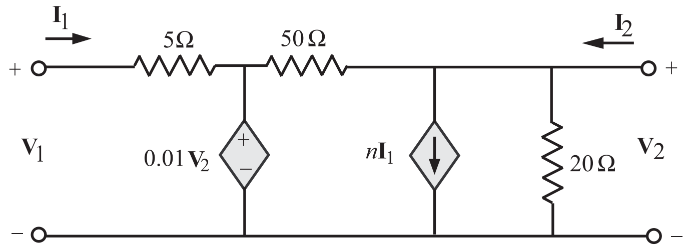


Figure R.P. 7.7

SOLUTION

Equations for $\mathbf{I}_1$ and $\mathbf{I}_2$ are

$$\mathbf{I}_1 = \frac{\mathbf{V}_1 - 0.01\mathbf{V}_2}{5}$$

$$= 0.2\mathbf{V}_1 - 0.002\mathbf{V}_2 \tag{7.73}$$

$$\mathbf{I}_2 = \frac{\mathbf{V}_2}{20} + n\mathbf{I}_1 + \frac{\mathbf{V}_2 - 0.01\mathbf{V}_2}{50} \tag{7.74}$$

Substituting the value of $\mathbf{I}_1$ from equation (7.73) in equation (7.74), we get

$$\mathbf{I}_2 = n(0.2\mathbf{V}_1 - 0.002\mathbf{V}_2) + \frac{\mathbf{V}_2}{20} + \frac{\mathbf{V}_2 - 0.01\mathbf{V}_2}{20} \tag{7.75}$$

Simplifying the above equation with n = 10,

$$\mathbf{I}_2 = 2\mathbf{V}_1 + 0.0498\mathbf{V}_2 \tag{7.76}$$

From equation (7.73), $\mathbf{y}_{12} = -0.002$

and from equation (7.75), $\mathbf{y}_{21} = 0.2n$

For reciprocity $\mathbf{y}_{12} = \mathbf{y}_{21}$

$\Rightarrow$ $-0.002 = 0.2n$

Hence, $n = -0.01$

R.P 7.8

Find $\mathbf{T}$ parameters (ABCD) for the two-port network shown in Fig. R.P. 7.8.

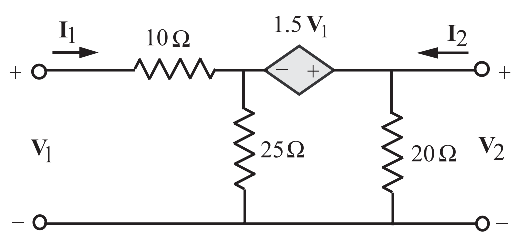


Figure R.P. 7.8

SOLUTION

Network equations are

$$\mathbf{V}_1 - 10\mathbf{I}_1 = \mathbf{V}_2 - 1.5\mathbf{V}_1 \tag{7.77}$$

$$\mathbf{I}_1 - \frac{\mathbf{V}_2 - 1.5\mathbf{V}_1}{25} + \mathbf{I}_2 - \frac{\mathbf{V}_2}{20} = 0 \tag{7.78}$$

Simplifying,

$$2.5\mathbf{V}_1 - 10\mathbf{I}_1 = \mathbf{V}_2$$

$$0.06\mathbf{V}_1 + \mathbf{I}_1 = 0.09\mathbf{V}_2 - \mathbf{I}_2$$

In matrix form,

$$\begin{bmatrix} 2.5 & -10 \\ 0.06 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{I}_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0.09 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{V}_2 \\ -\mathbf{I}_2 \end{bmatrix}$$

Therefore

$$\mathbf{T} = \begin{bmatrix} 2.5 & -10 \\ 0.06 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0.09 & 1 \end{bmatrix} = \begin{bmatrix} 0.613 & 3.23 \\ 0.053 & 0.806 \end{bmatrix}$$

R.P 7.9

(a) Find **T** parameters for the active two port network shown in Fig. R.P. 7.9.

(b) Find new **T** parameters if a 20 Ω resistor is connected across the output.

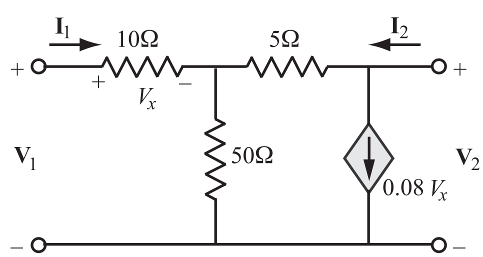


Figure R.P. 7.9

SOLUTION

(a) With $V_x = 10\mathbf{I}_1$,

$$0.08V_x = 0.8\mathbf{I}_1$$

Therefore,

$$\mathbf{V}_1 - 10\mathbf{I}_1 = \mathbf{V}_2 - 5(\mathbf{I}_2 - 0.08V_x)$$

$$= \mathbf{V}_2 - 5\mathbf{I}_2 + 4\mathbf{I}_1 \tag{7.79}$$

$$\mathbf{I}_1 + \mathbf{I}_2 - 0.8\mathbf{I}_1 = \frac{\mathbf{V}_1 - 10\mathbf{I}_1}{50} \tag{7.80}$$

Simplying the equations (7.79) and (7.80),we get

$$\mathbf{V}_1 - 14\mathbf{I}_1 = \mathbf{V}_2 - 5\mathbf{I}_2$$

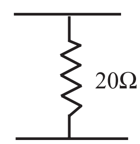
and

$$-\mathbf{V}_1 + 20\mathbf{I}_1 = -50\mathbf{I}_2$$

Therefore,

$$\mathbf{T} = \begin{bmatrix} 1 & -14 \\ -1 & 20 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 5 \\ 0 & 50 \end{bmatrix} = \begin{bmatrix} 3.33 & 133.33 \\ 0.167 & 9.17 \end{bmatrix}$$

(b) Treating 20 Ω across the output as a second T network for which



$$\mathbf{T} = \begin{bmatrix} 1 & 0 \\ \frac{1}{20} & 1 \end{bmatrix}$$

Then new **T**-parameters,

$$\mathbf{T} = \begin{bmatrix} 3.33 & 133.33 \\ 0.167 & 9.17 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1}{20} & 1 \end{bmatrix} = \begin{bmatrix} 10 & 133.33 \\ 0.625 & 9.17 \end{bmatrix}$$

R.P 7.10

Obtain **z** parameters for the network shown in Fig. R.P. 7.10.

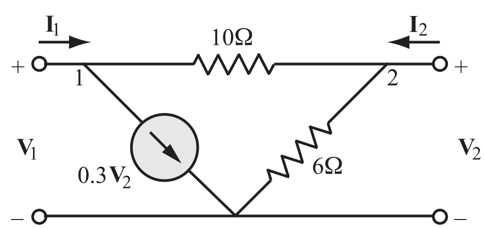


Figure R.P. 7.10

SOLUTION

At node 1,

$$\begin{aligned} \mathbf{V}_1 &= (\mathbf{I}_1 - 0.3\mathbf{V}_2)10 + \mathbf{V}_2 \\ &= 10\mathbf{I}_1 - 2\mathbf{V}_2 \end{aligned} \tag{7.81}$$

At node 2,

$$\mathbf{V}_2 = \left(\mathbf{I}_2 - \frac{\mathbf{V}_2}{6}\right)10 + \mathbf{V}_1 = 10\mathbf{I}_2 + \mathbf{V}_1 - \frac{5}{3}\mathbf{V}_2$$

$$\Rightarrow \quad \frac{8}{3}\mathbf{V}_2 = \mathbf{V}_1 + 10\mathbf{I}_2 \tag{7.82}$$

Putting in matrix form,

$$\begin{bmatrix} 1 & 2 \\ 1 & \dfrac{-8}{3} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & -10 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix}$$

Therefore,

$$\mathbf{z} = \begin{bmatrix} 1 & 2 \\ 1 & \dfrac{-8}{3} \end{bmatrix}^{-1} \begin{bmatrix} 10 & 0 \\ 0 & -10 \end{bmatrix} = \begin{bmatrix} 5.71 & -4.286 \\ 2.143 & 2.143 \end{bmatrix}$$

R.P 7.11

Obtain **z** and **y** parameters for the network shown in Fig. R.P. 7.11.

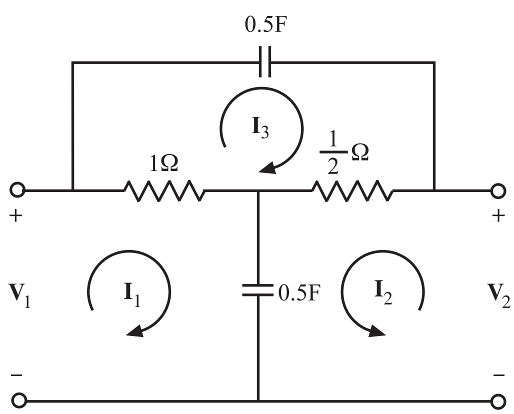


Figure R.P. 7.11

SOLUTION

For the meshes indicated, the equations in matrix form is

$$\left[\begin{array}{c} \mathbf{V}_1 \\ \mathbf{V}_2 \\ \hline 0 \end{array}\right] = \left[\begin{array}{cc:c} 1+\dfrac{2}{s} & \dfrac{2}{s} & -1 \\ \dfrac{2}{s} & \dfrac{1}{2}+\dfrac{2}{s} & \dfrac{1}{2} \\ \hdashline -1 & \dfrac{1}{2} & \dfrac{3}{2}+\dfrac{2}{s} \end{array}\right] \left[\begin{array}{c} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \hline \mathbf{I}_3 \end{array}\right]$$

By matrix partitioning,

$$\mathbf{z} = \begin{bmatrix} \dfrac{s+2}{s} & \dfrac{2}{s} \\ \dfrac{2}{s} & \dfrac{s+4}{2s} \end{bmatrix} - \left[\dfrac{2s}{3s+4}\right] \begin{bmatrix} -1 \\ \dfrac{1}{2} \end{bmatrix} \begin{bmatrix} -1 & \dfrac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{s+2}{s} & \dfrac{2}{s} \\ \dfrac{2}{s} & \dfrac{s+4}{2s} \end{bmatrix} - \left[\dfrac{2s}{3s+4}\right] \begin{bmatrix} 1 & \dfrac{-1}{2} \\ -\dfrac{1}{2} & \dfrac{1}{4} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{s+2}{s} & \dfrac{2}{s} \\ \dfrac{2}{s} & \dfrac{s+4}{2s} \end{bmatrix} - \begin{bmatrix} \dfrac{2s}{3s+4} & \dfrac{-s}{3s+4} \\ \dfrac{-s}{3s+4} & \dfrac{-8s}{3s+4} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{s^2+10s+8}{s(3s+4)} & \dfrac{s^2+6s+8}{s(3s+4)} \\ \dfrac{s^2+6s+8}{s(3s+4)} & \dfrac{s^2+8s+8}{s(3s+4)} \end{bmatrix}$$

R.P 7.12

Find **z** and **y** parameters at $\omega = 10^8$ rad/sec for the transistor high frequency equivalent circuit shown in Fig. R.P. 7.12.

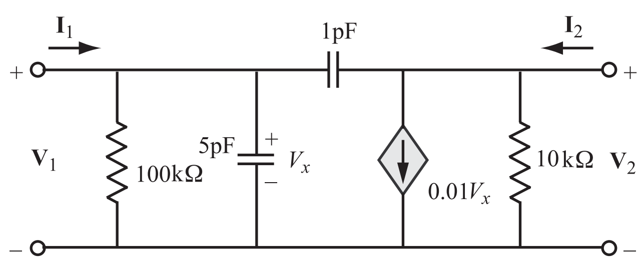


Figure R.P. 7.12

SOLUTION

In the circuit, $V_x = \mathbf{V}_1$. Therefore the node equations are

$$\mathbf{I}_1 = (10^{-5} + j6 \times 10^{-4})\mathbf{V}_1 - j10^{-4}\mathbf{V}_2$$
$$\mathbf{I}_2 = -j10^{-4}\mathbf{V}_1 + 0.01\mathbf{V}_1 + 10^{-4}(1+j)\mathbf{V}_2$$

Simplifying the above equations,

$$\mathbf{I}_1 = 10^{-4}[(0.1 + j6)\mathbf{V}_1 - j1\mathbf{V}_2]$$
$$\mathbf{I}_2 = 10^{-4}[(100 + j1)\mathbf{V}_1 + (1 + j)\mathbf{V}_2]$$

Therefore, $$\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 0.1 + j6 & -j1 \\ 100 - j1 & 1 + j1 \end{bmatrix} \times 10^{-4} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix}$$

$$\omega C_1 = 10^8 \times 5 \times 10^{-12} = 5 \times 10^{-4}$$
$$\omega C_2 = 10^8 \times 10^{-12} = 10^{-4}$$
$$\Delta = 10^{-8}[(0.1 + j6)(1 + j) + (100 - j1)(j1)]$$
$$= 10^{-8} \times 106.213 \underline{/92.64^\circ}$$

Therefore, $$\mathbf{y} = \begin{bmatrix} 6 \underline{/89^\circ} & -j1 \\ 100 \underline{/-0.6^\circ} & \sqrt{2} \underline{/45^\circ} \end{bmatrix} \times 10^{-4}$$

Then, $$\mathbf{z} = \mathbf{y}^{-1} = \begin{bmatrix} \sqrt{2} \underline{/45^\circ} & j1 \\ 100 \underline{/-180.6^\circ} & 6 \underline{/89^\circ} \end{bmatrix} \times 10^{-4} \div \Delta$$

$$= \begin{bmatrix} \sqrt{2} \underline{/45^\circ} & j1 \\ 100 \underline{/-180.6^\circ} & 6 \underline{/89^\circ} \end{bmatrix} \times \frac{10^{-4}}{10^{-8} \times 106.213 \underline{/92.64^\circ}}$$

$$= \begin{bmatrix} 133.15 \underline{/-47.64^\circ} & 94.16 \underline{/-2.64^\circ} \\ 94.16 \underline{/86.8^\circ} & 565 \underline{/-31.6^\circ} \end{bmatrix}$$

R.P 7.13

Obtain $\mathbf{T}_A$, $\mathbf{T}_B$, $\mathbf{T}_C$ for the network shown in Fig. R.P. 7.13 and obtain overall $\mathbf{T}$.

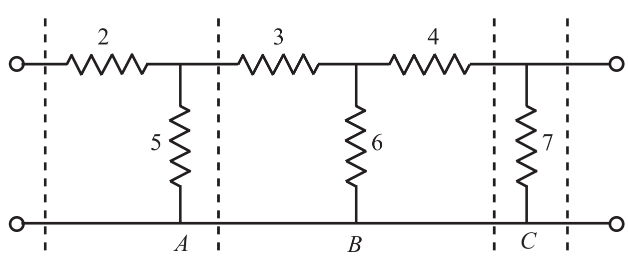
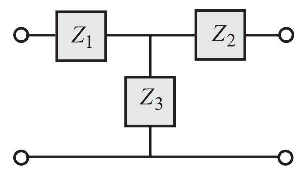


Figure R.P. 7.13

SOLUTION

Using the equation for **T**-parameters for a T-network

$$\mathbf{A} = \frac{Z_1 + Z_3}{Z_3}; \quad \mathbf{B} = \frac{\sum Z_1 Z_3}{Z_3}; \quad \mathbf{C} = \frac{1}{Z_3}; \quad \mathbf{D} = \frac{Z_2 + Z_3}{Z_3}$$



We have for A $$\mathbf{T}_A = \begin{bmatrix} \frac{7}{5} & 2 \\ \frac{1}{5} & 1 \end{bmatrix}$$

For B, $$\mathbf{T}_B = \begin{bmatrix} \frac{9}{6} & \frac{54}{6} \\ \frac{1}{6} & \frac{10}{6} \end{bmatrix}$$

For C, $$\mathbf{T}_C = \begin{bmatrix} 1 & 0 \\ \frac{1}{7} & 1 \end{bmatrix}$$

Overall **T** : $$\mathbf{T} = [\mathbf{T}_A][\mathbf{T}_B][\mathbf{T}_C] = \begin{bmatrix} 4.709 & 15.93 \\ 0.962 & 3.46 \end{bmatrix}$$

This dervation is left as an exercise to the reader.

Verification:

Using T $-$ Δ transformation, that is changing T (3, 4, 6) of Fig. R.P. 7.13, in to Δ,

$$Z_{xz} = 13.5$$
$$Z_{xy} = 9$$
$$Z_{yz} = 18$$

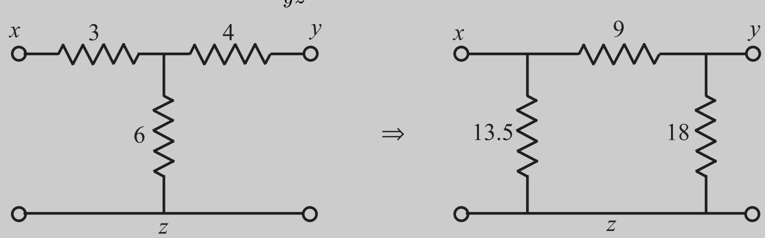


Figure R.P.7.13(a)

Putting the values in the circuit of Fig. R.P. 7.13, we get

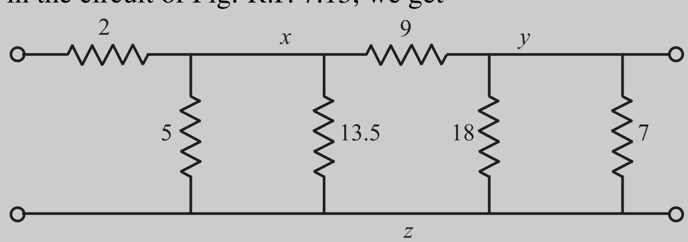


Figure R.P.7.13(b)

Reducing, the above circuit, we get the circuit shown in Fig. R.P. 7.13c.

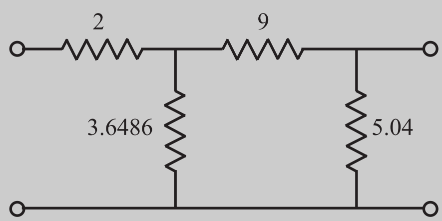


Figure R.P. 7.13(c)

Converting the circuit into T, we get the circuit shown in Fig. R.P. 7.13(d).
Now from Fig. R.P. 7.13(d),

$$\mathbf{A} = \frac{3.8564 + 1.0396}{1.0396} = 4.709$$

$$\mathbf{B} = \frac{1.0396(3.8564 + 2.5644) + 3.8564 \times 2.5644}{1.0396}$$

$$= 15.93\ \Omega$$

$$\mathbf{C} = \frac{1}{Z_p} = \frac{1}{1.0396} = 0.962$$

$$\mathbf{D} = \frac{2.5644 + 1.0396}{1.0396} = 3.46$$

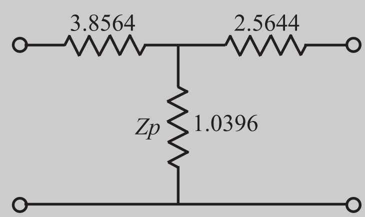


Figure R.P. 7.13(d)

Exercise Problems

E.P 7.1

Find the **y** parameters for the network shown in Fig. E.P. 7.1.

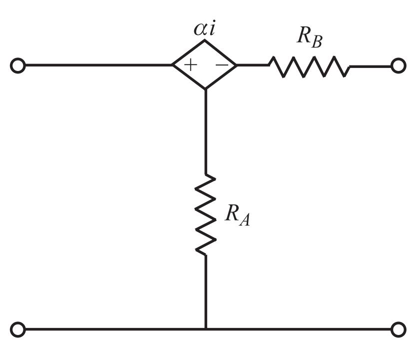


Figure E.P. 7.1

Ans: $\mathbf{y_{11}} = \dfrac{\boldsymbol{\alpha + R_A + R_B}}{\boldsymbol{R_A R_B}}, \mathbf{y_{12}} = \dfrac{\mathbf{-1}}{\boldsymbol{R_B}}, \mathbf{y_{21}} = \dfrac{\boldsymbol{-(\alpha + R_A)}}{\boldsymbol{R_A R_B}}, \mathbf{y_{22}} = \dfrac{\mathbf{1}}{\boldsymbol{R_B}}.$

E.P 7.2

Find the **z** parameters for the network shown in Fig. E.P. 7.2.

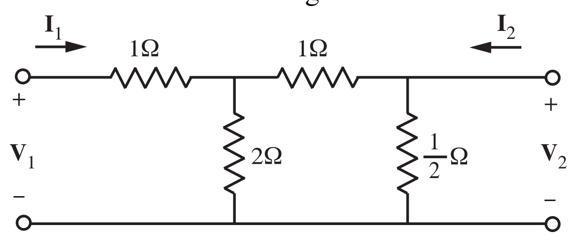


Figure E.P. 7.2

Ans: $\mathbf{z_{11} = \frac{13}{7}\Omega}$, $\mathbf{z_{12} = \frac{2}{7}\Omega}$, $\mathbf{z_{21} = \frac{2}{7}\Omega}$, $\mathbf{z_{22} = \frac{3}{7}\Omega}$.

E.P 7.3

Find the **h** parameters for the network shown in Fig. E.P. 7.3.

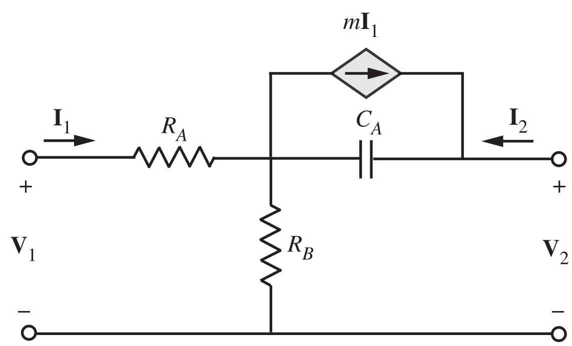


Figure E.P. 7.3

Ans:
$$\mathbf{h_{11}} = \frac{sC_AR_AR_B + R_A + (1-m)R_B}{sC_AR_B + 1}, \quad \mathbf{h_{21}} = \frac{sC_AR_B + m}{sC_AR_B + 1}.$$
$$\mathbf{h_{12}} = \frac{sC_AR_B}{sC_AR_B + 1}, \quad \mathbf{h_{22}} = \frac{sC_A}{sC_AR_B + 1}.$$

E.P 7.4

Find the **y** parameters for the network shown in Fig. E.P. 7.4.

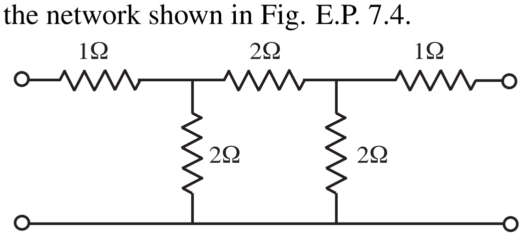


Figure E.P. 7.4

Ans: $\mathbf{y_{11} = y_{22} = \frac{7}{15}S}$, $\mathbf{y_{12} = y_{21} = \frac{-2}{15}S}$.

E.P 7.5

Find the **y** parameters for the network shown in Fig. E.P. 7.5. Give the result in s domain.

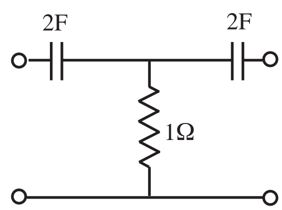


Figure E.P. 7.5

Ans: $\mathbf{y_{11} = y_{22} = \dfrac{2s(2s+1)}{4s+1}}$, $\quad \mathbf{y_{12} = y_{21} = \dfrac{-4s^2}{4s+1}}$.

E.P 7.6

Obtain the **y** parameters for the network shown in Fig. E.P. 7.6.

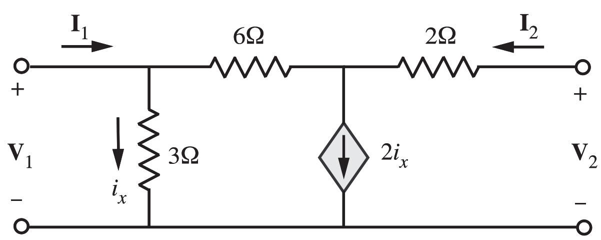


Figure E.P. 7.6

Ans: $\mathbf{y_{11} = 0.625\ S}$, $\quad \mathbf{y_{12} = -0.125\ S}$, $\quad \mathbf{y_{21} = 0.375\ S}$, $\quad \mathbf{y_{22} = 0.125\ S}$.

E.P 7.7

Find the **z** parameters for the two-port network shown in Fig. E.P. 7.7. Keep the result in s domain.

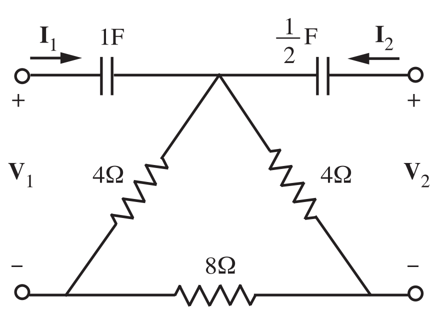


Figure E.P. 7.7

Ans: $\mathbf{z_{11} = \dfrac{2s+1}{s}}$, $\quad \mathbf{z_{12} = z_{21} = 2}$, $\quad \mathbf{z_{22} = \dfrac{2s+2}{s}}$.

E.P 7.8

Find the **h** parameters for the network shown in Fig. E.P. 7.8. Keep the result in s domain.

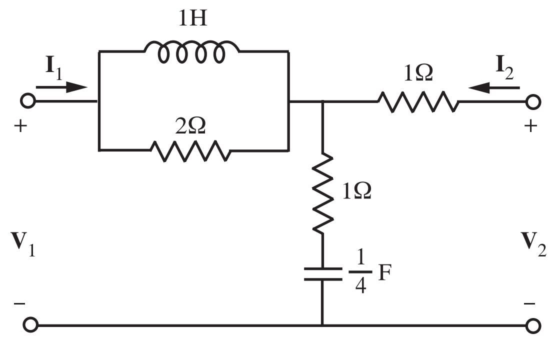


Figure E.P. 7.8

Ans: $\mathbf{h_{11}} = \dfrac{5s+4}{2(s+2)}, \quad \mathbf{h_{12}} = \dfrac{s+4}{2(s+2)}, \quad \mathbf{h_{21}} = \dfrac{-(s+4)}{2(s+2)}, \quad \mathbf{h_{22}} = \dfrac{s}{2(s+2)}.$

E.P 7.9

Find the transmission parameters for the network shown in Fig. E.P. 7.9. Keep the result in s domain.

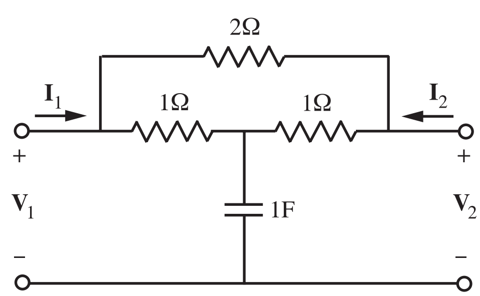


Figure E.P. 7.9

Ans: $\mathbf{A} = \dfrac{3s+4}{s+4}, \quad \mathbf{B} = \dfrac{2s+4}{s+4}, \quad \mathbf{C} = \dfrac{4s}{s+4}, \quad \mathbf{D} = \dfrac{3s+4}{s+4}.$

E.P 7.10

For the same network described in Fig. E.P. 7.9, find the **h** parameters using the defining equations. Then verify the result obtained using conversion formulas.

Ans: $\mathbf{h_{11}} = \dfrac{2s+4}{3s+4}, \quad \mathbf{h_{12}} = \dfrac{s+4}{3s+4}, \quad \mathbf{h_{21}} = \dfrac{-(s+4)}{3s+4}, \quad \mathbf{h_{22}} = \dfrac{4s}{3s+4}.$

E.P 7.11

Select the values of R_A, R_B, and R_C in the circuit shown in Fig. E.P. 7.11 so that **A** =1, **B** = 34 Ω, **C** = 20 mS and **D** = 1.4.

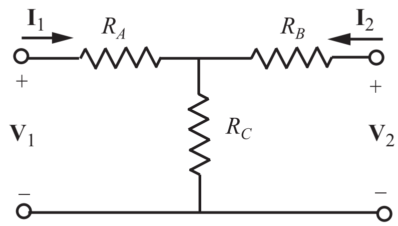


Figure E.P. 7.11

Ans: $\boldsymbol{R_A = 10\Omega}$, $\boldsymbol{R_B = 20\Omega}$, $\boldsymbol{R_C = 50\Omega}$.

E.P 7.12

Find the s domain expression for the **h** parameters of the circuit in E.P. 7.12.

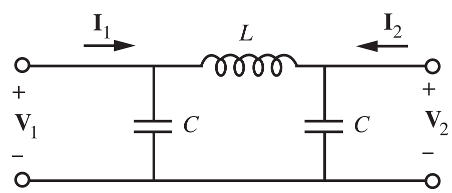


Figure E.P. 7.12

Ans: $$\mathbf{h_{11}} = \frac{\frac{1}{C}s}{s^2 + \frac{1}{LC}}, \qquad \mathbf{h_{12}} = \mathbf{h_{21}} = \frac{-\frac{1}{LC}}{s^2 + \frac{1}{LC}}, \qquad \mathbf{h_{22}} = \frac{Cs\left(s^2 + \frac{2}{LC}\right)}{s^2 + \frac{1}{LC}}.$$

E.P 7.13

Find the **y** parameters for the network shown in Fig. E.P. 7.13.

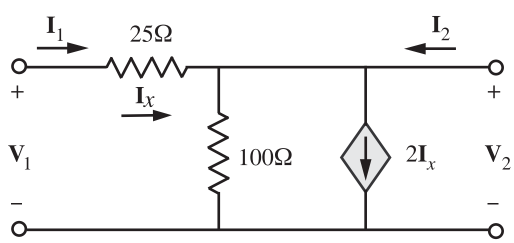


Figure E.P. 7.13

Ans: $\mathbf{y_{11}} = \mathbf{0.04S}$, $\mathbf{y_{12}} = \mathbf{-0.04S}$, $\mathbf{y_{21}} = \mathbf{0.04S}$, $\mathbf{y_{22}} = \mathbf{-0.03S}$.

E.P 7.14

Find the two-port parameters $\mathbf{h}_{12}$ and $\mathbf{y}_{12}$ for the network shown in Fig. E.P. 7.14.

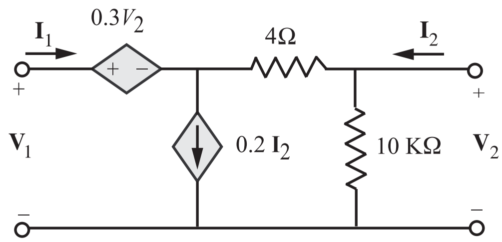


Figure E.P. 7.14

Ans: $\mathbf{h_{12} = 1.2}$, $\mathbf{y_{12} = 0.24S}$.

E.P 7.15

Find the **ABCD** parameters for the 4Ω resistor of Fig. E.P. 7.15. Also show that the **ABCD** parameters for a single 16Ω resistor can be obtained by **(ABCD)**^{4}.

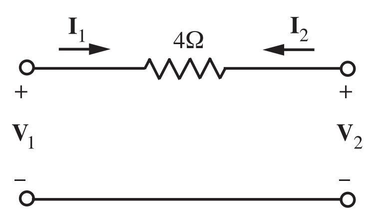


Figure E.P. 7.15

Ans: Verify your answer using the relation between the parameters.

E.P 7.16

For the T-network shown in Fig. E.P. 7.16. show that **AD** $-$ **BC** =1.

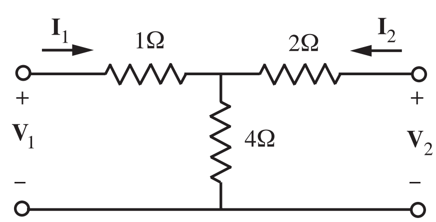


Figure E.P. 7.16

E.P 7.17

Find $\mathbf{y}_{21}$ for the network shown in Fig. E.P. 7.17.

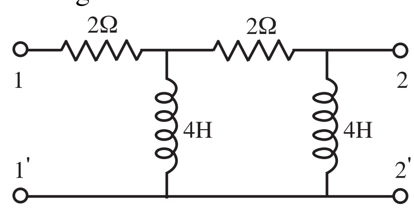


Figure E.P. 7.17

Ans: $\mathbf{y_{21}} = \dfrac{-s}{4s+1}$.

E.P 7.18

Determine the **y**-parameters for the network shown in Fig. E.P. 7.18.

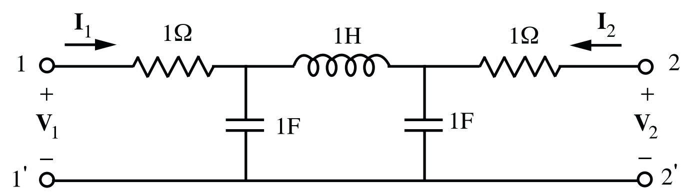


Figure E.P. 7.18

Ans: $\mathbf{y_{11}} = \dfrac{s^3+s^2+2s+1}{s(s^2+2)}, \quad \mathbf{y_{12}} = \mathbf{y_{21}} = \dfrac{-1}{s(s^2+2)}, \quad \mathbf{y_{22}} = \dfrac{s^3+s^2+2s+1}{s(s^2+2)}$.

E.P 7.19

Obtain the **h**-parameters for the network shown in Fig. 7.19.

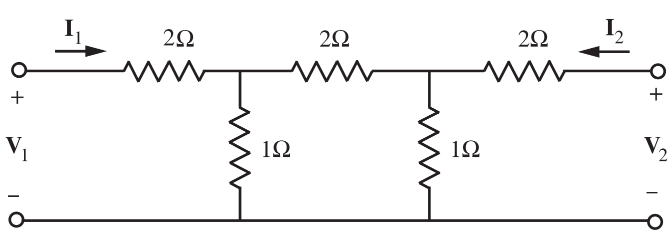


Figure E.P. 7.19

Ans: $\mathbf{h_{11}} = \dfrac{30}{11}\Omega, \qquad \mathbf{h_{21}} = \dfrac{-1}{11}, \qquad \mathbf{h_{12}} = \dfrac{1}{11}, \qquad \mathbf{h_{22}} = \dfrac{4}{11}℧$

E.P 7.20

The following equations are written for a two-port network. Find the transmission parameters for the network. (Hint: use relation between **y** and **T** parameters).

$$\mathbf{I_1 = 0.05V_1 - 0.4V_2 \qquad I_2 = -0.4V_1 + 0.1V_2}$$

E.P 7.21

Find the network shown in figure, determine the **z** and **y** parameters.

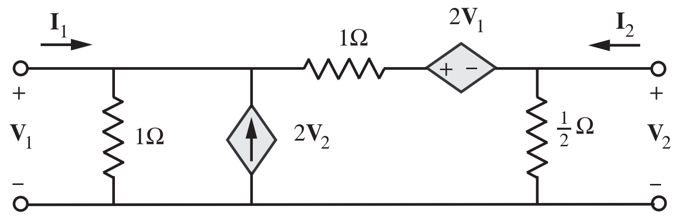


Figure E.P. 7.21

Ans: $\mathbf{y_{11}} = 4\mho^*$, $\quad \mathbf{y_{22}} = \mathbf{3}\mho$, $\quad \mathbf{y_{12}} = \mathbf{y_{21}} = -\mathbf{3}\mho$,

$\mathbf{z_{11}} = \mathbf{1\Omega}$, $\quad \mathbf{z_{22}} = \dfrac{\mathbf{4}}{\mathbf{3}}\mathbf{\Omega}$, $\quad \mathbf{z_{12}} = \mathbf{z_{21}} = \mathbf{1\Omega}$.

E.P 7.22

Determine the **z**, **y** and Transmission parameters of the network shown in Fig. 7.22.

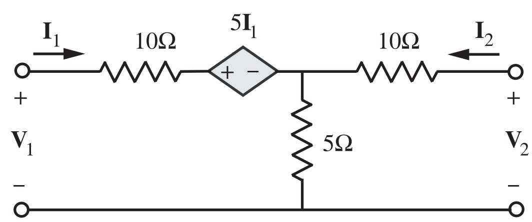


Figure E.P. 7.22

Ans: $\mathbf{y_{11}} = \dfrac{\mathbf{3}}{\mathbf{55}}\mho$, $\quad \mathbf{y_{12}} = \mathbf{y_{21}} = \dfrac{\mathbf{1}}{\mathbf{55}}\mho$, $\quad \mathbf{y_{22}} = \dfrac{\mathbf{4}}{\mathbf{55}}\mho$,

$\mathbf{z_{11}} = \mathbf{20\Omega}$, $\quad \mathbf{z_{12}} = \mathbf{z_{21}} = \mathbf{5\Omega}$ $\quad \mathbf{z_{22}} = \mathbf{15\Omega}$

$\mathbf{A} = \mathbf{55\Omega}$ $\quad \mathbf{B} = \mathbf{55\Omega}$ $\quad \mathbf{C} = \mathbf{0.2}\mho$, $\quad \mathbf{D} = \mathbf{3}$.

E.P 7.23

For the network shown in Fig. E.P. 7.23 determine **z** parameters.

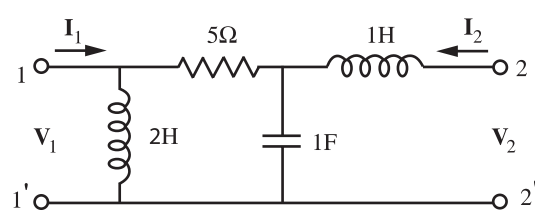


Figure E.P. 7.23

Ans: $\mathbf{z_{11}} = \dfrac{\mathbf{2}s(\mathbf{5}s+\mathbf{1})}{\mathbf{2}s^2+\mathbf{5}s+\mathbf{1}}$, $\mathbf{z_{12}} = \mathbf{z_{21}} = \dfrac{\mathbf{2}s}{\mathbf{2}s^2+\mathbf{5}s+\mathbf{1}}$, $\mathbf{z_{22}} = \dfrac{\mathbf{2}s^3+\mathbf{5}s^2+\mathbf{3}s+\mathbf{5}}{\mathbf{2}s^2+\mathbf{5}s+\mathbf{1}}$

* The unit ℧ and S are same

E.P 7.24

Determine the **y** parameters of the two-port network shown in Fig. E.P. 7.24.

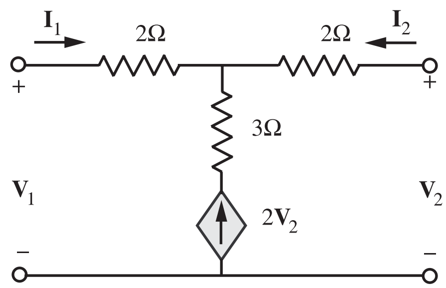


Figure E.P. 7.24

Ans: $\mathbf{y_{11}} = \dfrac{1}{4}℧$, $\mathbf{y_{21}} = \dfrac{-1}{4}℧$, $\mathbf{y_{12}} = \dfrac{-5}{4}℧$, $\mathbf{y_{22}} = \dfrac{-4}{3}℧$.